International Journal of Approximate Reasoning 166 (2024) 109126

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Moderated revision

Daniel Grimaldi [a,b], Maria Vanina Martinez [c,*], Ricardo O. Rodriguez [a,b]

[a] UBA, FCEyN, Departamento de Computacion, Int. Guiraldes 2160, Buenos Aires, 1428, Argentina
[b] UBA-CONICET, Instituto de Ciencias de la Computacion, Int. Guiraldes 2160, Buenos Aires, 1428, Argentina
[c] Artificial Intelligence Research Institute (IIIA-CSIC), Campus UAB, Bellaterra, 08193, Barcelona, Spain

ARTICLE INFO

Keywords:
Non-prioritized operators
Belief revision
Postulates and constructions

ABSTRACT

In this article, we provide a new kind of belief revision operator that we call Moderated Revision. At first glance, it is a non-prioritized operator that combines a basic classical AGM operator with a credibility-limited one. The underlying idea is this: when new observation μ is received, it is accepted but with doubts, i.e., uncertainty. We use a revision operator to model the accepted part and a credibility-limited one to represent uncertainty, whenever necessary. In the presence of uncertainty, a selection of the old knowledge balances the result of the revision through a disjunction, allowing the agent to accept part of the new observation and remain unsettled about the rest.

1. Introduction

In the different areas that study symbolic reasoning, there exist many different formalisms to represent and model the dynamics of knowledge, among which AGM [1] stands out, introducing belief contraction and revision operators. These operators model the basic decision-making of an agent when receiving as input a new observation. Screened Revision [2], Credibility-limited revision [3], shielded contraction [4], two-level credibility-limited revision [5] or filtered belief revision [6,7] extend these original operators by adding a choice mechanism, allowing the agent to accept or reject the observation before revising or contracting.

We argue that these mechanisms do not allow for proper reasoning in a variety of interesting situations such as the one following. Consider for example the case that, after winning the lottery, your ex-partner calls saying that s/he loves you and has always regretted breaking up with you; however, about a month ago s/he said that s/he did not love you anymore. You can definitely believe that s/he regrets the breaking up, but can not tell if s/he really loves you or if it is the case that s/he just wants to take advantage of your winning luck. In terms of knowledge representation, by considering the old knowledge obtained a month ago and the observation received by call, note that the new knowledge is not the result of rejecting the observation nor contracting the old knowledge, since the old knowledge is not the same as the new one and the new knowledge is not implied by the old one. It should not be the result of revising either, as the new knowledge does not imply the observation. Instead, the observation should be partially believed, while the rest of it becomes uncertain (neither believed nor disbelieved) based on the original knowledge.

In this work, we present a new operator for belief change called *moderated revision*. This proposal aims to add a feature of balance between certainty and complete doubt, where neither the piece of information nor its negation is believed by the agent. We consider this a more adequate and general model for belief change, where revision and contraction represent extreme situations for certainty and doubt, respectively. To do so, we combine a revision operator with a credibility-limited one, where its choice mechanism is used

* Corresponding author.
E-mail address: vmartinez@iiia.csic.es (M.V. Martinez).

https://doi.org/10.1016/j.ijar.2024.109126
Received 14 March 2023; Received in revised form 11 January 2024; Accepted 12 January 2024

Available online 17 January 2024

to model the possibility of doubt through a disjunction. This kind of combination is similar to the one seen in [8] for Promotion Operators.

This text is organized as follows. In Section 2, we present the framework of Katsuno and Mendelzon which our work is based on and discuss an alternative interpretation of the epistemic attitudes in terms of the Levi and Harper Identities. In Section 3, we adapt and generalize credibility-limited revision operators to our framework and present an important family for our proposal. In Section 4, we present our proposal, moderated revision: its motivation, constructive definition, axiomatization and a diverse genealogy. In Section 5, we compare our proposal with other known operators. Finally, in section 6, we summarize our results and present some future works. For better organization and to promote a more fluid reading, the proofs of the presented theorems will be included in an appendix at the end of the text.

2. Preliminaries

After the pioneering work of Alchourrón et al. [1], where the knowledge of the agent is represented as a set of formulas closed by logical consequence and the epistemic input is a formula in the language, different interpretations for the AGM model arise in order to be applied in a variety of contexts. Throughout this work, we consider the framework of Katsuno and Mendelzon [9] that differs from AGM in some key ideas: they work in a finitary propositional language $\mathcal{L}$, where both knowledge and observation are represented by formulas and, more importantly, both formulas are considered as parameters. These last technicalities allow us to treat them interchangeably as needed. In this section, the axiomatic approach of the AGM paradigm is briefly discussed in this context, including an interpretation of epistemic attitudes linked to the well-known Levi and Harper identities.

2.1. The Katsuno and Mendelzon's setting for AGM

In [9] Katsuno and Mendelzon proposed an alternative way for introducing the *AGM operators*. Their propositional approach consists of representing both the old knowledge and the observation by means of a formula each, instead of a set of formulas closed under consequence (a belief set) and a single formula. The essential idea behind this is to fix a way of representing any belief set K by a propositional formula ψ such that $K = \{\alpha \mid \psi \vDash \alpha\}$. This is always possible when the underlying propositional language $\mathcal{L}$ is finite. In particular, we say that a formula α is a *complete formula* if and only if for every $\varphi \in \mathcal{L}$, either $\alpha \vDash \varphi$ or $\alpha \vDash \neg\varphi$ and will denote $\bar{\alpha}$ as its canonical representation up to logical equivalence.

According to this assumption of this finitary model, they considered a revision operator $\psi * \mu$ where ψ is the knowledge of the agent and μ is the new observation and provided the following set of postulates.

Definition 2.1. A revision operator $*\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is defined by the following postulates for every $\psi, \mu \in \mathcal{L}$:

(R1) $\psi * \mu \vDash \mu$.
(R2) If $\psi \wedge \mu \nvDash \bot$ then $\psi * \mu \equiv \psi \wedge \mu$.
(R3) If $\mu \nvDash \bot$ then, $\psi * \mu \nvDash \bot$.
(R4) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$ then, $\psi_1 * \mu_1 \equiv \psi_2 * \mu_2$.
(R5) $(\psi * \mu) \wedge \nu \vDash \psi * (\mu \wedge \nu)$.
(R6) If $(\psi * \mu) \wedge \nu \nvDash \bot$ then $\psi * (\mu \wedge \nu) \vDash (\psi * \mu) \wedge \nu$.

Postulate **(R1)** is known as *success* and characterizes the main objective of the operator: the result must imply the observation. **(R2)** *vacuity* and **(R3)** *consistency* represent elemental constrains about the behavior of the operator. Vacuity states that the revision operator extends the conjunction operator (in the original framework, the extended operator is the expansion operator), while consistency forces the new knowledge to preserve the same consistency as the observation. Postulate **(R4)** is *extensionality* and it says that the revision operator is not influenced by the syntax of the formulas. When an operator satisfies **(R1)** $\sim$ **(R4)**, we may name it as "basic" revision. Katsuno and Mendelzon proved that these four postulates are equivalent in this finitary context to the original considered for Achourrón et al. They also proved that, if the previous postulates hold, **(R5)** and **(R6)** are equivalent to the original supplementary postulates. We will also use postulate **(R7)** as an alternative to the supplementary postulates:

(R7) If $(\psi * \mu) \wedge \nu \nvDash \bot$ then $(\psi * \mu) \wedge \nu \equiv \psi * (\mu \wedge \nu)$

In any case, when a revision operator satisfies all of these postulates we refer to it as "full" if the context needs it.

2.2. The AGM duality and their relation with the epistemic attitudes

When comes to the AGM model, revision and contraction operators arise, one as the dual of the other: given some prior knowledge and a new observation, revision is used to consistently append the new information to the knowledge, while contraction seeks to reduce the inference power of the knowledge in order to not imply the observation. Alchourrón et al. showed that the contraction and revision operators are interrelated, and this duality can be explicitly seen in Levi ([10]) and Harper's ([11]) identities:

Levi Identity: $\psi * \mu \equiv (\psi - \neg\mu) \wedge \mu$

Harper Identity: $\psi - \nu \equiv (\psi * \neg\nu) \vee \psi$

In terms of the three epistemic attitudes of the AGM framework [12] known as 'belief', 'disbelief' and 'unsettled', the success postulate of each operator establishes that: by applying revision operators, the agent believes in the new observation, or equivalently, disbelieves the negation of the new observation. Meanwhile, contraction makes the new observation uncertain, this is, if the agent consistently believed in the observation, after the contraction neither it nor its negation is believed. Revision can thus be seen as an operator pursuing *certainty* (either belief or disbelief), and contraction as an operator that pursues *doubt* (an unsettled state). In this work we seek to formalize the potential operators that lie in the middle of these extreme points.

Let us analyze the example given in the introduction in terms of epistemic attitudes. All the information at stake is composed of two propositions:

p: "Your ex-partner regrets the breakup".
q: "Your ex-partner loves you".
ψ: $\neg q$.
μ: $p \wedge q$.

Note that the behavior that we expect from the agent is neither to reject nor revise or contract the new observation: since the new observation is represented by a complete formula, there is one result for every revision by μ and contraction by $\neg\mu$, that is, $\psi * \mu = p \wedge q$ and $\psi - \neg\mu = (p \wedge q) \vee \neg q \equiv q \to p$. However, the knowledge we seek to incorporate, which is: "you can definitely believe that s/he regrets the breakup, but can not tell if s/he really loves you", is represented by formula p, which is between the revision and contraction operators: it is entailed by the knowledge resulting from the revision ($p \wedge q$) and it entails the knowledge resulting from the contraction ($q \to p$). We can understand this operator in a dual way, considering that the original knowledge was "moderately" revised, since p is now believed, or "moderately" contracted, considering that q is uncertain.

This analysis shows that the concepts of epistemic attitudes, certainty, unsettledness and duality are at the core of our work. We will revisit them in Sections 4 and 5. But first, we need to adapt the class of credibility-limited revision operators of Hansson et al. [3] to our context, in order to apply it to our proposal.

3. Credibility-limited revision operators

In [3], Hansson et al. proposed a non-prioritized revision operator named credibility-limited for a fixed knowledge K, considering an observation, represented by α, and a credible set of sentences C. In this section, we rewrite and expand some of their results following Katsuno and Mendelson's setup, representing both knowledge and observation as formulas ψ and μ, respectively, in a finitary language $\mathcal{L}$. One important remark is that our presentation is more general than the one given in [13] because, in their paper, the authors assume that the original knowledge is consistent and that the underlying revision operator is defined by a faithful assignment (i.e. it satisfies **(R1)** $\sim$ **(R6)**). For this reason, we need to make a more detailed presentation as follows.

The first modification that we are introducing is to make explicit that C may vary when ψ does. Therefore, we assume that the credible set depends on ψ and we define it as a function.

Definition 3.1. A credible function is a mapping $C : \mathcal{L} \to \mathcal{P}(\mathcal{L})$ associating a formula ψ to a set of credible sentences $C(\psi)$.

This idea was initially motivated by the representation theorem appearing in [3, Theorem 8], where the original knowledge is used to construct the credible set (i.e. $C = \{\alpha \mid \psi \circ \alpha \vDash \alpha\}$) and later found explicitly defined in [13] and [14]. With this in mind, we define the credibility-limited revision operator in our context.

Definition 3.2. A credibility-limited revision operator $\circ : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is defined by a revision operator $*$ satisfying **(R1)** $\sim$ **(R4)** and a credible function C such that for every $\psi, \mu \in \mathcal{L}$:

$$\psi \circ \mu = \begin{cases} \psi * \mu & \text{if } \mu \in C(\psi) \\ \psi & \text{otherwise} \end{cases}$$

With this previous definition, we extend the family of operators of the original proposal: we are allowing ψ to be inconsistent and $C(\psi)$ to not necessarily be closed under logical equivalence. As a result of this, we need to consider a restricted version of the original postulate of extensionality; we named this new postulate *relative extensionality*. This postulate helps us provide a representation theorem more suitable to the definition induced by $*$ and C.

Theorem 3.3. *The following conditions are equivalent:*

a. *Operator $\circ$ satisfies the following postulates:*

($\circ$**1**) $\psi \circ \mu = \psi$ *or* $\psi \circ \mu \vDash \mu$. *(relative success)*
($\circ$**2**) $\psi \wedge \mu \vDash \psi \circ \mu$. *(inclusion)*

(∘**3**) *If* $\psi \wedge (\psi \circ \mu) \nvDash \bot$ *then* $\psi \circ \mu \vDash \psi$. *(consistent expansion)*
(∘**4**) *If* $\psi \nvDash \bot$, $\mu \nvDash \bot$ *then* $\psi \circ \mu \nvDash \bot$. *(weak consistency preservation)*
(∘**5**) *If* $\psi \circ \mu \vDash \mu$, $\phi \circ \nu \vDash \nu$, $\psi \equiv \phi$ *and* $\mu \equiv \nu$ *then* $\psi \circ \mu \equiv \phi \circ \nu$. *(relative extensionality)*

b. *Operator* ∘ *is a credibility-limited revision induced by a revision operator* $*$ *satisfying* ***(R1)*** $\sim$ ***(R4)*** *and a credible function* C.

See proof in Appendix A.

It is worth noticing that the belief operator characterized in the previous theorem is more general than the ones appearing in Figure 1 from [3, pag. 1586].

Corollary 3.4. *Let* ∘ *be a credibility-limited revision operator induced by a revision operator* $*$ *and a credible function* C*, then there is an equivalent operator* $\circ'$ *induced by a revision operator* $*'$ *and a credible function* C' *such that* $C'(\psi) = Cn(\psi) \cup C(\psi)$ *when* $\psi \nvDash \bot$ *and* $C'(\psi) = C(\psi) \cap \{\alpha \mid \alpha \nvDash \bot\}$ *when* $\psi \vDash \bot$.

See proof in Appendix A.

The last corollary highlights the idea that the behavior of a credibility-limited operator is invariant by including in the credible set $C(\psi)$ any subsets of the consequence of ψ. This result is dual to Observation 9 in [3] where the authors show that the properties of ∘ only depend on $C(\psi) \setminus Cn(\psi)$.

In this way, we have characterized the most general credibility-limited operator that we are interested in. Following Hasson et al. [3], we could be able to reproduce the alternative constructions and provide axiomatic characterizations for all families of operators considered in Figure 1 from [3, pag. 1586]. We do not include such characterization here to preserve the readability of the manuscript.

To complete this section, we are focusing only on the following two properties of credibility-limited revisions because they have a particular role in our proposal.

Theorem 3.5. *Consider* ∘ *a credibility-limited revision operator satisfying* (∘**1**) $\sim$ (∘**5**) *where* C *is its credible function. Then, for every* $\psi \in \mathcal{L}$ *each of these properties of* C*:*

i. *Element Consistency: If* $\psi \nvDash \bot$ *and* $\mu \in C(\psi)$ *then* $\mu \nvDash \bot$
ii. *Expansive Credibility: If* $\psi \wedge \mu \nvDash \bot$ *and* $\psi \nvDash \mu$ *then* $\mu \in C(\psi)$

are respectively equivalent to these properties of ∘*:*

a. *Consistency Preservation: If* $\psi \nvDash \bot$ *then* $\psi \circ \mu \nvDash \bot$.
b. *Vacuity: If* $\psi \wedge \mu \nvDash \bot$ *then* $\psi \circ \mu \equiv \psi \wedge \mu$.

See proof in Appendix A.

We can understand these properties as complementary of the ones appearing in Corollary 3.4. Due to this corollary, for every credibility-limited revision operator, we can always require its credible function to satisfy $C(\psi) \subseteq \{\alpha \mid \alpha \nvDash \bot\}$ for every inconsistent ψ. Combining this with Element Consistency we have for every ψ that:

$$C(\psi) \subseteq \{\alpha \mid \alpha \nvDash \bot\}$$

Analogously, according to Corollary 3.4, for every credibility-limited revision operator, we can always require its credible function to satisfy $Cn(\psi) \subseteq C(\psi)$ for every consistent ψ. This property is referred to in [15] as *credibility lower bounding*. Combining this with Expansive Credibility we have for every consistent ψ that:

$$\{\alpha \mid \alpha \wedge \psi \nvDash \bot\} \subseteq C(\psi)$$

Note also that, if ψ is inconsistent, expansive credibility trivially holds and $\{\alpha \mid \alpha \wedge \psi \nvDash \bot\} = \emptyset$. Therefore, previous inclusion is trivially satisfied.

Thus, element consistency and expansive credibility can be understood as complementary extensions of properties appearing in Corollary 3.4, considering that they define a more canonical behavior of the credible function. A credibility-limited revision operator with its associated credible function satisfying only these properties has not been explicitly considered in [3].

Definition 3.6. A Credibility-limited Revision based on $*$ satisfying **(R1)** $\sim$ **(R4)** such that its credible function C satisfies the normal condition:

$$\{\alpha | \alpha \wedge \psi \nvDash \bot\} \subseteq C(\psi) \subseteq \{\alpha | \alpha \nvDash \bot\} \tag{NC}$$

is called a Normal Credibility-limited Revision and in the following we denoted it by ©.

Due to Corollary 3.4 we have that two normal credibility-limited revision operators are equivalent if they share the same credible function and have equivalent revision operators $*$ and $*'$ for every credible observation.

The following representation theorem is directly deduced from Theorem 3.5. Note that vacuity implies inclusion and consistent expansion when combined with relative success, allowing us to reduce the original axiomatic definition.

Corollary 3.7. *The following two conditions are equivalent:*

a. *Operator $\circledcirc$ satisfies the following postulates:*

- ($\circledcirc$**1**) $\psi \circledcirc \mu = \psi$ *or* $\psi \circledcirc \mu \vDash \mu$. *(relative success)*
- ($\circledcirc$**2**) *If* $\psi \wedge \mu \nvDash \bot$ *then* $\psi \circledcirc \mu \equiv \psi \wedge \mu$. *(vacuity)*
- ($\circledcirc$**3**) *If* $\psi \nvDash \bot$ *then* $\psi \circledcirc \mu \nvDash \bot$. *(consistency preservation)*
- ($\circledcirc$**4**) *If* $\psi \circledcirc \mu \vDash \mu$, $\phi \circledcirc \nu \vDash \nu$, $\psi \equiv \phi$ *and* $\mu \equiv \nu$ *then* $\psi \circledcirc \mu \equiv \phi \circledcirc \nu$. *(relative extensionality)*

b. *$\circledcirc$ is a credibility-limited revision induced by a revision operator $*$ satisfying **(R1)** $\sim$ **(R4)** and a credible function C satisfying the normal condition* (NC)*.*

Since $C(\psi)$ satisfies (NC) we can separate the credible set $C(\psi)$ into trivial credible formulas, i.e. the ones consistent with ψ, and the non-trivial credible formulas, noted as $C_{\bot}(\psi) = \{\mu \in C(\psi) \mid \psi \wedge \mu \vDash \bot\}$. Therefore, we can rewrite $C(\psi)$ as the union of $C_{\bot}(\psi)$ and $\{\mu \mid \psi \wedge \mu \nvDash \bot\}$. This representation allows us to explicit some easy observations that we show later in Proposition 3.8. However, the key property of this family is that the credible function C associated with the operator can be easily defined in terms of $\circledcirc$:

$$(\psi \circledcirc \mu) \wedge \mu \nvDash \bot \overset{Element}{\underset{Consistency}{\Longleftrightarrow}} \psi \wedge \mu \nvDash \bot \text{ or } \mu \in C(\psi) \overset{Expansive}{\underset{Credibility}{\Longleftrightarrow}} \mu \in C(\psi)$$

We sum up all of this in the following proposition.

Proposition 3.8. *Let $\circledcirc$ a normal credibility-limited revision operator satisfying* ($\circledcirc$**1**) $\sim$ ($\circledcirc$**4**) *and C its associated credible function satisfying* (NC)*, then the following properties hold:*

1. $\mu \in C(\psi) \Leftrightarrow (\psi \circledcirc \mu) \wedge \mu \nvDash \bot$;
2. *If* $\psi \vDash \phi$ *then* $\{\mu \mid \psi \wedge \mu \nvDash \bot\} \subseteq C(\phi)$;
3. *If* $\mu \in C_{\bot}(\psi)$ *then* $\mu \vDash \neg\psi$;
4. $C_{\bot}(\psi) \cap C_{\bot}(\neg\psi) = \emptyset$;
5. $C(\psi) \cup C(\neg\psi) = \{\mu \mid \mu \nvDash \bot\}$;

We are now in conditions to define moderated revision operators. Although we only focus on two of the properties considered in [3], the rest of them will appear representing new concepts for our proposal, and new properties will arise, giving us a diverse genealogy. But normal credibility-limited revision operators are going to have a key role in simplifying the connections between revision and credibility-limited revision on one side and moderated revision on the other.

4. Moderated revision

Before defining our proposal, we review what we have so far. In Section 2 we asserted that, in terms of epistemic attitudes, revision is an operator for certainty, while contraction is an operator for doubt (uncertainty).

Let us analyze these concepts explicitly by considering, as usual, ψ the knowledge of the agent and μ some new observation. When reading $\psi - \neg\mu$, this can be understood as the agent not being capable of telling if the observation should be disbelieved or not, while $\psi * \mu$ means that the agent is certain to believe the observation. It is important to remark that the new observation is always taken into consideration. Furthermore, through Levi and Harper's identities, we can see that a conjunction is used to assert certainty in the definition of the revision operator through contraction, while a disjunction models the uncertainty created by the contraction operator from revision.

Compare this behavior with a credibility-limited revision operator introduced in Section 3. For an agent with knowledge ψ, a new observation μ can be considered certain if $\mu \in C(\psi)$, or rejected otherwise. Then the emergent behavior of credibility-limited revision is either to completely accept the new observation or to stay as if the observation has never been received. Note that here "reject" does not mean to negate the new observation, but to completely disavow it. This could be considered a very drastic result.

Moderated revision recovers the acceptance behavior of both revision and contraction and takes into account the new observation, but aims to stay in between certainty and doubt. Inspired by Harper Identity, we go from certainty to an eventually partial doubt with a disjunction, defining the operator as a combination of a revision and a credibility-limited revision operators. The first one preserves the foretold acceptance behavior, while the latter is used to decide if the observation is doubtful or not. The idea of combining two operators with a disjunction has already been seen in [8] for Promotion Operators.

Definition 4.1. A moderated revision operator $\circledast : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is defined by a revision operator $*$ satisfying **(R1)** $\sim$ **(R4)** and a credibility limited revision operator $\circ$ satisfying ($\circ$**1**) $\sim$ ($\circ$**5**) such that for every formula $\psi, \mu \in \mathcal{L}$:

$$\psi \circledast \mu = (\psi * \mu) \vee ((\mu \circ \psi) \wedge \psi)$$

Our proposal models a balance of certainty, where extreme situations are represented by revision and its dual contraction respectively, giving a more suitable and general model for belief change, as entailed by the following observation:

$$\psi * \mu \vDash \psi \circledast \mu \vDash \psi - \neg\mu$$

Consider the case when ψ and μ are incompatible (i.e. $\psi \wedge \mu \vDash \bot$), and $\psi \in C(\mu)$ holds. Intuitively, this means that some situations (possible worlds of ψ) where $\neg\mu$ is believed are acceptable even in the μ-context. We are going to rephrase this situation by saying that the agent with knowledge ψ has some doubts about μ, formally represented by the result of the inducing revision in the credibility-limited. Note that in the case of incompatibility between ψ and μ but $\psi \notin C(\mu)$, the result of the credibility-limited revision is by definition μ and in this case the second argument of the disjunction of the moderated revision is equivalent to $\bot$, i.e. it is a neutral term with respect to $\vee$. A third case has to be considered when ψ and μ are compatible (i.e. $\psi \wedge \mu \nvDash \bot$). But by properties of both revision operators ($*$ and $\circ$), it is easy to verify that the obtained result for the moderated revision is $\psi \wedge \mu$.

In the situation when ψ is complete, the result of the moderated revision only has the following three possible values: if $\psi \wedge \mu \nvDash \bot$ then $\psi \circledast \mu \equiv \psi \wedge \mu \equiv \psi$ by vacuity, and if $\psi \wedge \mu \vDash \bot$ then $\psi \circledast \mu \equiv \psi * \mu$ or $\psi \circledast \mu \equiv (\psi * \mu) \vee \psi \equiv \psi - (\neg\mu)$ according to Harper identity.

Now that we formally defined our proposal, let us test it with the example considered at the beginning. We have already defined the propositional variables that model the situation and know the results of the revision and contraction operators:

p: "Your ex-partner regrets the breakup".
q: "Your ex-partner loves you".
ψ: $\neg q$.
μ: $p \wedge q$.
$\psi * \mu$: $p \wedge q$ since μ is a complete formula.
$\psi - \neg\mu$: $q \to p$.

On the other hand, the definition of moderated revision combines the results of revision and credibility-limited revision operators. For the sake of this example, let us take an arbitrary C. Since μ is complete, the result of the revision operator is $\mu = p \wedge q$ itself. But in $\mu \circ \psi$ we have four different results, one given by the fact that $\psi \notin C(\mu)$ and the rest by the result of the associated revision operator $*'$ of the credibility-limited when $\psi \in C(\mu)$. Note that by Definition 2.1, the possible result of $\mu *' \psi$ could be:

$$\neg q \wedge p \quad , \quad \neg q \wedge \neg p \quad , \quad \neg q$$

Therefore, applying these results to our proposal, if the agent does not doubt μ (i.e. $\psi \notin C(\mu)$) then $\psi \circledast \mu = \mu \vee (\psi \wedge \mu) \equiv p \wedge q$, emulating the revision operator. If the agent doubts μ (i.e. $\psi \in C(\mu)$) then:

- $\psi \circledast \mu = (p \wedge q) \vee ((\neg q \wedge p) \wedge \neg q) \equiv p$ that can be understood as "you can definitely believe that s/he regrets the breakup, but cannot tell if s/he really loves you".
- $\psi \circledast \mu = (p \wedge q) \vee ((\neg q \wedge \neg p) \wedge \neg q) \equiv q \leftrightarrow p$ that can be understood as "You can believe that s/he regrets the breakup as a proof the s/he loves you, but there is not enough information about your ex-partner true feelings".
- $\psi \circledast \mu = (p \wedge q) \vee (\neg q \wedge \neg q) \equiv q \to p$ that can be understood as emulating the contraction operator: "The only thing certain is that if s/he loves you, then s/he regrets the breakup".

The original example has the extra-logic knowledge of "winning the lottery", which allows the agent not only to doubt the observation but to consider p as the more suitable result. However, the other options are also valid by changing this extra-logic situation. Maybe the call came after meditating for some time about their complex relationship. In any case, these extra-logic effects are modeled by the credibility-limited revision.

This example shows how moderated revision can represent a diversity of situations, including the ones modeled by revision and contraction (when the knowledge and the observation are not consistent) and intermediate situations considering these classical operators as extreme cases of certainty and doubt. The credibility-limited revision operator plays a key role in this diversity: the credible function is used to decide whether to believe in the certainty of the observation ($\psi \notin C(\mu)$) or doubt it ($\psi \in C(\mu)$). And, in case of doubt, the associated revision operator is in charge of choosing which piece of the original knowledge should be used to weaken the result of the revision part.

After making an intuitive presentation of our proposal, from now on in this section we aim to develop a formal conceptualization of moderated revision operators. We start by providing some elemental ideas that help us to get a better interpretation of the representation theorem, which links the first induced definition of moderated revision to an equivalent definition given by postulates.

4.1. Main representation theorem

We start our road to the representation theorem by first pointing out that we can use moderated revision to characterize its inducing operators. Since $\psi \circledast \mu \vDash \mu \vee \psi$ by definition and the success postulate of its revision operator, we have:

$$\psi \circledast \mu \equiv ((\psi \circledast \mu) \wedge \mu) \vee ((\psi \circledast \mu) \wedge \psi)$$

Intuitively, we can perceive that there is a correspondence between the disjunction that defines our proposal and this equivalence. The following theorem formalizes this intuition.

Theorem 4.2 *(Moderated identities). Let $\circledast$ a moderated revision, induced by $*$ revision operator and $\circ$ credibility-limited revision operator, where C is the credible function of $\circ$ and $*'$ is its associated revision operator. Then:*

a. $\psi * \mu \equiv (\psi \circledast \mu) \wedge \mu$.
b. *If* $\psi \in C(\mu)$ *then* $\mu *' \psi \equiv (\psi \circledast \mu) \wedge \psi$.
c. $C(\mu) \cap \{\psi \nvDash \bot \mid \psi \wedge \mu \vDash \bot\} = \{\psi \mid (\psi \circledast \mu) \wedge \psi \nvDash \bot$ *and* $\psi \wedge \mu \vDash \bot\}$.

See proof in Appendix A.

Theorem 4.2 presents some identities between the moderated revision and its inducing operators in a direct way. But in fact, it can be reinterpreted as a separation theorem: given a moderated revision operator, it is possible to recover the AGM revision operator and (partially) the credibility-limited revision that induces it. More precisely, given a moderated revision operator induced by $*$ and $\circ$, it is possible to recover exactly the AGM revision operator $*$ that has been used to define it and to obtain the part of the credibility-limited revision operator $\circ$ when $\psi \nvDash \bot$, $\psi \wedge \mu \vDash \bot$ and originally $\psi \in C(\mu)$. Note that when this is not the case we have that $\psi \circledast \mu \equiv \psi * \mu$.

The following corollary states that if we restrict ourselves to normal credibility-limited revision operators, then for each moderated revision operator there is a unique credible function, thus determining a unique normal credibility-limited revision that induced it. We also characterize $C_{\bot}(\mu)$ as the subset of formulas of $C(\mu)$ appearing in identity *3* from Theorem 4.2.

Corollary 4.3. *There is a one-to-one correspondence between the set of moderated revision operators $\circledast$ and the set of pairs $(*, ©)$ where $*$ is a revision operator satisfying **(R1)** $\sim$ **(R4)**, © is a normal credibility-limited revision operator satisfying* $(©\mathbf{1}) \sim (©\mathbf{4})$ *and $\circledast$ is its induced moderated revision.*

Theorem 4.2 and Corollary 4.3 allow us to uniquely define $C(\mu)$ in terms of $\circledast$ as $C_{\bot}(\mu) \cup \{\psi \mid \mu \wedge \psi \nvDash \bot\}$. This also means that relation (**1**.) in Proposition 3.8 of the normal credibility-limited revision operators is inherited by the moderated revision $\circledast$ in the following way:

$$(\mu © \psi) \wedge \psi \nvDash \bot \quad \Leftrightarrow \quad \psi \in C(\mu) \quad \Leftrightarrow \quad (\psi \circledast \mu) \wedge \psi \nvDash \bot \tag{1}$$

Relation (1) makes a clear connection between the normal credibility-limited revision operator and its induced moderated revision, by showing that left and right expressions are not only simultaneously consistent but equivalent, due to Theorem 4.2.

In the rest of this section, we focus on the relationship between the properties of credibility-limited revision operators and moderated revision postulates. Some of these properties are the ones appearing in [3], but we also consider new ones that are relevant to the current setting. This allows us to characterize several families of our proposal.

We start by showing a characterization theorem of our moderated revision in terms of the following list of postulates.

Theorem 4.4. *The following conditions are equivalent:*

a. *Operator $\circledast$ satisfies the following postulates:*

($\circledast$**1**) $\psi \circledast \mu \vDash \psi \vee \mu$. *(shared success)*
($\circledast$**2**) *If* $\psi \wedge \mu \nvDash \bot$ *then* $\psi \circledast \mu \equiv \psi \wedge \mu$. *(vacuity)*
($\circledast$**3**) *If* $\mu \nvDash \bot$ *then* $(\psi \circledast \mu) \wedge \mu \nvDash \bot$. *(R-consistency)*
($\circledast$**4**) *If* $\psi \equiv \phi$, $\mu \equiv \nu$ *then* $(\psi \circledast \mu) \wedge \mu \equiv (\phi \circledast \nu) \wedge \nu$. *(R-extensionality)*
($\circledast$**5**) *If* $(\psi \circledast \mu) \wedge \psi \nvDash \bot$, $(\phi \circledast \nu) \wedge \phi \nvDash \bot$, $\psi \equiv \phi$ *and* $\mu \equiv \nu$ *then* $(\psi \circledast \mu) \wedge \psi \equiv (\phi \circledast \nu) \wedge \phi$. *(CL-relative extensionality)*

b. *$\circledast$ is a moderated revision operator induced by a revision operator $*$ satisfying **(R1)** $\sim$ **(R4)** and a credibility-limited revision operator $\circ$ satisfying* $(\circ\mathbf{1}) \sim (\circ\mathbf{5})$.

See proof in Appendix A.

Let us analyze each postulate separately. Shared success determines that the new knowledge is constrained by what the original knowledge or the new observation says, giving the disjoint structure. Vacuity, as in the case of revision, says that the operator extends the conjunction in the consistent case. Postulate R-consistency guarantee the consistency of the result of the revision operator if $\mu \nvDash \bot$. Lastly, R-extensionality and CL-relative extensionality preserve known extensionality from the inducing operators.

One remarkable observation regarding these postulates is the modularity of the presentation, where postulates shared success and vacuity play a fundamental role. While vacuity determines the case when the observation is consistent with the original knowledge, shared success separates the result of moderated revision as the disjunction of the inducing operators in the inconsistent case. These properties are therefore the core of its modularity because the rest is about adapting the revision postulates that define $(\psi \circledast \mu) \wedge \mu$ as a revision (consistency and extensionality) and the credibility-limited revision postulates that define $(\psi \circledast \mu) \wedge \psi$ as a normal credibility-limited with a conjunction (relative extensionality). This modularity concept and its consequence of adapting postulates are central to how known properties and postulates of revision and credibility-limited revision operators can be easily adapted to moderated operators.

Shared success has appeared when defining postulates for arbitration [16] and merging operators [17–20], linking these operators with moderated revision. It also marks a main difference from other non-prioritized operators, such as credibility-limited or selective revision. These connections are presented in Section 5.

Inspired by Corollary 4.3, it is easy to see that every moderated revision can be rewritten using a normal credibility-limited revision instead of a generic one.

Corollary 4.5. *Every moderated revision operator $\circledast$ induced by a revision operator $*$ satisfying* ***(R1)*** $\sim$ ***(R4)*** *and a credibility-limited revision operator $\circ$ satisfying* $(\circ\mathbf{1}) \sim (\circ\mathbf{5})$ *is equivalent to another moderated revision operator $\circledast'$ induced by the same revision operator $*$ but with a normal credibility-limited revision operator © satisfying* (©**1**) $\sim$ (©**4**).

See proof in Appendix A.

So far, we have two equivalent presentations of a moderated operator: induced (Definition 4.1) and as a list of postulates (Theorem 4.4). In order to show the different families of moderated revision and its representation theorems, we are separating the rest of the section into two parts by taking advantage of working with a credible function instead of a credible set in the context of credibility-limited revision operators. The first part is related to the branches that inherit the classical point of view of the credibility-limited revision In the second part, we explore new properties that emerged from studying relations in the domain of the credible function, and that led us to other branches of families. Since all these branches represent complementary actions that can be combined, it is possible to get multiple hybrid families. So we also analyze the main properties of these combinations and their interactions.

4.2. Properties inherited from the credibility-limited revision

Moderated revision operators can be classified in terms of the properties that both operators $*$ and $\circ$ satisfy. We are going to first focus on the branch of moderated revision families where its associated credibility-limited revision operator is classified in terms of Hansson et al. [3]. Recall that expansive credibility and element consistency are already considered since we can always think that the operator is a normal credibility-limited revision, as has been proven in Corollary 4.5.

As in credibility-limited, properties that enrich the choice mechanism are viewed as ways of how the agent interacts with the observation. The following theorems show how the properties in Hansson et al. [3] are rephrased into postulates by applying Representation Theorem 4.4 and relation (1).

Theorem 4.6. *Let $\circledast$ be a moderated revision operator, satisfying* $(\circledast\mathbf{1})$*~*$(\circledast\mathbf{5})$*, and C the credible function of its associated normal credibility-limited revision. Then, each pair of property and postulate defines an equivalence for every formula $\psi, \phi, \mu \in \mathcal{L}$:*

C property		$\circledast$ ***postulate***	
$(\mathbf{1}C)$	*$\psi \in C(\mu)$ iff $\neg\neg\psi \in C(\mu)$* *(closure under double negation)*	$(\circledast\mathbf{6})$	*$(\psi \circledast \mu) \wedge \psi \equiv ((\neg\neg\psi) \circledast \mu) \wedge \neg\neg\psi$* *(CL-double negation closure)*
$(\mathbf{2}C)$	*If $\psi \equiv \phi$ and $\psi \in C(\mu)$ then $\phi \in C(\mu)$* *(closure under logical equivalence)*	$(\circledast\mathbf{7})$	*If $\psi \equiv \phi$ then $(\psi \circledast \mu) \wedge \psi \equiv (\phi \circledast \mu) \wedge \phi$* *(CL-logical equivalence closure)*
$(\mathbf{3}C)$	*If $\psi \in C(\mu)$ then $Cn(\psi) \subseteq C(\mu)$* *(single sentence closure)*	$(\circledast\mathbf{8})$	*If $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ and $\psi \vDash \phi$ then $(\phi \circledast \mu) \wedge \phi \nvDash \bot$* *(CL-strict improvement)*
$(\mathbf{4}C)$	*If $\psi \vee \phi \in C(\mu)$ then* *either $\psi \in C(\mu)$ or $\phi \in C(\mu)$* *(disjunctive completeness)*	$(\circledast\mathbf{9})$	*If $((\psi \vee \phi) \circledast \mu) \wedge (\psi \vee \phi) \nvDash \bot$ then* *either $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ or $(\phi \circledast \mu) \wedge \phi \nvDash \bot$* *(CL-disjunctive distribution)*
$(\mathbf{5}C)$	*If $\mu \nvDash \bot$ then either $\psi \in C(\mu)$ or $\neg\psi \in C(\mu)$* *(negation completeness)*	$(\circledast\mathbf{10})$	*If $\mu \nvDash \bot$ then either $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ or $(\neg\psi \circledast \mu) \wedge \neg\psi \nvDash \bot$* *(CL-disjunctive success)*

Let us analyze each property according to our proposed interpretation, where $\psi \in C(\mu)$ is understood as if the agent with knowledge ψ doubts μ. The first two properties are related to the syntax of the original knowledge ψ, where they say that the agent is (partially) immune to its syntax. *Single sentence closure* assumes that if the agent has knowledge ψ and doubts μ, then the agent with a knowledge $\phi \in Cn(\psi)$ would also doubt μ. *Disjunctive completeness* works as a dual version of single sentence closure, where if the knowledge of an agent can be expressed as a disjunction of two pieces of knowledge and the agent doubts μ, then the agent would doubt μ with one of these pieces. Lastly, *negation completeness* affirms that if the agent with knowledge ψ does not doubt μ, then the agent with knowledge $\neg\psi$ would doubt μ.

Continuing with [3], the authors present a family of credibility-limited revision operators known as Endorsed Core Belief Revision, the subfamily of normal credibility-limited revision where its credible function C also satisfies $(1C) \sim (5C)$. This family captures Makinson's screened revision idea [2]. A core belief revision is related to a set of formulas named *core belief*, that in our context depends on ψ, it is noted by $\mathcal{A}$ and satisfies:

$$\alpha \in C(\psi) \Leftrightarrow \mathcal{A}(\psi) \nvDash \neg\alpha \tag{2}$$

Since the credibility-limited revision operator is normal, $C(\psi)$ is a finite set of consistent formulas up to logical equivalence, thus $\mathcal{A}(\psi)$ can always be defined. Also, having $\mathcal{A}(\psi) \nvDash \neg\alpha$ is the same as having $Cn(\mathcal{A}(\psi)) \nvDash \neg\alpha$. Therefore we can assume $\mathcal{A}(\psi)$ to be a belief set and represent it as a formula as we did with the knowledge K since we are working with a finite language.

Definition 4.7. A core belief function is a mapping $\mathcal{A} : \mathcal{L} \to \mathcal{L}$ that, given a formula ψ, associates it with a formula $\mathcal{A}(\psi)$ known as its *core belief*.

Thus, the credible function C is univocally defined by a core belief function $\mathcal{A}$ where $C(\psi) = \{\alpha \mid \alpha \wedge \mathcal{A}(\psi) \nvDash \bot\}$. Also, since we are working with a credible function belonging to normal credibility-limited revision operators, we have that $\psi \vDash \mathcal{A}(\psi)$. In a moderated revision context, this core belief approach leads us to the case where, if the agent originally believes in ψ, then for each new observation μ the agent selects a formula $\mathcal{A}(\mu)$ such that $\mu \vDash \mathcal{A}(\mu)$ to determine if μ should be doubted by checking whether $\mathcal{A}(\mu) \wedge \psi$ is consistent or not.

Formula $\mathcal{A}(\mu)$ is used in this context as a mere choice mechanism characterized by Theorem 4.6. The following theorems continue with the representation of our proposal when its associated credibility-limited revision satisfies outcome credibility or strong outcome credibility, giving a semantic interpretation to $\mathcal{A}$.

Theorem 4.8. *Let $\circledast$ be a moderated revision operator, satisfying $(\circledast\mathbf{1}) \sim (\circledast\mathbf{10})$, and $\mathcal{A}$ the core belief function of its associated normal credibility-limited revision operator $\copyright$. Then each pair of property and postulate defines an equivalence for every formula $\psi, \phi, \mu \in \mathcal{L}$:*

	***C* property**		**$\circledast$ postulate**
$(\mathbf{6}C)$	*If $\mu \copyright \psi \nvDash \bot$ then $(\mu \copyright \psi) \wedge \mathcal{A}(\mu) \nvDash \bot$ (outcome credibility)*	$(\circledast\mathbf{11})$	*If $(\psi \circledast \mu) \wedge \psi \vDash \phi$ and $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ then $(\phi \circledast \mu) \wedge \phi \nvDash \bot$ (CL-regularity)*
$(\mathbf{7}C)$	*$\mu \copyright \psi \vDash \mathcal{A}(\mu)$ (strong outcome credibility)*	$(\circledast\mathbf{12})$	*If $(\psi \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$ then $(\phi \circledast \mu) \wedge \phi \nvDash \bot$ (CL-strong regularity)*

See proof in Appendix A.

When considering *regularity* or *strong regularity*, $\mathcal{A}$ is used to decide whether μ would be doubted or not, and also to select which information of μ is doubtful. By definition, $(\psi \circledast \mu) \wedge \psi$ is the information from the original belief used to represent the uncertainty of μ. If $\psi \wedge \mu \vDash \bot$, this is consistent when $\psi \wedge \mathcal{A}(\mu) \nvDash \bot$. Therefore, if regularity holds, then $(\psi \circledast \mu) \wedge \psi \wedge \mathcal{A}(\mu) \nvDash \bot$, meanwhile if strong regularity is true, we have that $(\psi \circledast \mu) \wedge \psi \vDash \psi \wedge \mathcal{A}(\mu)$.

This theorem ends the characterization of this branch of moderated revision operators. Let us name the families we have until now.

Definition 4.9. Consider a moderated revision satisfying $(\circledast\mathbf{1})$~$(\circledast\mathbf{5})$ and C the credible function of its associated normal credibility-limited revision. If its credible function satisfies:

- $(\mathbf{1}C) \sim (\mathbf{2}C)$, we call it a *Credible Extensional Moderated Revision*;
- $(\mathbf{1}C) \sim (\mathbf{5}C)$, we call it an *Endorsed Moderated Revision*.

Moreover, we say that an endorsed moderated revision operator:

- satisfies *Outcome Credibility* (OC) if $(\circledast\mathbf{11})$ holds;
- satisfies *Strong Outcome Credibility* (SOC) if $(\circledast\mathbf{12})$ holds.

Following [3], the classification of this branch of moderated revision operators focuses on the image of the credible function, that is, the properties of the set defined by $C(\psi)$ for each ψ. The branch that is presented in the next subsection, focuses mainly on the domain of the credible function.

4.3. New properties inherited from the credibility-limited revision

Until now, every property shown for C was adapted from [3], where they only add features to the image of C for each μ, or to the associated revision operator of ©. In this section, we explore relations that can be defined in terms of the domain of C.

The following properties aim to model the agent's decisions in terms of the new observation (the parameter of the credible function) instead of properties of how the set is defined (the image of the credible function), that is, how the agent behavior changes when the observation changes. The first property we consider is related to *extensionality*.

Theorem 4.10. *Let $\circledast$ be a moderated revision operator, satisfying ($\circledast$**1**)~($\circledast$**5**) , and C the credible function of its associated normal credibility-limited revision. Then, each pair of property and postulate defines an equivalence for every formula $\psi, \mu, \nu \in \mathcal{L}$:*

C **property**		$\circledast$ **postulate**	
(**8**C)	$C(\mu) = C(\neg\neg\mu)$ *(Irrelevance of double negation syntax)*	($\circledast$**13**)	$(\psi \circledast \mu) \wedge \psi \equiv (\psi \circledast \neg\neg\mu) \wedge \psi$ *(Double negation extensionality)*
(**9**C)	*If* $\mu \equiv \nu$ *then* $C(\mu) = C(\nu)$ *(Irrelevance of syntax)*	($\circledast$**14**)	*If* $\mu \equiv \nu$ *then* $(\psi \circledast \mu) \wedge \psi \equiv (\psi \circledast \nu) \wedge \psi$ *(Doubt extensionality)*

Note that by applying relation (1), both equivalencies are directly deduced, except for the irrelevance of syntax when trying to prove the validity of the postulate if the property is satisfied. In this case, the consistent situation holds by **(R4)** of the inducing revision operators. The irrelevance of syntax and its weaker version, the irrelevance of double negation syntax, say that the agent does not (partially) care about the syntax of the observations. Note that *doubt extensionality* is the missing extensionality property that we need to formulate for the classic version of the postulate to hold for moderated revision, as shown in the next lemma.

Lemma 4.11. *Consider $\circledast$ a moderated revision satisfying ($\circledast$**1**)~($\circledast$**5**).*[1] *The following conditions are equivalent:*

a. *$\circledast$ satisfies **(R4)** (extensionality).*
b. *$\circledast$ satisfies ($\circledast$**7**) and ($\circledast$**14**).*

See proof in Appendix A.

Postulate ($\circledast$**14**) also gives a uniform perspective of regularity ($\circledast$**11**) and strong regularity ($\circledast$**12**). To understand this uniformity, let us first analyze these two postulates when combined only with ($\circledast$**5**):

CL-Regularity implies that when $\psi \wedge \mathcal{A}(\mu) \nvDash \bot$, for every $\mu \equiv \nu$ we have that if $\psi \wedge \mathcal{A}(\nu) \nvDash \bot$ then $(\mu © \psi) \wedge \mathcal{A}(\nu) \nvDash \bot$.
CL-Strong Regularity implies that when $\psi \wedge \mathcal{A}(\mu) \nvDash \bot$, we have $\mu © \psi \vDash \bigwedge_{\nu\equiv\mu\,,\,\mathcal{A}(\nu)\wedge\psi\nvDash\bot} \mathcal{A}(\nu) \nvDash \bot$.

Now, if ($\circledast$**14**) also holds, then $\mathcal{A}(\mu) \equiv \mathcal{A}(\nu)$ for every $\nu \equiv \mu$. Therefore:

CL-Regularity is equivalent to if $\psi \wedge \mathcal{A}(\mu) \nvDash \bot$ and $\mu \equiv \nu$ then $(\mu © \psi) \wedge \mathcal{A}(\nu) \nvDash \bot$.
CL-Strong Regularity is equivalent to $\mu © \psi \vDash \mathcal{A}(\nu)$ if $\mu \equiv \nu$.

The characterizations given by assuming ($\circledast$**14**) allow us to consider a name for each sub-family of operators that satisfies it.

Definition 4.12. A moderated revision operator satisfying ($\circledast$**1**) ~ ($\circledast$**5**) is called Doubt Extensional Moderated Revision if satisfies doubt extensionality ($\circledast$**14**). A Doubt Extensional Moderated Revision that:

- it is also Credible Extensional (i.e. ($\circledast$**6**) ~ ($\circledast$**7**)), it is known as Extensional Moderated Revision;
- it is also Endorsed (i.e. ($\circledast$**6**) ~ ($\circledast$**10**)), it is known as Triggered Moderated Revision;

Moreover, a Triggered Moderated Revision that:

- it also satisfies OC (i.e. ($\circledast$**11**)), it is known as Regular Moderated Revision;
- it also satisfies SOC (i.e. ($\circledast$**12**)), it is known as Basic Promotion.

Doubt extensionality can be also understood as a postulate that says that the credible function preserves the logical equivalence of its parameters, the observations in our context. This idea leads us to consider other properties where the credible function reflects not only the logical equivalence relation, but also the strength of the observations, using unions or intersections.

[1] This lemma can be generalized to operators satisfying only ($\circledast$**1**), where **(R4)** holds if and only if ($\circledast$**4**), ($\circledast$**7**) and ($\circledast$**14**) hold.

We have already seen some properties for normal credibility-limited revision operator in Proposition 3.8 related to unions and intersections. Recall also that $\psi \wedge \mu \nvDash \bot$ is a special case where the credible function does not reflect doubt but complete the expansion case. Therefore, the elements in $C_\bot(\mu)$ are the pieces of knowledge that actually makes the agent doubt μ. Taking these into consideration, we separate these properties into two types.

On one side we have *imprudent postulates*, the ones that tend to associate bigger credible sets to weaker observations, and are related to the agents that tend to be certain of strong observations and doubt weak observations. So, the stronger the observations are, the easier for these agents to commit themselves to it, convinced that what they perceive is true. On the other side, we named *wise postulates* the ones that consider bigger credible sets the stronger the observation is, representing the agents that tend to doubt strong observations and trust weak observations. So, the stronger the observation is, the harder for these agents to commit themselves to it, leaving always something still unsettled. The names of these properties come from many quotes about wisdom, certainty and doubt. We start characterizing the properties of the imprudent agents.

Theorem 4.13. *Let $\circledast$ be a moderated revision operator, satisfying ($\circledast$**1**)~($\circledast$**5**), and C the credible function of its associated normal credibility-limited revision. Then, each pair of property and postulate defines an equivalence for every formula $\psi, \mu, \nu \in \mathcal{L}$:*

	***C* property**		**$\circledast$ postulate**
(**10***C*)	*If $\mu \vDash \nu$ then $C(\mu) \subseteq C(\nu)$* *(monotony)*	($\circledast$**15**)	*If $\mu \vDash \nu$ and $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ then $(\psi \circledast \nu) \wedge \psi \nvDash \bot$* *(Imprudent monotony)*
(**11***C*)	$C(\mu \vee \nu) = C(\mu) \cup C(\nu)$ *($\bigcup - \bigvee$ Equivalence)*	($\circledast$**16**)	$(\psi \circledast (\mu \vee \nu)) \wedge \psi \nvDash \bot$ *iff* $((\psi \circledast \mu) \vee (\psi \circledast \nu)) \wedge \psi \nvDash \bot$ *(Imprudent Join distribution)*
(**12***C*)	$\bigcap_{\mu \in \mathcal{L}} C(\mu) = \emptyset$ *($\bigcap$ −Completeness)*	($\circledast$**17**)	*If $\mu \vDash \bot$ then $(\psi \circledast \mu) \wedge \psi \vDash \bot$* *(Imprudent Consistency)*

Imprudent properties are well defined by Proposition 3.8 and deduced from relation (1). This is also true for wise postulates. Due to the many similarities between imprudent and wise properties, we continue the analysis after presenting the properties related to wise agents.

Theorem 4.14. *Let $\circledast$ be a moderated revision operator, satisfying ($\circledast$**1**)~($\circledast$**5**), and C the credible function of its associated normal credibility-limited revision. Then, each pair of property and postulate defines an equivalence for every formula $\psi, \mu, \nu \in \mathcal{L}$:*

	***C* property**		**$\circledast$ postulate**
(**13***C*)	*If $\mu \vDash \nu$ then $C_\bot(\nu) \subseteq C_\bot(\mu)$* *(inverted monotony)*	($\circledast$**18**)	*If $\mu \vDash \nu$, $\nu \wedge \psi \vDash \bot$ and $(\psi \circledast \nu) \wedge \psi \nvDash \bot$ then $(\psi \circledast \mu) \wedge \psi \nvDash \bot$* *(Wise monotony)*
(**14***C*)	$C_\bot(\mu \vee \nu) = C_\bot(\mu) \cap C_\bot(\nu)$ *($\bigcap - \bigvee$ Equivalence)*	($\circledast$**19**)	*If $\psi \wedge (\mu \vee \nu) \vDash \bot$ then $(\psi \circledast (\mu \vee \nu)) \wedge \psi \nvDash \bot$ iff $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ and $(\psi \circledast \nu) \wedge \psi \nvDash \bot$* *(Wise Join distribution)*
(**15***C*)	*If $\psi \nvDash \bot$ then $\psi \in \bigcup_{\mu \in \mathcal{L}} C_\bot(\mu)$* *($\bigcup$ −Completeness)*	($\circledast$**20**)	*If $\psi \nvDash \bot$ then there exists $\mu \in \mathcal{L}$ s.t. $\psi \wedge \mu \vDash \bot$ and $\psi \wedge \psi \circledast \mu \nvDash \bot$* *(Wise consistency)*

Let us now analyze all these properties together, taking advantage of the resemblance they have. Imprudent and wise postulates can be considered simultaneously similar and opposite. For example, ($\circledast$**15**) and ($\circledast$**18**) independently imply ($\circledast$**14**), the postulate that allows us to think of these properties without worrying about the syntax of the observations. Also, if ($\circledast$**14**) holds, then ($\circledast$**15**) can be deduced from ($\circledast$**16**) and ($\circledast$**18**) from ($\circledast$**19**).

As said before, any agent represented by an imprudent operator has more certainty about the observation the stronger the observation is, while a wise agent goes the other way around, being certain of observation near tautologies and doubting stronger observations. These behaviors are reflected by ($\circledast$**15**) and ($\circledast$**18**). Recalling that $\psi \in C_\bot(\mu)$ are the pieces of knowledge that actually make the agent doubts μ, we can rewrite them in a more intuitive way:

($\circledast$**15**)': $C(\mu) = \bigcap_{\mu \vDash \nu} C(\nu)$ ($\circledast$**18**)': $C(\mu) = \{\psi \mid \psi \wedge \mu \nvDash \bot\} \cup \bigcup_{\mu \vDash \nu} C_\bot(\nu)$

The rest of the properties are refinements of these behaviors. For example, Join Postulates ($\circledast$**16**) and ($\circledast$**19**) allow us to write $C(\mu)$ or $C_\bot(\mu)$ in terms of the complete formulas that imply μ, noted as α, when μ is consistent. If ($\circledast$**16**) holds, then $C(\mu) = \bigcup C(\alpha)$. Therefore, the weaker the formula, the more complete formulas are considered in the union, making the set bigger. Analogously, if ($\circledast$**19**) holds, then $C_\bot(\mu) = \bigcap C_\bot(\alpha)$. So, the weaker the formula, the smaller the set, since there are more complete formulas considered in the intersection.

Consistency Postulates ($\circledast$**17**) and ($\circledast$**20**) tell how the agents deal with inconsistent observations. Applying the alternative version of ($\circledast$**15**) and ($\circledast$**18**), it is clear that when observation $\gamma \vDash \bot$ we have that for imprudent agents, inconsistent observations are always certain ($C(\gamma) = \emptyset$), but wise agents always doubt inconsistent observation ($C(\gamma) = \{\psi \mid \psi \nvDash \bot\}$).

Although each postulate can stand on its own, clearly these properties behave well when ($\circledast$**14**) holds. That is why we are going to define two new branches from Doubt Extensional Moderated Revision.

Definition 4.15. A Doubt Extensional Moderated Revision satisfying:

- ($\circledast$**15**) is called Imprudent Moderated Revision;
- ($\circledast$**15**) and ($\circledast$**16**) is called Join Imprudent Moderated Revision;
- ($\circledast$**15**) and ($\circledast$**17**) is called Complete Imprudent Moderated Revision;
- ($\circledast$**15**) $\sim$ ($\circledast$**17**) is called Complete Join Imprudent Moderated Revision;
- ($\circledast$**18**) is called Wise Moderated Revision;
- ($\circledast$**18**) and ($\circledast$**19**) is called Join Wise Moderated Revision;
- ($\circledast$**18**) and ($\circledast$**20**) is called Complete Wise Moderated Revision;
- ($\circledast$**18**) $\sim$ ($\circledast$**20**) is called Complete Join Wise Moderated Revision;

Combining these properties with the previous ($\circledast$**6**) $\sim$ ($\circledast$**12**) affect how the subset $C_\bot(\mu)$ is defined, since originally we still lack, for example, *closure under logical equivalence* or *single sentence closure*. Some interesting relations appear when analyzing these new properties in the context of triggered moderated revision operators 4.12. The next proposition shows how the core belief function interacts with wise and imprudent properties.

Theorem 4.16. *Let $\circledast$ be a triggered moderated revision operator, satisfying ($\circledast$**1**) $\sim$ ($\circledast$**10**) and ($\circledast$**14**), and $\mathcal{A}$ the core belief function of its associated normal credibility-limited revision operator ©. Then $C_\bot(\mu) = \{\psi \mid \mathcal{A}(\mu) \wedge \neg\mu \wedge \psi \nvDash \bot,\ \psi \wedge \mu \vDash \bot\}$, and it also satisfies:*

1. ($\circledast$**15**)*, then $\circledast$ satisfies that if $\mu \vDash \nu$ then $\mathcal{A}(\mu) \vDash \mathcal{A}(\nu)$.*
2. ($\circledast$**16**)*, then $\mathcal{A}(\mu) = \bigvee \mathcal{A}(\bar{\alpha})$ being α the complete formulas s.t. $\alpha \vDash \mu$.*[2]
3. ($\circledast$**17**)*, then $\mathcal{A}$ preserves consistency: $\mu \nvDash \bot$ iff $\mathcal{A}(\mu) \nvDash \bot$.*
4. ($\circledast$**18**)*, then $\circledast$ satisfies that if $\mu \vDash \nu$ then $\mathcal{A}(\nu) \wedge \neg\nu \vDash \mathcal{A}(\mu)$.*
5. ($\circledast$**19**)*, then each $\psi \in C_\bot(\mu)$ satisfies $\mathcal{A}(\alpha) \wedge \psi \nvDash \bot$ and $\alpha \wedge \psi \vDash \bot$ for every α complete formula that implies μ.*
6. ($\circledast$**18**) *and* ($\circledast$**20**)*, then $\vDash \mathcal{A}(\mu)$ for every inconsistent μ.*

See proof in Appendix A.

Postulates ($\circledast$**11**) and ($\circledast$**12**) do not add properties to C but to the associated revision operator of the inducing credibility-limited revision operator. Therefore, they do not have any interaction with wise or imprudent postulates. Nevertheless, by the end of this section, we provide new properties that do relate to these kinds of postulates.

Surprisingly, wise and imprudent properties can also be combined between them, but it leads us to a fixed characterization of the credible function.

Theorem 4.17. *Let $\circledast$ be a doubt extensional moderated revision, satisfying ($\circledast$**1**)~($\circledast$**5**) and ($\circledast$**14**). If ($\circledast$**15**) and ($\circledast$**18**) also hold, then the operator satisfies ($\circledast$**16**), ($\circledast$**19**) and the following equality for every $\mu \in \mathcal{L}$:*

$$C(\mu) = C(\bot) \cup \{\psi \mid \psi \wedge \mu \nvDash \bot\}$$

See proof in Appendix A.

The set $C(\bot)$ is well defined by ($\circledast$**14**) and it is an arbitrary subset of $\mathcal{L}$ of consistent formulas. In this case we have that $C_\bot(\nu) \subseteq C(\bot)$ for every $\nu \in \mathcal{L}$. That is if there is a knowledge ψ that makes the agent doubt some arbitrary observation μ and $\psi \wedge \mu \vDash \bot$ (i.e. $\psi \in C_\bot(\mu)$), then this knowledge would make the agent doubt for every $\nu \in \mathcal{L}$ (i.e. $\psi \in C_\bot(\nu)$ for every $\nu \in \mathcal{L}$).

This is a new family of Doubt Extensional moderated revision, called Wise-Imprudent Moderated Revision

Definition 4.18. A $\circledast$ a moderated revision satisfying ($\circledast$**1**)~($\circledast$**5**), ($\circledast$**14**), ($\circledast$**15**) and ($\circledast$**18**) it is known as a Wise-Imprudent Moderated Revision.

Due to its characterization, some of these Wise-Imprudent Moderated Revision operators are already known or easier to define. For example, all the core belief functions of the Endorsed Wise-Imprudent Moderated Revision operators satisfy $\mathcal{A}(\mu) = \mathcal{A}(\bot) \vee \mu$. In particular, all the Wise-Imprudent Moderated Revision operators that also satisfy ($\circledast$**17**), hence $C(\bot) = \emptyset$, are a very particular kind of basic promotion operators: revision operators. Similarly, we have the family of Wise-Imprudent Moderated Revision where the inducing credibility-limited revision operator reduces to its revision operator, i.e. satisfying ($\circledast$**20**), which is also a basic promotion.

[2] $\bar{\alpha}$ is the canonical representation of α and the set of these representations is finite.

Corollary 4.19. *Let $\circledast$ be a moderated revision, satisfying* ($\circledast$**1**)∼($\circledast$**5**)*,* $*$ *the associated revision operator and* $*'$ *the revision operator that induces the associated credibility limited revision. Then, the following are equivalents:*

a. *$\circledast$ satisfies tautological credibility:*

($\circledast$**21**) *If ψ is consistent then $(\psi \circledast \mu) \wedge \psi \nvDash \bot$.*

b. *$\circledast$ is a Wise-Imprudent that satisfies* ($\circledast$**20**).
1. *$\circledast$ is a Basic Promotion such that $\vDash \mathcal{A}(\mu)$.*
2. *$\psi \circledast \mu = (\psi * \mu) \vee (\mu *' \psi)$ for every $\psi, \mu \in \mathcal{L}$.*
3. *$\circledast$ satisfies* ($\circledast$**6**) ∼ ($\circledast$**16**) *and* ($\circledast$**18**) ∼ ($\circledast$**20**).

See proof in Appendix A.

We can easily see that ($\circledast$**21**) is an adaptation of **(R3)** for the credibility-limited associated to the moderated revision operator, and since it is a basic promotion, we also have **(R4)**. Let us name this new family.

Definition 4.20. A $\circledast$ a moderated revision satisfying ($\circledast$**1**)∼($\circledast$**5**) and ($\circledast$**21**) is called Consensual Basic Promotion.

Postulate ($\circledast$**21**) leads us to a new idea for properties: the ones that put the old knowledge and the new observation at the same priority level as if it was a merging operator [18–20]. In this case, the fact that we are using the old knowledge to doubt the new observation means that they are both equally important for the agent, but the agent treats them differently, in the sense that two different revisions are being used. The next postulate characterizes the situation of equal treatment (i.e. with the same revision operator) of old knowledge and observation, but only in case of doubt.

Theorem 4.21. *Let $\circledast$ be a moderated revision, satisfying* ($\circledast$**1**)∼($\circledast$**5**)*,* $*$ *the associated revision operator and* $*'$ *the revision operator that induces the associated credibility-limited revision. Then for every $\psi, \mu \in \mathcal{L}$, we have that $\mu *' \psi \equiv \mu * \psi$ when $\psi \in \mathcal{C}(\mu)$ if and only if $\circledast$ satisfies:*

($\circledast$**22**) *If $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ then $\psi \circledast \mu) \wedge \psi \equiv (\mu \circledast \psi) \wedge \psi$.*

See proof in Appendix A.

The previous theorem gives us a new family of moderated revision, where the inducing operators use the same revision for both old knowledge and new observation in case of doubt, giving a more balanced result in terms of how the new knowledge is accepted.

Definition 4.22. A $\circledast$ a moderated revision satisfying ($\circledast$**1**)∼($\circledast$**5**) and ($\circledast$**22**) is called Balanced Moderated Revision.

Postulate ($\circledast$**22**), as postulates ($\circledast$**11**) or ($\circledast$**12**), does not affect the credible function but defines how the associated revision operator of the inducing credibility-limited revision acts. Clearly, it can be independently combined with postulates ($\circledast$**6**) ∼ ($\circledast$**10**) and ($\circledast$**13**) ∼ ($\circledast$**20**). When combined with ($\circledast$**11**) or ($\circledast$**12**) it implies that the inducing revision operator also reacts to the core belief function. Lastly, the combination of both postulates ($\circledast$**21**) and ($\circledast$**22**) is equivalent to having commutativity, thus satisfying many of the presented postulates, as the following result formalizes.

Corollary 4.23. *Let $\circledast$ be a moderated revision, satisfying* ($\circledast$**1**)∼($\circledast$**5**)*. The following are equivalents:*

a. *$\circledast$ satisfies commutativity:*

($\circledast$**23**) *$\psi \circledast \mu \equiv \mu \circledast \psi$.*

b. *$\circledast$ satisfies* ($\circledast$**21**) *and* ($\circledast$**22**).
c. *$\psi \circledast \mu = (\psi * \mu) \vee (\mu * \psi)$ for every $\psi, \mu \in \mathcal{L}$,*

These subsections conclude the classification of the moderated revision families in terms of the properties of the credible function. In the following subsection, we study the case where the revision operators related to moderated revision satisfy the supplementary postulates.

4.4. Supplementary postulates for moderated revision

In previous subsections, we classified moderated revision in terms of the properties of the credible function. In this section, we focus on the families of moderated revision families whose associated revision operators satisfy the supplementary postulates.

Definition 4.24. A moderated revision in which its associated revision operator satisfies **(R1)** ∼ **(R6)** is known as a Full Moderated Revision.

Let us add this property of the associated revision operator of a moderated revision in terms of postulates. Recall that adding **(R5)** and **(R6)** to **(R1)** ∼ **(R4)** is equivalent as adding **(R7)**. As in the case of extensionality and consistency in Theorem 4.4, the adaptation of this postulate is straightforward due to Theorem 4.2.

Theorem 4.25. *Let* ⊛ *be a moderated revision, satisfying* (⊛**1**)∼(⊛**5**)*, and* $*$ *its associated revision operator. Then operator* $*$ *satisfies* ***(R1)*** ∼ ***(R6)*** *if and only if* ⊛ *satisfies* ***R-minimality****:*

(⊛**24**) *If* $(\psi \circledast \mu) \wedge \mu \wedge \nu \nvDash \bot$ *then* $(\psi \circledast \mu) \wedge \mu \wedge \nu \equiv (\psi \circledast (\mu \wedge \nu)) \wedge \mu$

See proof in Appendix A.

In [3], the authors show that the revision associated to the credibility-limited can satisfy the supplementary postulates in the context of Endorsed Core Belief Revision that satisfies Strong Outcome Credibility. This family is known as Sphere-based Credibility-Limited Revision. The same can be done for Endorsed Moderated Revision that satisfies Strong Outcome Credibility.

Definition 4.26. An Endorsed Moderated Revision with Strong Outcome Credibility in which the associated revision operator of its credibility-limited satisfies **(R1)** ∼ **(R6)** is known as a Sphere-based Moderated Revision.

A Triggered Moderated Revision with Strong Outcome Credibility in which the associated revision operator of its credibility-limited satisfies **(R1)** ∼ **(R6)** is known as a Sphere-based Basic Promotion.

Recall that Triggered Moderated Revision operators are Endorsed Moderated Revision operators that also satisfy (⊛**14**). In the following representation theorems, we are only considering Endorsed Moderated Revision operators, but the same can be applied to Triggered Moderated Revision.

Theorem 4.27. *Consider* ⊛ *an endorsed moderated revision with strong outcome credibility, that is satisfying* (⊛**1**) ∼ (⊛**12**)*. Then* ⊛ *is a sphere-based moderated revision operator if and only if it satisfies* ***CL-disjunctive factoring:***

(⊛**25**) *Either* $((\psi \vee \phi) \circledast \mu) \wedge (\psi \vee \phi)$ *is equivalent to* $((\psi \circledast \mu) \wedge \psi) \vee ((\phi \circledast \mu) \wedge \phi)$, $(\psi \circledast \mu) \wedge \psi$ *or* $(\phi \circledast \mu) \wedge \phi$

See proof in Appendix A.

Note that combining (⊛**22**) with (⊛**24**) is the same as combining it with (⊛**25**), since both define the supplementary postulates for the revision operators.

Defining a sphere-based moderated revision consists of a long list of postulates. Also, the fact that property (⊛**25**) is not similar to (⊛**24**) tells us that there may be another simplified way to represent this family.

The following lemma, besides reducing the list of postulates needed to characterize sphere-based moderated revision, also considers an alternative axiomatic presentation more related to our setting in terms of the adaptation done in Theorem 4.25 of the Katsuno and Mendelzon's supplementary postulates **(R5)** and **(R6)**.

Lemma 4.28. *Consider* ⊛ *a moderated revision satisfying* (⊛**1**)∼(⊛**5**)*. The following conditions are equivalent:*

a. ⊛ *is a Sphere-based Moderated Revision, i.e. satisfying* (⊛**6**) ∼ (⊛**12**) *and* (⊛**25**)*.*
b. ⊛ *satisfies* (⊛**12**) *and* (⊛**25**)*.*
c. ⊛ *satisfies* (⊛**7**)*,* (⊛**8**) *and* (⊛**25**)*.*
d. ⊛ *satisfies:*

(⊛**26**) *If* $(\psi \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$ *then* $(\psi \circledast \mu) \wedge \psi \wedge \phi \equiv ((\psi \wedge \phi) \circledast \mu) \wedge \psi$
(⊛**27**) *If* $((\psi \wedge \phi) \circledast \mu) \wedge \psi \nvDash \bot$ *then* $(\psi \circledast \mu) \wedge \psi \nvDash \bot$.

See proof in Appendix A.

Previous theorem shows us that the new postulates capture not only the supplementary ones for the associated revision of the credibility-limited associated to the moderated revision but also all the core belief approach. Note that postulate (⊛**26**) is an adaptation of **(R7)** for the associated revision operator used in the credibility-limited revision of the ⊛ operator.

In this section, we presented a complete classification of the diverse family of moderated revision operators. We postpone the discussion of the family tree (Fig. 1) that summarizes all the connections to the conclusions since we still need to present another member of the family: Schwind, Konieczny and Marquis' belief promotion operator [21]. This family corresponds to the intersection between Full Moderated Revision and Sphere-based Basic Promotion families which in the following we call *Full Sphere-based Basic Promotion*. We do this in Section 5, where we ask ourselves where moderated revision fits in the known world of change operators. For example, we compare it with selective revision, one of the most general operators of the classical family of non-prioritized revision, as presented in [22]. We also show that, in fact, our proposal is more than a non-prioritized revision family by presenting an equivalent definition of moderated revision induced by contraction operators, that allows us to give a dual version of Theorem 4.2.

5. Relationship with other operators

In this section, we compare moderated revision with other known operators. The closest one to our proposal is *promotion* [21]. In fact, we show that it is a particular case of moderated revision. Next, we consider *selective revision*, the most general non-prioritized operator, and show that although our proposal is related, they are in fact complementary concepts in terms of revision. Lastly, we analyzed moderated revision in terms of Levi and Harper identities, where *contraction and shielded contraction operators* appear.

5.1. Promotion

In [21], Schwind, Konieczny and Marquis' presented *belief promotion operators*, a non-prioritized kind of revision with the following rationale: an agent may prioritize new information μ, but sometimes she will consider preserving some of its original beliefs ψ if they are "close enough" to the new data.

Definition 5.1. A belief promotion operator $\ominus$ is defined by the postulates:

$\ominus$ **1.** $\psi \ominus \mu \vDash \psi \vee \mu$.
$\ominus$ **2.** If $\psi \wedge \mu \nvDash \bot$ then $\psi \ominus \mu \equiv \psi \wedge \mu$.
$\ominus$ **3.** If $\mu \nvDash \bot$ then $(\psi \ominus \mu) \wedge \mu \nvDash \bot$.
$\ominus$ **4.** If $\psi \equiv \psi'$ and $\mu \equiv \mu'$then, $\psi \ominus \mu \equiv \psi' \ominus \mu'$.
$\ominus$ **5.** If $(\psi \ominus \mu) \wedge \mu \wedge \nu \nvDash \bot$ then $(\psi \ominus (\mu \wedge \nu)) \wedge \mu \equiv (\psi \ominus \mu) \wedge \mu \wedge \nu$.
$\ominus$ **6.** If $(\psi \ominus \mu) \wedge \psi \wedge \varphi \nvDash \bot$ then $((\psi \wedge \varphi) \ominus \mu) \wedge \psi \equiv (\psi \ominus \mu) \wedge \psi \wedge \varphi$.
$\ominus$ **7.** If $((\psi \wedge \nu) \ominus \mu) \wedge \psi \nvDash \bot$ then $(\psi \ominus \mu) \wedge \psi \nvDash \bot$.

As can be seen, promotion satisfies (⊛**1**)~(⊛**3**), **(R4)** (or equivalently, (⊛**4**), (⊛**7**) and (⊛**14**)), (⊛**24**), (⊛**26**) and (⊛**27**). Since (⊛**26**) and (⊛**7**) imply (⊛**5**), we have that promotion is a moderated revision. In fact, the family of Promotion Operators corresponds to Full Sphere-based Basic Promotion. Nevertheless, Schwind et al. took a different approach in terms of the representation theorem, giving us an interesting comparison. To accomplish the representation, they introduced the concept of the trigger function.

Definition 5.2. A trigger function is a mapping $\sigma : \mathcal{L} \to \mathcal{L}$ associating any formula μ with a formula $\sigma(\mu)$ such that for every formula μ, μ':

1. $\mu \vDash \sigma(\mu)$;
2. If $\mu \equiv \mu'$ then $\sigma(\mu) \equiv \sigma\left(\mu'\right)$.

The formula $\sigma(\mu)$ establishes which formulas are preservable according to μ, i.e. we are going to say that a formula φ is preservable with respect to μ, if $\|\varphi \wedge \sigma(\mu)\| \neq \emptyset$. Then, their trigger function can be understood as the core belief function of the moderated revision operator.

Theorem 5.3. *([21]) A promotion operator $\ominus$ satisfies conditions $(\ominus 1) \sim (\ominus 7)$ if and only if there exist two revision operators $*_1$, $*_2$ satisfying **(R1)** ~ **(R6)** and a trigger function σ, such that for every formula ψ and μ:*

$$\psi \ominus \mu \equiv \left(\psi *_1 \mu\right) \vee \left(\mu *_2 (\psi \wedge \sigma(\mu))\right)$$

By combining our representation theorem and theirs, we can see that $(\mu © \psi) \wedge \psi$ can be rewritten as $\left(\mu *_2 (\psi \wedge \sigma(\mu))\right)$. Our representation gains in interest since the trigger function does not appear i.e. it is not explicitly mentioned which is the case in the representation theorem for $\ominus$ in Proposition 5 of [21]. Since their trigger function is our core belief function, this means that, in terms of the credibility-limited revision operator, σ is "hidden" in the definition of the credible function C.

The authors also added some new postulates to promotion, giving extra properties for the trigger function (a.k.a. core belief function) and the revision operators. We have already defined them before, where the trigger function is a tautology, that is (⊛**21**), and when the operator is commutative, i.e. (⊛**23**). They called them Consensual Promotion and Commutative Promotion respectively. They used the latter to show that Liberatore and Schaerf's [17] Commutative Revision operators or Arbitration and a particular case of merging operators are special cases of Commutative Promotion, therefore they are also special cases of Moderated Revision. Moreover, more combinations and generalizations can be achieved since we separate all the properties into different families. For example, it is no longer necessary to work with full revision or trigger functions, and many families that we have presented are different generalizations of these operators.

In the following subsection, we compare our proposal with Two-Level Credibility-Limited Revision, whose aim resembles ours.

5.2. Two-level credibility-limited revision

In [5], the author proposes a family of non-prioritized revision operators. This kind of revision has two levels of credibility. When an old knowledge ψ is revised by a new observation μ, then the degree of credibility of this observation is analyzed. If the observation belongs to the first and highest level of credibility, then the revision works as a standard AGM operator. If otherwise, it is considered to be at the second level of credibility, then that sentence is not incorporated in the revision process, however, its negation is removed from the original knowledge. The underlying intuition is that the observation by which the old knowledge is revised is not credible enough to be incorporated but, it creates in the agent sufficient doubt that forces the agent to remove the beliefs that are inconsistent with it. When revised by a non-credible sentence, the operator leaves the original belief set unchanged. Both the general idea and the intuition have some resemblance to the ones of moderated revision and this is our motivation for comparing them.

The following definition formalizes this concept adapted to our framework[3]:

Definition 5.4. [5, Definition 3.7] Given a fixed knowledge ψ, a two-level credibility-limited revision operator $\odot : \mathcal{L} \to \mathcal{L}$ for a knowledge ψ is defined by a revision operator $*$ satisfying **(R1)** $\sim$ **(R4)** and two credible sets C_H and C_L such that for every $\mu \in \mathcal{L}$:

$$\psi \odot \mu = \begin{cases} \psi * \mu & \text{if } \mu \in C_H \\ (\psi * \mu) \vee \psi & \text{if } \mu \in C_L \\ \psi & \text{otherwise} \end{cases} \tag{3}$$

Note that according to the Harper identity $(\psi * \mu) \vee \psi$ coincides with the contraction of ψ by $\neg\mu$. In [6], a very similar operator is also divided into the same three parts but the interpretation for each case is a little different and corresponds to credible, allowable, and rejectable, respectively.

It is easy to see that given an AGM revision $*$, if we fix C_L and take $C_H = \mathcal{L} \setminus C_L$, then the operator defined above is a moderated revision by taking $C(\mu) = C_L$, and the underlying credibility-limited revision is defined as:

$$\mu \circ \psi = \begin{cases} \psi & \text{if } \psi \in C(\mu) \\ \mu & \text{otherwise} \end{cases}$$

Essentially because, $\psi \in C_H$ is equivalent to $\psi \notin C_L$, and then, $\psi \odot \mu = (\psi * \mu) \vee ((\mu \circ \psi) \wedge \psi)$. Note that according to this construction, the last case of (3) will never happen for $\odot$ since C_H and C_L are complementary. That means that the intersection of moderated revisions and two-levels credibility-limited revisions correspond to the class of operators defined by (3) for any AGM revision operator, any C_L and fixing $C_H = \mathcal{L} \setminus C_L$.

Now we are going to compare our proposal with Selective Revision, where we show that Moderated Revision can not be classified as a subfamily of it, making it one of a kind as a non-prioritized revision.

5.3. Selective revision

The selective revision operator appeared in [23] as a third option between full acceptance or full rejection of the new information μ over a knowledge ψ.[4] This operator is formally defined by a revision operator $*$ satisfying **(R1)** $\sim$ **(R4)** and a Transformation Function $f : \mathcal{L} \to \mathcal{L}$ that represents how the new information is handled before the revision operator is applied:

$$\psi \circ_f \mu \equiv \psi * f(\mu)$$

The main idea of selective revision is that the new information has to be partially rejected or changed in order to be properly accepted. This is the main difference from moderated revision since it does not reject nor change the observation given. It is accepted as it is, but eventually with doubts. In fact, the following theorem shows that the family of moderated revision operators and selective revision operators only share the family of classical revision operators.

Theorem 5.5. *Let $\circ_f$ be a selective revision induced by revision operator $*$ and a transformation function $f : \mathcal{L} \to \mathcal{L}$ over a knowledge ψ. If $\circ_f$ is also a moderated revision according to Definition 4.1 then, for every $\mu \in \mathcal{L}$ with μ consistent:*

$$\psi \circ_f \mu = \psi * \mu$$

and $*$ is the revision that induces $\circ_f$ as a moderated revision operator.

See proof in Appendix A.

Since it is known that credibility-limited revision is a particular case of selective revision [22], it is important to remark that Theorem 5.5 also holds for credibility-limited revision, i.e. the only credibility-limited revision operators that behave like a moderated

[3] Originally the knowledge is a fixed belief set K, but we adapted the definition to our finite context by considering a fixed formula ψ.

[4] As happened with Two-Level Credibility-Limited Revision, we adapted the definition to our finite context.

revision are revision operators. This observation must not be confused with the fact that it is possible to recover the credibility-limited revision associated with a moderated revision.

Now that we know that moderated revision is a unique kind of non-prioritized revision, we are going one step further by presenting a whole new idea of what really moderated revision is. Moderated revision has some interesting properties related to the dynamics of epistemic attitudes and Levi and Harper's identities, linking it also with contraction operators.

5.4. Contraction operators and Levi and Harper's Identities

Just by considering its induced definition, moderated revision has some interesting properties related to Levi and Harper's identities and the dynamics of epistemic attitudes. In fact, Theorem 4.2 can be understood from this perspective as a Levi Moderated Identities. In order to continue with this analysis, we need first to talk about contraction and shielded contraction operators.

Although we first presented credibility-limited revision, this operator originally appears as the dual of shielded contraction in [4]. An induced definition for shielded contraction can be given following Theorem 3.3 and the Harper identity. However, since we are only interested in this operator due to duality properties, our adaptation is based directly on the Harper Identity, i.e. as induced by a credibility-limited revision operator.

Definition 5.6. A shielded contraction operator $\div$ is defined by a credibility-limited revision $\circ$ satisfying $(\circ\mathbf{1}) \sim (\circ\mathbf{5})$ such that for every $\psi, \mu \in \mathcal{L}$:

$$\psi \div \mu \equiv (\psi \circ \neg\mu) \vee \psi$$

The following theorem is the adaptation of Levi identity for credibility-limited revision operators, which shows how to recover the credibility-limited revision from a shielded contraction.

Theorem 5.7 *([4]). Let $\div$ be a shielded contraction. Then, the following construction satisfies $(\circ\mathbf{1}) \sim (\circ\mathbf{5})$:*

$$\psi \circ \mu = \begin{cases} (\psi \div \neg\mu) \wedge \mu & \text{if } (\psi \div \neg\mu) \wedge \mu \nvdash \bot \\ \psi & \text{otherwise} \end{cases}$$

Also, $\circ$ is equivalent to the credibility-limited revision that defines the $\div$.

Now we show in the following result that moderated revision is not just a non-prioritized revision operator, but also a non-prioritized contraction operator which is its own dual in terms of Levi and Harper's Identities.

Theorem 5.8. *Let $\circledast$ a moderated revision, induced by $*$ revision operator and $\circ$ credibility-limited revision operator, and consider also $-$ the contraction operator associated to $*$ and $\div$ the shielded contraction associated to $\circ$. Then:*

$$\psi \circledast \mu \equiv (\psi - \neg\mu) \wedge (\mu \div \neg\psi)$$

See proof in Appendix A.

Due to this newly induced definition in terms of contractions, a natural question is to confirm if the dual version of Theorem 4.2 holds. This is shown in the following result.

Theorem 5.9 *(Moderated Harper identities). Let $\circledast$ a moderated revision, induced by $*$ revision operator and $\circ$ credibility-limited revision operator, and consider also $-$ the contraction operator associated to $*$ and $\div$ the shielded contraction associated to $\circ$. Then:*

1. $\psi - \mu \equiv (\psi \circledast \neg\mu) \vee \psi$.
2. $\mu \div \psi \equiv (\neg\psi \circledast \mu) \vee \mu$.

See proof in Appendix A.

Let us analyze Theorem 4.2 a bit further. Note that if we compare these identities with classical and credibility-limited Levi Identities, moderated revision, represented by $\psi \circledast \mu$, is in some way replacing the contraction operator appearing as $\psi - \neg\mu$. In a similar way, we can see that in Theorem 5.9 moderated revision, represented by $\psi \circledast \mu$, is replacing the revision operator appearing as $\psi * \neg\mu$. Therefore, combining these theorems with Representation Theorem 4.4 allows us to think that we can characterize revisions, contractions, credibility-limited revisions and shielded contractions in terms of moderated revision operators.

Concluding this section, we can say that Theorems 5.8, 5.9 and 4.2 represent the key concepts of moderated revision: it models a position between contraction and revision, in the sense that we can recover both operators by applying conjunction or disjunction respectively, as in Levi and Harper Identities. Moreover, the choice mechanism can also be recovered using the same idea, considering the general version of the identities. Lastly, the fact that it can be represented with (credibility-limited) revisions on one side or (shielded) contractions on the other side tells us that in terms of Levi and Harper's identities, moderated revision is its own dual. Finally, these special properties make the operator capable of traveling through all the epistemic attitudes.

6. Conclusions and future work

In this work, we have proposed moderated revision operators, a new family of non-prioritized operators. Our proposal is originally presented as induced by a revision and a credibility-limited revision, in order to model an operator that stands between revision and contraction. While the revision operator preserves the relevance of the new observation, the credibility-limited revision is in charge of deciding if the observation is doubtful or not through its credible function and considering the original knowledge. We also show that both inducing operators can be recovered from moderated revision in a similar way that they are recovered in the Levi identities.

We have chosen Katsuno and Mendelzon's settings for our work since we needed to logically represent the original knowledge and the new observation the same way, and both of them are represented through formulas in this context. This choice forces us to rewrite some properties of revision and credibility-limited revision to this setup. The many known properties of revision and credibility-limited revision grant us the possibility of knowing numerous types of moderated revision, and we added some other properties that let us expand the family even more. For each property, we have presented an axiomatic representation and we also explore the combination of them, completing the comparison of the induced definition moderated revision and the one using postulates. This led us to a list of more than thirty different kinds of moderated revision, including promotion, revision, consensual promotion and commutative promotion as already known operators of the family. A summary of this "family tree" can be seen from Fig. 1 with some detailed branches from Figs. 2 and 3.

We continued by analyzing what kind of non-prioritized revision operator moderated revision is, and compared it with the most general known non-prioritized revision operator: selective revision. We concluded not only that they are completely different kinds of non-prioritized revision operators, but that revision of consistent observations is the only operator that is at the same time a moderated and a selective revision.

Lastly, we studied the connections between moderated revision and contraction and found out that our proposal is also a non-prioritized contraction operator and its own dual operator. We presented a new induced definition of moderated revision using a contraction and a shielded contraction and, analogously from the revision case, both operators were recovered in a similar way that they are recovered in the Harper identities.

We would like to conclude our paper by mentioning some future work:

- In [13], the authors also make a definition of credibility-limited revisions into the Katsuno-Mendelzon (KM) framework which is less general than the one introduced in Section 3. However, they also give a characterization of credibility-limited revisions into the Darwiche and Pearl (DP) framework ([24]) which have not been considered here. We would like to look at their last approach for studying an iterated version of moderated revision in the sense of the Darwiche-Pearl framework.
- We would like to explore an extension of our approach by using theories and bases instead of formulas. We think the most appropriate framework for that is given by multiple revision [25,26].
- In this paper, we introduce Moderated Revision by combining a belief revision operator and a credibility-limited one. We think that it is possible to extend our proposal to combine any two belief revision operators. In particular, multiple levels of credibility, as in [5–7], or in [8], a dynamic-limited revision operator is considered, which are generalizations of credibility-limited revision.

CRediT authorship contribution statement

Daniel Grimaldi: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Maria Vanina Martinez:** Writing – review & editing, Writing – original draft, Funding acquisition, Formal analysis, Conceptualization. **Ricardo O. Rodriguez:** Writing – review & editing, Writing – original draft, Supervision, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

We wish to thank the two referees of the IJAR for their valuable comments on the first version of this paper which have significantly contributed to improving its readability. Martinez was funded in part by Secretaría de Ciencia y Técnica de la Universidad de Buenos Aires (RESCS-2020-345-E-UBA-REC), CONICET under the PIP (grant 112-2020-0101408CO), Agencia Nacional de Promoción Científica y Tecnológica, Argentina under grants PICT-2018-0475 (PRH-2014-0007) and PICT-2020-SERIEA-01481. Rodriguez acknowledges partial support of Argentinean projects: PICT-2019-2019-00882, UBA-CyT-20020190100021BA and PIP 112-2015-0100412CO. Finally, the three authors want to thank the partial support of The EU-funded MOSAIC project (Grant agreement ID: 101007627). The authors also acknowledge support by the Spanish project PID2022-139835NB-C21 funded by MCIN MCIN/AEI/10.13039/501100011033 and by "European Union NextGenerationEU/PRTR".

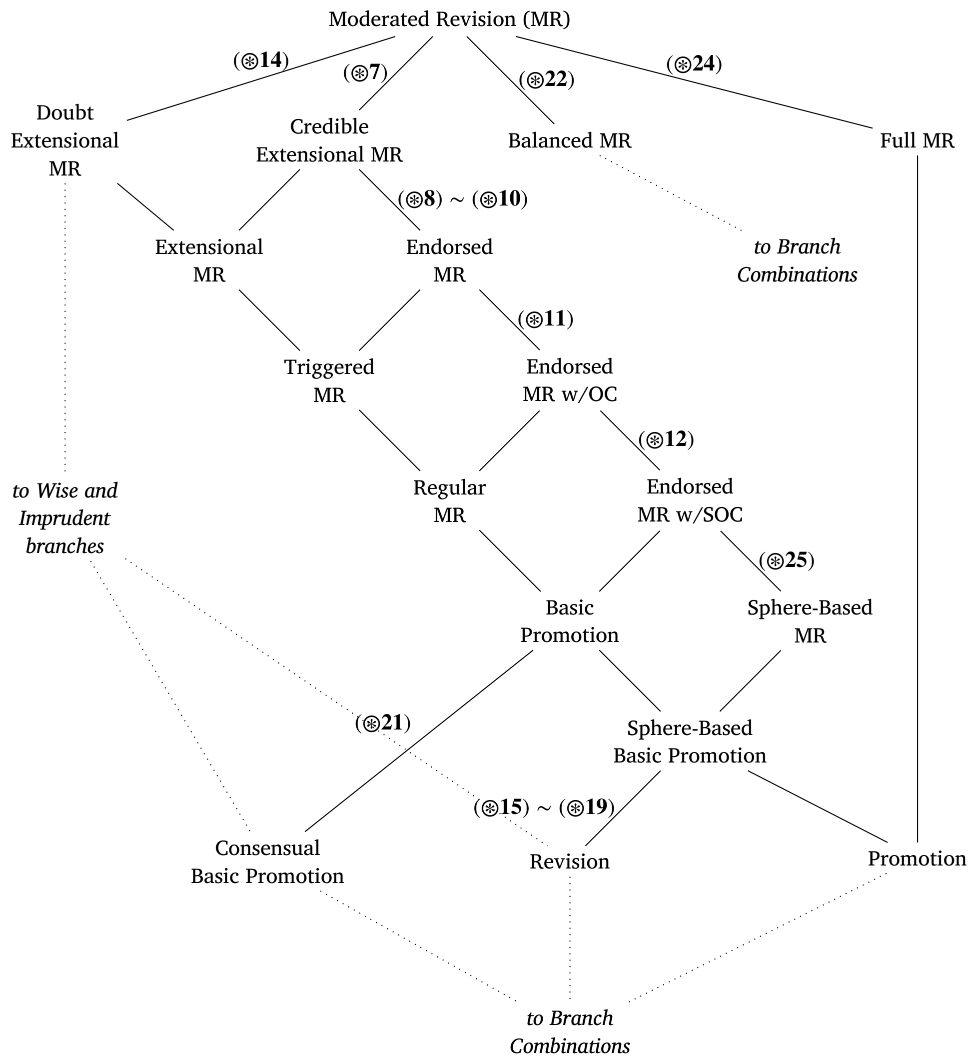

Fig. 1. Main tree of moderated revision.

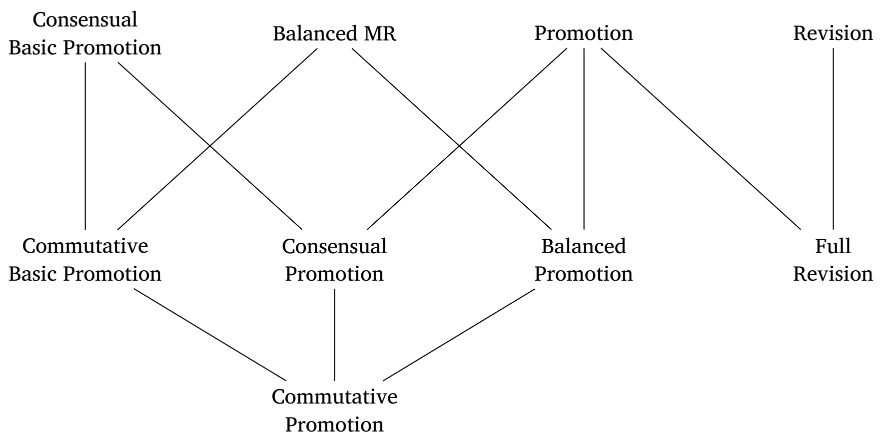

Fig. 2. Branch Combinations from Main Tree.

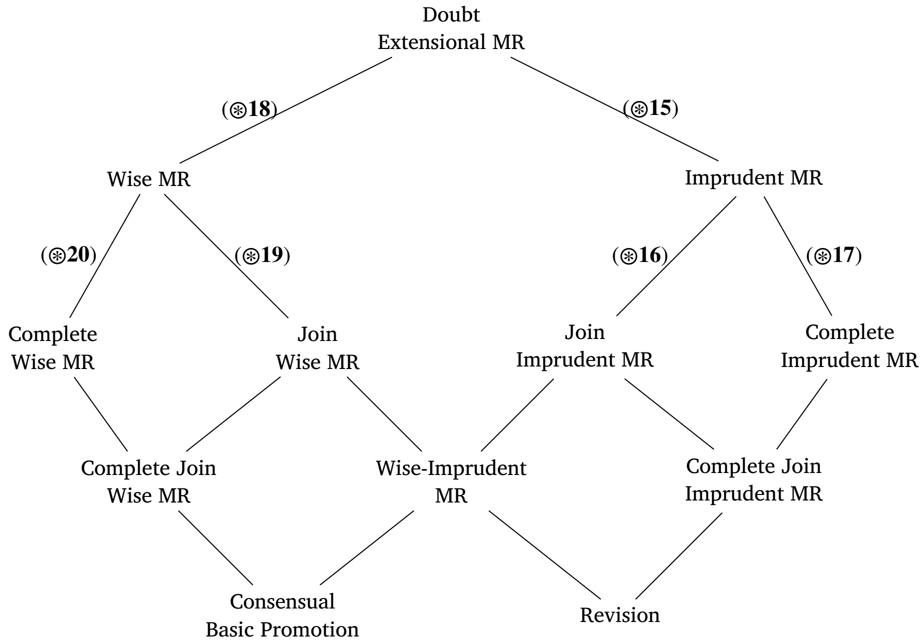

Fig. 3. Imprudent and Wise Branches from Main Tree.

Appendix A. Proofs

Proof of Theorem 3.3. **(a. ⇒ b.)** Consider an arbitrary revision operator $*'$ satisfying **(R1)** ∼ **(R4)** and define:

$$\psi * \mu \equiv \begin{cases} \phi \circ \nu & \text{if } \exists \phi,\, \nu \text{ s.t. } \phi \circ \nu \models \nu,\, \phi \circ \nu \not\models \bot,\, \psi \equiv \phi,\, \mu \equiv \nu \\ \psi *' \mu & \text{otherwise} \end{cases}$$

$$C(\psi) = \begin{cases} \{\alpha \in \mathcal{L} \mid \psi \circ \alpha \models \alpha\} & \text{if } \psi \not\models \bot \\ \{\alpha \in \mathcal{L} \mid \psi \circ \alpha \not\models \bot\} & \text{if } \psi \models \bot \end{cases}$$

After noting that $*$ is well defined by relative extensionality, let us show that it is a revision operator. First, $*$ satisfies **(R1)** and **(R3)** by definition. If $\psi \wedge \mu \not\models \bot$ then for every $\phi \equiv \psi$ and every $\nu \equiv \mu$ we have $\phi \wedge (\phi \circ \nu) \not\models \bot$ by inclusion, and hence by consistent expansion $\phi \circ \nu \models \phi$. Therefore if $\phi \circ \nu \models \nu$ then $\psi * \mu \equiv \phi \circ \nu \equiv \psi \wedge \mu$ by inclusion; and if not, $\psi * \mu \equiv \psi \wedge \mu$ by vacuity of $*'$. In any case, $*$ satisfies **(R2)**. Lastly, assume that $\psi \equiv \psi'$ and $\mu \equiv \mu'$. Note that the condition of $*$ is true for ψ and μ if and only if it is also true for ψ' and μ', both satisfied by the same $\phi \equiv \psi \equiv \psi'$ and $\nu \equiv \mu \equiv \mu'$. Therefore $\psi * \mu \equiv \phi \circ \nu \equiv \psi' * \mu'$ where $\phi \equiv \psi \equiv \psi'$, $\nu \equiv \mu \equiv \mu'$ or $\psi * \mu \equiv \psi *' \mu \equiv \psi' *' \mu' \equiv \psi * \mu'$. Either way, $*$ satisfies **(R4)**.

Now, consider the credibility-limited revision induced by $*$ and $C(\psi)$:

$$\psi \bar{\circ} \mu \equiv \begin{cases} \psi * \mu & \text{if } \mu \in C(\psi) \\ \psi & \text{otherwise} \end{cases}$$

Let us show that $\bar{\circ}$ defines the same operator as $\circ$. We separate into cases:

Case I: $\psi \models \bot$ *and* $\mu \models \bot$. By definition of $\bar{\circ}$ we know that $\psi \bar{\circ} \mu \models \bot$ in any case, and by relative success $\psi \circ \mu \models \bot$.

Case II: $\psi \models \bot$ *and* $\mu \not\models \bot$. If $\mu \in C(\psi)$ then $\psi \circ \mu \not\models \bot$ and $\psi \circ \mu \models \mu$ by relative success. Therefore $\psi \bar{\circ} \mu \equiv \psi * \mu \equiv \psi \circ \mu$ by definition of $\bar{\circ}$. If $\mu \notin C(\psi)$ then both $\psi \bar{\circ} \mu$ and $\psi \circ \mu$ are inconsistent.

Case III: $\psi \not\models \bot$ *and* $\mu \models \bot$. In this case $\psi \circ \mu \equiv \psi$ if $\mu \notin C(\psi)$ and $\psi \circ \mu \models \bot$ if $\mu \in C(\psi)$, by relative success. This is also true for $\psi \bar{\circ} \mu$ by definition and $*$ success.

Case IV: $\psi \not\models \bot$ *and* $\mu \not\models \bot$. Note that for every $\phi \equiv \psi$ and $\nu \equiv \mu$, we have that $\phi \circ \nu \not\models \bot$ by weak consistency preservation. If $\mu \in C(\psi)$ then $\psi \circ \mu \models \mu$ and $\psi \bar{\circ} \mu \equiv \psi * \mu$. Therefore $\psi \bar{\circ} \mu \equiv \psi \circ \mu$. If $\mu \notin C(\psi)$ then $\psi \bar{\circ} \mu \equiv \psi$ by definition and $\psi \circ \mu \equiv \psi$ by relative success.

These four cases combined show that $\psi \bar{\circ} \mu \equiv \psi \circ \mu$ for every $\psi, \mu \in \mathcal{L}$.

(b. ⇒ a.) Consider the $\circ$ operator induced by the $*$ and $C(\psi)$ of the hypothesis as follows:

$$\psi \circ \mu \equiv \begin{cases} \psi * \mu & \text{if } \mu \in C(\psi) \\ \psi & \text{otherwise} \end{cases}$$

Then $\circ$ trivially satisfies relative success, inclusion and weak consistency preservation. Assume now that $\psi \wedge (\psi \circ \mu) \nvDash \bot$. Clearly, if $\mu \notin C(\psi)$ then $\psi \circ \mu \vDash \psi$. If $\mu \in C(\psi)$ then $\psi \wedge (\psi * \mu) \nvDash \bot$ and by success this implies that $\psi \wedge \mu \nvDash \bot$. Then $\psi * \mu \equiv \psi \wedge \mu$ by vacuity and therefore $\psi \circ \mu \vDash \psi$. We showed that $\circ$ satisfies consistent expansion.

Lastly, consider $\psi \equiv \phi$ and $\mu \equiv \nu$ such that $\psi \circ \mu \vDash \mu$ and $\phi \circ \nu \vDash \nu$. Note that if $\psi \vDash \mu$ then, in any case, $\psi \circ \mu \equiv \psi \equiv \phi \circ \nu$, so assume that $\psi \nvDash \mu$. Hence, $\mu \in C(\psi)$ and $\nu \in C(\phi)$, therefore $\psi \circ \mu \equiv \psi * \mu \equiv \phi * \nu \equiv \phi \circ \nu$, satisfying relative extensionality. □

Proof of Corollary 3.4. Observe that for every $\circ$ induced by $*$ and a credible function C satisfies postulates $(\circ\mathbf{1}) \sim (\circ\mathbf{5})$ by the proof of **(b. ⇒ a.)** in Theorem 3.3, and we can redefine it by using the construction done in **(a. ⇒ b.)** in Theorem 3.3. Clearly, the condition when $\psi \vDash \bot$ holds by definition. Let us check that the construction of this C' satisfies that if $\psi \vDash \mu$ and $\psi \nvDash \bot$ then $\mu \in C'(\psi)$. In this case, we have that $\psi \vDash \psi \circ \mu$ by inclusion. Also, since $\psi \nvDash \bot$, the premise of consistent expansion holds, therefore $\psi \circ \mu \vDash \psi$. Hence $\psi \circ \mu \equiv \psi \vDash \mu$, that is, $\mu \in C'(\psi)$ by definition. The rest of the proof of **(a. ⇒ b.)** in Theorem 3.3 shows the equivalence between these operators. □

Proof of Theorem 3.5. **(i. ⇔ a.)** If C satisfies element consistency and $\psi \nvDash \bot$ then $\psi \circ \mu \nvDash \bot$ by definition of $\circ$ and **(R3)** of the associated revision operator. If $\circ$ satisfies consistency preservation and $\mu \vDash \bot$ then $\mu \notin C(\psi)$ or $\psi \vDash \bot$.

(ii. ⇔ b.) Assume $\psi \wedge \mu \nvDash \bot$, then $\psi \circ \mu = \psi$ or $\psi \circ \mu = \psi \wedge \mu$ by definition of $\circ$ and **(R2)** of the associated revision operator. We separate in two cases: if $\psi \vDash \mu$ then vacuity is trivially satisfied, since $\psi \wedge \mu \equiv \psi$. Therefore $\psi \circ \mu \equiv \psi$ either $\mu \in C(\psi)$ or not. If $\psi \nvDash \mu$ then $\psi \wedge \mu \not\equiv \psi$, hence $\psi \circ \mu \equiv \psi \wedge \mu$ if and only if $\mu \in C(\psi)$. □

Proof of Theorem 4.2. Using the fact that $\psi \circledast \mu \equiv ((\psi \circledast \mu) \wedge \mu) \vee ((\psi \circledast \mu) \wedge \psi)$, note that each identity trivially holds if $\psi \vDash \bot$. A similar situation occurs when $\psi \wedge \mu \nvDash \bot$, since $\psi \circledast \mu \vDash (\psi \wedge \mu) \vee (\psi \wedge \mu)$ and also $\psi \wedge \mu \vDash \psi \circledast \mu$ both by definition, hence $\psi \circledast \mu \equiv \psi \wedge \mu$. Therefore each identity holds by revision vacuity.

So, the interesting case is when $\psi \wedge \mu \vDash \bot$ and $\psi \nvDash \bot$. Looking back at $\psi \circledast \mu \equiv ((\psi \circledast \mu) \wedge \mu) \vee ((\psi \circledast \mu) \wedge \psi)$, note that in this case, the conjunction of the credibility-limited revision with μ is inconsistent, thus $(\psi \circledast \mu) \wedge \mu \equiv \psi * \mu$ and the first identity holds. The same idea works with the revision term and ψ, so $(\psi \circledast \mu) \wedge \psi \equiv (\mu \circ \psi) \wedge \psi$. Note also that the consistency of $(\psi \circledast \mu) \wedge \psi$ is given by the belonging of ψ in $C(\mu)$. Therefore, we can say that in this particular case, $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ if and only if $\psi \in C(\mu)$, giving us the third identity. Moreover, when $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ we know that $(\mu \circ \psi) \wedge \psi \equiv \mu *' \psi$, giving us the equivalence we want to prove for the second identity. □

Proof of Corollary 4.3. Clearly, the correspondence $(*, ©) \mapsto \circledast$ is given by Definition 4.1. Let us define the correspondence $\circledast \mapsto (*, ©)$. Theorem 4.2 allows to fully recover the revision operator $*$ used to define $\circledast$ and to obtain the parts of the credibility-limited revision operator $\circ$ that are relevant to $\circledast$. Also, if $*'$ is the revision operator associated to $\circ$, $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ and $\psi \wedge \mu \vDash \bot\}$ then $(\psi \circledast \mu) \wedge \psi \equiv \mu *' \psi$. Define then the following operator

$$\mu © \psi = \begin{cases} (\psi \circledast \mu) \wedge \psi & \text{if } (\psi \circledast \mu) \wedge \psi \nvDash \bot \text{ and } \psi \wedge \mu \vDash \bot \\ \psi \wedge \mu & \text{if } \psi \wedge \mu \nvDash \bot \\ \mu & \text{if } (\psi \circledast \mu) \wedge \psi \vDash \bot \text{ and } \psi \wedge \mu \vDash \bot \end{cases}$$

Note that it is a credibility-limited revision operator with its credible function defined by $C'(\mu) = C(\mu) \cap \{\psi \nvDash \bot \mid \psi \wedge \mu \vDash \bot\} \cup \{\psi \wedge \mu \nvDash \bot\}$. Therefore $\{\psi \wedge \mu \nvDash \bot\} \subseteq C'(\mu) \subseteq \{\psi \nvDash \bot\}$. We can conclude then that it is a normal credibility-limited revision operator.

Now we prove that the result of the composition of these correspondences is the identity $(*, ©) \mapsto (*, ©)$. This is true for the $*$ operator by Theorem 4.2. If the induced credibility-limited revision of the moderated revision is normal, then $C(\mu) \cap \{\psi \nvDash \bot \mid \psi \wedge \mu \vDash \bot\} = C_{\bot}(\mu)$. Therefore, $C(\mu) = C'(\mu)$ for every $\mu \in \mathcal{L}$ by definition of C in terms of $C_{\bot}$ and the definition of C'. Since the credible functions are the same, then the associated revision operators are equivalents due to the construction of the $\circledast \mapsto (*, ©)$ correspondence. Recall that two normal operators are equivalent if they share the same credible function and have equivalent revision operators $*$ and $*'$ for every credible observation. □

Proof of Theorem 4.4. **(a. ⇒ b.)** Assume we have an operator $\circledast$ that satisfies $(\circledast\mathbf{1})\sim(\circledast\mathbf{5})$ and define:

$$\phi * \gamma \equiv (\phi \circledast \gamma) \wedge \gamma \qquad \phi \circ \gamma \equiv \begin{cases} \phi *' \gamma & \text{if } \gamma \in C(\phi) \\ \phi & \text{otherwise} \end{cases}$$

where $C(\phi) = \{\alpha \in \mathcal{L} \mid (\alpha \circledast \phi) \wedge \alpha \nvDash \bot\}$ and:

$$\phi *' \gamma \equiv \begin{cases} (\alpha \circledast \beta) \wedge \alpha & \text{if } \exists \alpha, \beta \text{ s.t. } (\alpha \circledast \beta) \wedge \alpha \nvDash \bot, \alpha \equiv \gamma, \beta \equiv \phi \\ \gamma & \text{otherwise} \end{cases}$$

Clearly, $*$ satisfies **(R1)** by definition and **(R2)** $\sim$ **(R4)** by $(\circledast\mathbf{2})\sim(\circledast\mathbf{4})$ respectively. On the other hand, $*'$ satisfies **(R1)** $\sim$ **(R3)** by definition and $(\circledast\mathbf{2})$, and **(R4)** holds by $(\circledast\mathbf{5})$.

Let us show that $\psi \circledast \mu \equiv (\psi * \mu) \vee ((\mu \circ \psi) \wedge \psi)$. Note first that if $\psi \wedge \mu \nvDash \bot$ then the equivalence trivially holds by vacuity of all the operators. So assume $\psi \wedge \mu \vDash \bot$. By $(\circledast\mathbf{1})$ we can rewrite $\psi \circledast \mu$ as $\psi \circledast \mu \equiv ((\psi \circledast \mu) \wedge \mu) \vee ((\psi \circledast \mu) \wedge \psi)$, therefore $\psi \circledast \mu \equiv (\psi * \mu) \vee ((\psi \circledast \mu) \wedge \psi)$ by definition. Now we only need to prove that $((\psi \circledast \mu) \wedge \psi) \equiv ((\mu \circ \psi) \wedge \psi)$. If $(\psi \circledast \mu) \wedge \psi \vDash \bot$ then $\psi \notin C(\mu)$ and therefore $(\mu \circ \psi) \wedge \psi \equiv \mu \wedge \psi \vDash \bot$. Since both are inconsistent, in particular, they are equivalent. If $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ then $\psi \in C(\mu)$ and $\mu \circ \psi \equiv (\psi \circledast \mu) \wedge \psi$, validating what we wanted to prove, thus proving $\psi \circledast \mu \equiv (\psi * \mu) \vee ((\mu \circ \psi) \wedge \psi)$.

(b. ⇒ a.) Consider a revision operator $*$ satisfying **(R1)** ∼ **(R4)** and a credibility-limited revision $\circ$ satisfying ($\circ$**1**) ∼ ($\circ$**5**), and define:

$$\psi \circledast \mu = (\psi * \mu) \vee ((\mu \circ \psi) \wedge \psi)$$

$\psi \circledast \mu \vDash \psi \vee \mu$ by definition and **(R1)**. If $\psi \wedge \mu \nvDash \bot$ then expand the definition of $\circledast$, where C is the induced credible function associated to $\circ$ and $*'$ is its induced revision operator:

$$\psi \circledast \mu = \begin{cases} (\psi * \mu) \vee ((\mu *' \psi) \wedge \psi) & \text{if } \psi \in C(\mu) \\ (\psi * \mu) \vee (\mu \wedge \psi) & \text{if } \psi \notin C(\mu) \end{cases}$$

Then, by revision vacuity of both operators, the cases reduce to $\psi \wedge \mu$. Therefore $\psi \circledast \mu \equiv \psi \wedge \mu$. If $\mu \nvDash \bot$ then $\psi * \mu \nvDash \bot$ by **(R3)** and by **(R1)** we conclude that $(\psi \circledast \mu) \wedge \mu \nvDash \bot$. Recall that in Theorem 4.2 we have $(\psi \circledast \mu) \wedge \mu \equiv \psi * \mu$. Therefore, by **(R4)** of $*$ it is directly deduced that if $\psi \equiv \phi$ and $\mu \equiv \nu$ then $(\psi \circledast \mu) \wedge \mu \equiv (\phi \circledast \nu) \wedge \nu$. Lastly, assume that $(\psi \circledast \mu) \wedge \psi \nvDash \bot$, $(\phi \circledast \nu) \wedge \phi \nvDash \bot$, $\psi \equiv \phi$ and $\mu \equiv \nu$. Note that if $\psi \wedge \mu \nvDash \bot$ then $(\psi \circledast \mu) \wedge \psi \equiv \psi \wedge \mu \equiv (\phi \circledast \nu) \wedge \phi$. Consider then that $\psi \wedge \mu \vDash \bot$. Clearly, this is also true for ϕ and ν. Hence, $(\mu \circ \psi) \wedge \psi \nvDash \bot$ and $(\nu \circ \phi) \wedge \phi \nvDash \bot$, which means that $\mu \circ \psi \vDash \psi$ and $\nu \circ \phi \vDash \phi$. So, applying ($\circ$**5**) we conclude $\mu \circ \psi \equiv \nu \circ \phi$. Therefore $(\psi \circledast \mu) \wedge \psi \equiv (\phi \circledast \nu) \wedge \phi$. □

Proof of Corollary 4.5. For the last part of Theorem 4.4, observe that for every $\circledast$ induced by $*$ and a $\circ$ satisfies postulates ($\circledast$**1**)∼($\circledast$**5**) by the proof of **(b. ⇒ a.)**, and we can redefine it by using the construction done in **(a. ⇒ b.)**. Then, the defined C satisfies expansive credibility by $\circledast$ vacuity and element consistency by definition. □

Proof of Theorem 4.8. Note that both properties and postulates are individually satisfied if $\psi \wedge \mu \nvDash \bot$ due to ($\circledast$**2**) and the normal condition (NC). So assume that $\psi \wedge \mu \vDash \bot$.

((6C) ⇒ ($\circledast$11)) Assume $(\psi \circledast \mu) \wedge \psi \vDash \phi$ and $(\psi \circledast \mu) \wedge \psi \nvDash \bot$. This means that $\psi \in C(\mu)$ by relation (1). Since in this case $(\psi \circledast \mu) \wedge \psi \equiv \mu \copyright \psi$ we have $\mu \copyright \psi \nvDash \bot$ and $\mu \copyright \psi \vDash \phi$. By outcome credibility $(\mu \copyright \psi) \wedge \mathcal{A}(\mu) \nvDash \bot$, hence $\phi \wedge \mathcal{A}(\mu) \nvDash \bot$. Which means $\phi \in C(\mu)$, therefore $(\phi \circledast \mu) \wedge \phi \nvDash \bot$ by relation (1).

(($\circledast$11) ⇒ (6C)) Assume that $\mu \copyright \psi \nvDash \bot$ and $(\mu \copyright \psi) \wedge \mathcal{A}(\mu) \vDash \bot$. Then, $(\mu \copyright \psi) \vDash \neg\mathcal{A}(\mu)$ and since $\mu \vDash \mathcal{A}(\mu)$ by (NC), $(\mu \copyright \psi) \wedge \mu \vDash \bot$. Therefore $\psi \in C(\mu)$ because $\psi \wedge \mu \vDash \bot$. This means $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ by relation (1), so we can affirm that $(\psi \circledast \mu) \wedge \psi \equiv \mu \copyright \psi$. Applying CL-regularity taking $\neg\mathcal{A}(\mu$ as ϕ we have $(\neg\mathcal{A}(\mu) \circledast \mu) \wedge \neg\mathcal{A}(\mu) \nvDash \bot$, but this means that $\mathcal{A}(\mu) \wedge \neg\mathcal{A}(\mu) \nvDash \bot$, a contradiction.

((7C) ⇒ ($\circledast$12)) Assume that $(\psi \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$. Then again by relation (1) $\psi \in C(\mu)$ and $(\psi \circledast \mu) \wedge \psi \equiv \mu \copyright \psi$. By strong outcome credibility, $(\psi \circledast \mu) \wedge \psi \vDash \mathcal{A}(\mu)$ then by hypothesis $\phi \wedge \mathcal{A}(\mu) \nvDash \bot$, which means that $(\phi \circledast \mu) \wedge \phi \nvDash \bot$.

(($\circledast$12) ⇒ (7C)) Assume that $\mu \copyright \psi \nvDash \mathcal{A}(\mu)$. Since $\mu \vDash \mathcal{A}(\mu)$ by (NC), $\mu \copyright \psi \not\equiv \mu$, hence $\mu \copyright \psi \vDash \psi$. Since $\mu \copyright \psi \nvDash \bot$ then $\mu \copyright \psi \equiv (\mu \copyright \psi) \wedge \psi \nvDash \bot$ and we can apply relation (1) to rewrite this in terms of $\circledast$ operator. Rewriting the hypothesis, we have $(\psi \circledast \mu) \wedge \psi \wedge \neg\mathcal{A}(\mu) \nvDash \bot$. By CL-strong regularity, taking $\neg\mathcal{A}(\mu$ as ϕ, we have $(\neg\mathcal{A}(\mu) \circledast \mu) \wedge \neg\mathcal{A}(\mu) \nvDash \bot$, but this means that $\mathcal{A}(\mu) \wedge \neg\mathcal{A}(\mu) \nvDash \bot$, a contradiction. □

Proof of Lemma 4.11. Note that the conditions are equivalent if $\psi \wedge \mu \nvDash \bot$ by ($\circledast$**2**), so assume $\psi \wedge \mu \vDash \bot$.

(a. ⇒ b.) If $\circledast$ satisfies revision extensionality **(R4)**, then the equivalence remains when applying the conjunction equivalent formulas, as happens in ($\circledast$**7**) and ($\circledast$**14**).

(b. ⇒ a.) Note that If $\circledast$ satisfies ($\circledast$**7**) and ($\circledast$**14**), it means that if $\psi \equiv \phi$ and $\mu \equiv \nu$ then $(\psi \circledast \mu) \wedge \psi \equiv (\phi \circledast \nu) \wedge \phi$. Combining this result with ($\circledast$**4**) and ($\circledast$**1**) we have that:

$$\psi \circledast \mu \equiv [(\psi \circledast \mu) \wedge \mu] \vee [(\psi \circledast \mu) \wedge \psi] \equiv [(\phi \circledast \nu) \wedge \nu] \vee [(\phi \circledast \nu) \wedge \phi] \equiv \phi \circledast \nu$$

Thus $\circledast$ satisfies **(R4)**. □

Proof of Theorem 4.16. First of all, recall that in this case $C(\mu) = \{\phi \mid \phi \wedge \mathcal{A}(\mu) \nvDash \bot\}$ and that $C_{\bot}(\mu) = C(\mu) \cap \{\phi \mid \phi \wedge \mu \vDash \bot\}$. Note that if $\phi \wedge \mu \vDash \bot$ then $\phi \equiv \phi \wedge \neg\mu$. Therefore:

$$C_{\bot}(\mu) = \{\phi \mid \phi \wedge \mathcal{A}(\mu) \nvDash \bot \text{ and } \phi \wedge \mu \vDash \bot\} = \{\phi \mid \phi \wedge \neg\mu \wedge \mathcal{A}(\mu) \nvDash \bot\}$$

if also satisfies:

1. ($\circledast$**15**) and $\mu \vDash \nu$, assume that $\mathcal{A}(\mu) \wedge \neg\mathcal{A}(\nu) \nvDash \bot$. Then, $\neg\mathcal{A}(\nu) \in C(\mu)$, hence $\neg\mathcal{A}(\nu) \in C(\nu)$, which means that $\neg\mathcal{A}(\nu) \wedge \mathcal{A}(\nu) \nvDash \bot$, a contradiction. Therefore $\mathcal{A}(\mu) \vDash \neg\mathcal{A}(\nu)$.

2. ($\circledast$**16**), since ($\circledast$**14**) holds, we recover ($\circledast$**15**). Then, $\mathcal{A}(\alpha) \vDash \mathcal{A}(\mu)$ for every α complete formula s.t. $\alpha \vDash \mu$ by previous item. Due to ($\circledast$**14**), $\mathcal{A}(\alpha) = \mathcal{A}(\bar{\alpha})$. Therefore $\bigvee \mathcal{A}(\bar{\alpha})$ is a well-defined formula and $\bigvee \mathcal{A}(\bar{\alpha}) \vDash \mathcal{A}(\mu)$. Assume now that $\mathcal{A}(\mu) \nvDash \bigvee \mathcal{A}(\bar{\alpha})$, then $\mathcal{A}(\mu) \wedge \neg \bigvee \mathcal{A}\bar{\alpha}) \nvDash \bot$. Since $C(\mu) = \bigcup C(\bar{\alpha})$ by hypothesis, this means that there is an $\bar{\alpha}_0$ such that $\mathcal{A}(\bar{\alpha}_0) \wedge \neg \bigvee \mathcal{A}(\bar{\alpha}) \nvDash \bot$. In particular, $\mathcal{A}(\bar{\alpha}_0) \wedge \neg \mathcal{A}(\bar{\alpha}_0) \nvDash \bot$, a contradiction.
3. ($\circledast$**17**) and $\mu \vDash \bot$, then $(\psi \circledast \mu) \wedge \psi \vDash \bot$. Due to relation (1), this means that $\psi \notin C(\mu)$ for every $\psi \in \mathcal{L}$. By definition of $\mathcal{A}(\mu)$ this could only happen if $\mathcal{A}(\mu) \vDash \bot$. If $\mu \nvDash \bot$ it is already known that $\mathcal{A}(\mu) \nvDash \bot$ by the fact that $\mu \vDash \mathcal{A}(\mu)$.
4. ($\circledast$**18**) and $\mu \vDash \nu$, assume $\mathcal{A}(\nu) \wedge \neg\nu \wedge \neg\mathcal{A}(\mu) \nvDash \bot$. This means that $\neg\nu \wedge \neg\mathcal{A}(\mu) \in C_{\bot}(\nu)$ by the first part of the theorem, since $\neg\nu \wedge \neg\mathcal{A}(\mu) \wedge \nu \vDash \bot$, implying that $\neg\nu \wedge \neg\mathcal{A}(\mu) \in C(\mu)$ by hypothesis. But this says that $\neg\nu \wedge \neg\mathcal{A}(\mu) \wedge \mathcal{A}(\mu) \nvDash \bot$, a contradiction. Therefore $\mathcal{A}(\nu) \wedge \neg\nu \vDash \mathcal{A}(\mu)$.
5. ($\circledast$**19**), since ($\circledast$**14**) holds, we recover ($\circledast$**18**). If $\psi \in C_{\bot}(\mu)$ we have that $\mathcal{A}(\mu) \wedge \neg\mu \wedge \psi \nvDash \bot$ and $\psi \wedge \mu \vDash \bot$. Then $\mathcal{A}(\alpha) \wedge \psi \nvDash \bot$ for every complete formula α s.t. $\alpha \vDash \mu$ by the previous item, and since $\alpha \vDash \mu$, we have that $\psi \wedge \alpha \vDash \bot$.
6. ($\circledast$**18**) and ($\circledast$**20**), and $\mu \vDash \bot$, we have seen that $C_{\bot}(\mu) = \{\psi \nvDash \bot$. Therefore, the only possible formula for $\mathcal{A}(\mu)$ is a tautological one. □

Proof of Theorem 4.17. Due to ($\circledast$**15**) and ($\circledast$**18**), if $\gamma \vDash \mu$ then

$$C_{\bot}(\mu) \subseteq C(\gamma) \subseteq C(\mu)$$

Which means that $C(\mu) = C(\gamma) \cup \{\psi \mid \psi \wedge \mu \nvDash \bot\}$. Since $\bot \vDash \mu$ for every $\mu \in \mathcal{L}$, if $\gamma \vDash \bot$ we have that $C(\mu) = C(\bot) \cup \{\psi \mid \psi \wedge \mu \nvDash \bot\}$ and $C_{\bot}(\mu) = C(\bot) \cap \{\psi \mid \psi \wedge \mu \vDash \bot\}$.

With this characterization, we can directly deduce ($\circledast$**16**) and ($\circledast$**19**):

$$\begin{aligned} C_{\bot}(\mu) \cap C_{\bot}(\nu) &= C(\bot) \cap \{\psi \mid \psi \wedge \mu \vDash \bot\} \cap \{\psi \mid \psi \wedge \nu \vDash \bot\} \\ &= C(\bot) \cap \{\psi \mid \psi \wedge (\mu \vee \nu) \vDash \bot\} \\ &= C_{\bot}(\mu \vee \nu) \end{aligned}$$

$$\begin{aligned} C(\mu) \cup C(\nu) &= C(\bot) \cup \{\psi \mid \psi \wedge \mu \nvDash \bot\} \cup \{\psi \mid \psi \wedge \nu \nvDash \bot\} \\ &= C(\bot) \cup \{\psi \mid \psi \wedge (\mu \vee \nu) \nvDash \bot\} \\ &= C_{\bot}(\mu \vee \nu) \quad \square \end{aligned}$$

Proof of Corollary 4.19. **(a. ⇒ b.)** The hypothesis combined with relation (1) tells us that $C(\mu) = \{\psi \mid \psi \nvDash \bot\}$ for every $\mu \in \mathcal{L}$. Therefore, $C(\mu) = C(\nu)$ if $\mu \vDash \nu$ thus satisfying ($\circledast$**14**), ($\circledast$**15**) and ($\circledast$**18**). Also, if $\psi \nvDash \bot$ then $\psi \in C(\bot)$, therefore $\psi \in \bigcup C_{\bot}(\mu)$ for every $\mu \in \mathcal{L}$, satisfying ($\circledast$**20**).

(b. ⇒ a.) Since $\circledast$ is a Wise-Imprudent, we can apply Theorem 4.17. Note that $C(\bot) = \{\psi \mid \psi \nvDash \bot\}$ due to ($\circledast$**20**), so we have that $C(\mu) = \{\psi \mid \psi \nvDash \bot\}$. Combining this with relation (1) tells us that $\psi \nvDash \bot$ for every $\mu \in \mathcal{L}$, therefore $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ for every $\mu \in \mathcal{L}$.

(a. ⇒ c.) By relation (1) we know that $C(\mu) = \{\psi \mid \psi \nvDash \bot\}$. Note that this characterization trivially implies ($\circledast$7) ∼ ($\circledast$13) and ($\circledast$18), therefore $\circledast$ is a basic promotion. Then, since $C(\mu) = \{\psi \mid \psi \wedge \mathcal{A}(\mu) \nvDash \bot\}$, the only possibility of $\mathcal{A}(\mu)$ is to be a tautology.

(c. ⇒ a.) If $\vDash \mathcal{A}(\mu)$, then $C(\mu) = \{\psi \mid \psi \nvDash \bot\}$ by definition. Then by relation (1) be recover ($\circledast$25).

(a. ⇒ d.) Clearly, by relation (1) and Theorem 4.2, $\mu *' \psi \equiv (\mu \circ \psi) \wedge \psi$ if $\psi \nvDash \bot$. If $\psi \vDash \bot$ then $\mu *' \psi \vDash \bot$ and $(\mu \circ \psi) \wedge \psi \vDash \bot$. Therefore $\mu *' \psi \equiv (\mu \circ \psi) \wedge \psi$ in any case.

(d. ⇒ a.) If $\psi \nvDash \bot$ then $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ since $(\mu *' \psi) \wedge \psi \nvDash \bot$.

(b. ⇒ e.) Note that by assuming **(b.)** we have postulates ($\circledast$**13**) ∼ ($\circledast$**16**) and ($\circledast$**18**) ∼ ($\circledast$**20**). Then, through **(a.)**, we can have **(c.)**, therefore having postulates ($\circledast$**6**) ∼ ($\circledast$**12**).

(e. ⇒ b.) Trivial. □

Proof of Theorem 4.21. **(From construction to postulate)** Consider $(\psi \circledast \mu) \wedge \psi \nvDash \bot$, then $\psi \in C(\mu)$ by relation (1), and hence $(\psi \circledast \mu) \wedge \psi \equiv \mu *' \psi$. Therefore, by hypothesis and item 1 of Theorem 4.2, $(\psi \circledast \mu) \wedge \psi \equiv \mu * \psi \equiv (\mu \circledast \psi) \wedge \psi$.

(From postulate to construction) Suppose $\psi \in C(\mu)$, we know that $(\psi \circledast \mu) \wedge \psi \equiv \mu *' \psi$ and that is consistent. By hypothesis and item 1 of Theorem 4.2, $\mu *' \psi \equiv (\psi \circledast \mu) \wedge \psi \equiv (\mu \circledast \psi) \wedge \psi \equiv \mu * \psi$. □

Proof of Theorem 4.25. Recall that, if **(R1)** ∼ **(R4)** holds, **(R5)** and **(R6)** are equivalent to **(R7)**. We are also going to use item **(a.)** of Theorem 4.2 repeatedly.

(If) Suppose that the operator $*$ satisfies **(R1)** ∼ **(R6)** and assume $(\psi \circledast \mu) \wedge \mu \wedge \nu \nvDash \bot$. Recall that adding **(R5)** and **(R6)** to **(R1)** ∼ **(R4)** is equivalent as adding **(R7)**. Therefore, by item **(a.)**, $(\psi * \mu) \wedge \nu \nvDash \bot$, hence by **(R7)** we have $(\psi * \mu) \wedge \nu \equiv \psi * (\mu \wedge \nu)$. Therefore applying item **(a.)** again, $(\psi \circledast \mu) \wedge \mu \wedge \nu \equiv (\psi \circledast (\mu \wedge \nu)) \wedge \mu \wedge \nu$. Note that if $\mu \wedge \psi \vDash \bot$ then $(\psi \circledast (\mu \wedge \nu)) \wedge \mu \wedge \nu \equiv (\psi \circledast (\mu \wedge \nu)) \wedge \mu$ and if not, then $\psi \wedge \mu \wedge \nu \nvDash \bot$ by vacuity and the hypothesis. Therefore, $(\psi \circledast (\mu \wedge \nu)) \wedge \mu \wedge \nu \equiv \psi \wedge \mu \wedge \nu \equiv (\psi \circledast (\mu \wedge \nu)) \wedge \mu$.

(Only if) Assume the operator $\circledast$ satisfies R-minimality and suppose $(\psi * \mu) \wedge \nu \nvDash \bot$. Then by item **(a.)** $(\psi \circledast \mu) \wedge \mu \wedge \nu \nvDash \bot$, and applying ($\circledast$**24**) we have $(\psi \circledast \mu) \wedge \mu \wedge \nu \equiv (\psi \circledast (\mu \wedge \nu)) \wedge \mu$. Hence, by again item **(a.)**, $(\psi * \mu) \wedge \nu \equiv (\psi * (\mu \wedge \nu)) \wedge (\mu \wedge \nu) \wedge \mu$, implying the result. □

Proof of Theorem 4.27. Note that from item **b.** of Theorem 4.2, $(\psi \circledast \mu) \wedge \psi \equiv \mu *' \psi$ when $\psi \in C(\mu)$, otherwise $(\psi \circledast \mu) \wedge \psi$ is inconsistent. On the other hand, $*'$ satisfying **(R5)** and **(R6)** is equivalent as satisfying disjunctive factoring:

$$\text{Either } \mu *' (\psi \vee \phi) \text{ is equivalent to } (\mu *' \psi) \vee (\mu *' \phi),\ \mu *' \psi \text{ or } \mu *' \phi.$$

If $\psi \vee \phi \notin C(\mu)$ then both postulates trivially hold. So assume $\psi \vee \phi \in C(\mu)$, that is, either $\psi \in C(\mu)$ or $\phi \in C(\mu)$. Then, what follows is to check every case directly with the previous equivalences. □

Proof of Lemma 4.28. First note that combining (⊛**7**) and (⊛**25**) we have the following: If $(\psi \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$, consider $\alpha = \psi \wedge \phi$, $\beta = \psi \wedge \neg\phi$. By (⊛**7**) we have that $[(\alpha \vee \beta) \circledast \mu] \wedge (\alpha \vee \beta) \equiv (\psi \circledast \mu) \wedge \psi$. thus applying (⊛**25**) we deduce that $(\psi \circledast \mu) \wedge \psi$ is equivalent to $((\alpha \circledast \mu) \wedge \alpha) \vee ((\beta \circledast \mu) \wedge \beta)$, $(\alpha \circledast \mu) \wedge \alpha$ or $(\beta \circledast \mu) \wedge \beta$. Therefore, $(\psi \circledast \mu) \wedge \psi \wedge \phi \equiv (\alpha \circledast \mu) \wedge \alpha$, since is consistent and $\alpha \wedge \beta \vDash \bot$. This is used more than once in this proof.

(a. ⇒ b.) Trivial.

(b. ⇒ a. & c.) Clearly, (⊛**12**) ⇒ (⊛**11**) ⇒ (⊛**8**). Also, combining (⊛**12**) with (⊛**5**) we have (⊛**7**), which we know implies (⊛**6**). On the other hand, (⊛**25**) ⇒ (⊛**9**), and by using (⊛**9**) with $\phi = \neg\psi$, we have by vacuity that $((\psi \vee \neg\psi) \circledast \mu) \wedge (\psi \vee \neg\psi) \equiv \mu$. Therefore, if $\mu \nvDash \bot$ we deduce (⊛**10**).

(c. ⇒ b.) We only need to see that (⊛**12**) holds. Consider $(\psi \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$. Then, by the observation that uses (⊛**7**) and (⊛**25**), we have $((\psi \wedge \phi) \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$. Therefore, $(\phi \circledast \mu) \wedge \phi \nvDash \bot$ by (⊛**8**).

(c. ⇒ d.) Let us show that (⊛**26**) is satisfied. Note that if $\psi \wedge \mu \nvDash \bot$ then the postulate is trivial by (⊛**2**), so assume $\psi \wedge \mu \vDash \bot$. If $(\psi \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$, then applying (⊛**7**) and (⊛**25**) as in the observation we have $((\psi \wedge \phi) \circledast \mu) \wedge \psi \wedge \phi \equiv (\psi \circledast \mu) \wedge \psi \wedge \phi$. The last part is to show that $((\psi \wedge \phi) \circledast \mu) \wedge \psi \vDash \phi$. Consider then that $((\psi \wedge \phi) \circledast \mu) \wedge \psi \wedge \neg\phi \nvDash \bot$. Then, by (⊛**1**) we have that $(\psi \wedge \phi) \vee \mu) \wedge \psi \wedge \neg\phi \nvDash \bot$, which can not occur. Therefore $((\psi \wedge \phi) \circledast \mu) \wedge \psi \vDash \phi$ and the equivalence of (⊛**26**) holds.

Now we prove (⊛**27**) by contradiction. Assume $((\psi \wedge \phi) \circledast \mu) \wedge \psi \nvDash \bot$ and $(\psi \circledast \mu) \wedge \psi \vDash \bot$. Clearly, ψ is consistent and, by vacuity, $\mu \wedge \psi \vDash \bot$. Note that $((\psi \wedge \phi) \circledast \mu) \wedge (\psi \wedge \phi) \vDash \bot$, otherwise $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ by (⊛**8**). Therefore, $((\psi \wedge \phi) \circledast \mu) \wedge \psi \vDash \neg\phi$. Hence, $((\psi \wedge \phi) \circledast \mu) \wedge \psi \vDash ((\psi \wedge \phi) \vee \mu) \wedge \psi \wedge \neg\phi$ by (⊛**1**). But then, the hypothesis implies that $(\psi \wedge \phi \wedge \psi \wedge \neg\phi) \vee (\mu \wedge \psi \wedge \neg\phi)$ is consistent, which is a contradiction.

(d. ⇒ c.) We need to see that (⊛**7**), (⊛**8**), and (⊛**25**) are deduced from (⊛**26**) and (⊛**27**). If $\psi \equiv \phi$, in order to prove (⊛**7**), we only need to show that $(\psi \circledast \mu) \wedge \psi \nvDash \bot$ if and only if $(\phi \circledast \mu) \wedge \phi \nvDash \bot$, and let (⊛**5**) deal with the rest. Assume that $(\psi \circledast \mu) \wedge \psi \nvDash \bot$, therefore $(\psi \circledast \mu) \wedge \psi \wedge \phi \nvDash \bot$. By (⊛**26**) we have $((\psi \wedge \phi) \circledast \mu) \wedge \psi \equiv (\psi \circledast \mu) \wedge \psi \wedge \phi$. Hence, $((\psi \wedge \phi) \circledast \mu) \wedge \phi$ is consistent, just by replacing ψ by ϕ at the conjunction. Applying (⊛**27**) to this expression we conclude that $(\phi \circledast \mu) \wedge \phi \nvDash \bot$. The "only if" part is identical, replacing ψ by ϕ at every step. Postulate (⊛**8**) is a direct implication of (⊛**26**) now that we have (⊛**7**), so the remaining postulate is (⊛**25**).

Consider $\gamma = ((\alpha \vee \beta) \circledast \mu) \wedge (\alpha \vee \beta)$. If $\gamma \wedge \alpha \nvDash \bot$, then $\gamma \wedge \alpha \equiv (\alpha \circledast \mu) \wedge \alpha$ by (⊛**26**) and (⊛**7**). Analogously, $\gamma \wedge \beta \equiv (\beta \circledast \mu) \wedge \beta$. To sum up, if $\gamma \vDash \bot$, then by (⊛**27**) and (⊛**7**) both $(\alpha \circledast \mu) \wedge \alpha$ and $(\beta \circledast \mu) \wedge \beta$ are inconsistent. If $\gamma \nvDash \bot$, then:

$$\gamma \equiv \begin{cases} (\alpha \circledast \mu) \wedge \alpha & \text{if } \gamma \wedge \beta \vDash \bot \\ (\beta \circledast \mu) \wedge \beta & \text{if } \gamma \wedge \alpha \vDash \bot \\ ((\alpha \circledast \mu) \wedge \alpha) \vee ((\beta \circledast \mu) \wedge \beta) & \text{otherwise} \end{cases}$$ □

Proof of Theorem 5.5. Recall that we have:

$$\psi \circledast \mu \equiv ((\psi \circledast \mu) \wedge \mu) \vee ((\psi \circledast \mu) \wedge \psi)$$

Considering this decomposition, we separate the proof into three cases:

1. If $\psi \wedge \mu \nvDash \bot$, we know that $\psi \circ_f \mu \equiv \psi \wedge \mu$ due to moderated revision definition, which is trivially equivalent to $\psi * \mu$.
2. If $\psi \wedge \mu \vDash \bot$ and $(\psi \circ_f \mu) \wedge \psi \vDash \bot$, we know that $\psi \circ_f \mu \equiv \psi * \mu$ by definition of moderated revision.
3. If $\psi \wedge \mu \vDash \bot$ and $(\psi \circ_f \mu) \wedge \psi \nvDash \bot$, let us rewrite as $(\psi * f(\mu)) \wedge \psi \nvDash \bot$. By $*$ success, we have that $f(\mu) \wedge \psi \nvDash \bot$, therefore $\psi * f(\mu) \equiv \psi \wedge f(\mu)$ by $*$ vacuity, or equivalently $\psi \circ_f \mu \equiv \psi \wedge f(\mu)$. But since $\mu \nvDash \bot$ and $\circ_f$ is a moderated revision operator, by (⊛**3**), $(\psi \circ_f \mu) \wedge \mu \nvDash \bot$, hence $\psi \wedge \mu \nvDash \bot$, a contradiction. Thus, this case is not possible.

As a result, the only real possibilities are the first two, implying that f is the identity function, thus $\psi \circ_f \mu = \psi * \mu$. Also, if $\circ_f$ is viewed as a moderated revision operator, then $*$ is its inducing revision. □

Proof of Theorem 5.8. We use the Harper Identity to rewrite the conjunction expression and distribute every term we have:

$$(\psi - \neg\mu) \wedge (\mu \div \neg\psi)$$
$$[(\psi * \mu) \vee \psi] \wedge [(\mu \circ \psi) \vee \mu]$$
$$[[(\psi * \mu) \vee \psi] \wedge (\mu \circ \psi)] \vee [[(\psi * \mu) \vee \psi] \wedge \mu]$$
$$[(\psi * \mu) \wedge (\mu \circ \psi)] \vee [\psi \wedge (\mu \circ \psi)] \vee [(\psi * \mu) \wedge \mu] \vee [\psi \wedge \mu]$$

Since both $(\psi * \mu) \wedge (\mu \circ \psi)$ and $(\psi \wedge \mu)$ imply $(\psi * \mu)$ and $(\psi * \mu) \wedge \mu \equiv \psi * \mu$ by **(R1)**, then:

$$[(\psi * \mu) \wedge (\mu \circ \psi)] \vee [(\psi * \mu) \wedge \mu] \vee [\psi \wedge \mu] \equiv \psi * \mu$$

Therefore $(\psi - \neg\mu) \wedge (\mu \div \neg\psi) \equiv (\psi * \mu) \vee ((\mu \circ \psi) \wedge \psi) = \psi \circledast \mu$. □

Proof of Theorem 5.9. 1. This is directly deduced from definition and Harper identity: $(\psi \circledast \neg\mu) \vee \psi \equiv (\psi * \neg\mu) \vee \psi \equiv \psi - \mu$.

2. By $\circledast$ definition, $(\neg\psi \circledast \mu) \vee \mu \equiv \mu \vee [(\mu \circ \neg\psi) \wedge \neg\psi]$. Note that either $\mu \circ \neg\psi = \mu$ or $\mu \circ \neg\psi \vDash \neg\psi$, the previous expression is always equivalent to $\mu \vee (\mu \circ \neg\psi)$ which is $\mu \div \psi$ by definition. □

References

[1] C.E. Alchourrón, P. Gärdenfors, D. Makinson, On the logic of theory change: partial meet contraction and revision functions, J. Symb. Log. 50 (1985) 510–530.
[2] D. Makinson, Screened revision, Theoria 63 (1997) 14–23, https://doi.org/10.1111/j.1755-2567.1997.tb00737.x.
[3] S. Hansson, E. Fermé, J. Cantwell, M. Falappa, Credibility limited revision, J. Symb. Log. 66 (2001) 1581–1596.
[4] E. Fermé, S. Hansson, Shielded contraction, Appl. Log. Ser. 22 (2001), 85–10.
[5] M. Garapa, Two level credibility-limited revisions, Rev. Symb. Log. 15 (2022) 388–408, https://doi.org/10.1017/S1755020320000283.
[6] G. Bonanno, Filtered belief revision and generalized choice structures, SSRN Electron. J. (2020), https://doi.org/10.2139/ssrn.3527743.
[7] G. Bonanno, Filtered belief revision: syntax and semantics, J. Log. Lang. Inf. 31 (2022) 645–675, https://doi.org/10.1007/s10849-022-09374-x.
[8] K. Sauerwald, G. Kern-Isberner, C. Beierle, On limited non-prioritised belief revision operators with dynamic scope, CoRR, arXiv:2108.07769 [abs], 2021, https://arxiv.org/abs/2108.07769.
[9] H. Katsuno, A. Mendelzon, Propositional knowledge base revision and minimal change, Technical Report, Technical Report n°3, Department of Computer Science, University of Toronto, 1991.
[10] I. Levi, Subjunctives, dispositions and chances, Synthese 34 (1977) 423–455, https://doi.org/10.1007/bf00485649.
[11] W.L. Harper, Rational conceptual change, in: PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association 1976, 1976, pp. 462–494.
[12] E. Fermé, S. Hansson, Belief Change, Springer, Cham, Switzerland, 2018.
[13] R. Booth, E. Fermé, S. Konieczny, R. Pino Pérez, Credibility-limited revision operators in propositional logic, in: 13th International Conference on the Principles of Knowledge Representation and Reasoning, KR 2012, 2012.
[14] R. Booth, E. Fermé, S. Konieczny, R. Pino Pérez, Credibility-limited improvement operators, in: European Conference on Artificial Intelligence, 2014, pp. 123–128, https://api.semanticscholar.org/CorpusID:7640654.
[15] M. Garapa, E. Fermé, M. Reis, Credibility-limited base revision: new classes and their characterization, J. Artif. Intell. Res. 69 (2020) 1023–1075.
[16] P. Liberatore, M. Schaerf, Arbitration: a commutative operator for belief revision, in: Proceedings of the Second World Conference on the Fundamentals of Artificial Intelligence (WOCFAI'95), Angkor Press, 1995, pp. 217–228.
[17] P. Liberatore, M. Schaerf, Arbitration (or how to merge knowledge bases), IEEE Trans. Knowl. Data Eng. 10 (1998) 76–90.
[18] S. Konieczny, R. Pino Pérez, Merging information under constraints: a logical framework, J. Log. Comput. 12 (2002) 773–808, https://doi.org/10.1093/logcom/12.5.773.
[19] S. Konieczny, R. Pino Pérez, Propositional belief base merging or how to merge beliefs/goals coming from several sources and some links with social choice theory, Eur. J. Oper. Res. 160 (2005) 785–802, https://ideas.repec.org/a/eee/ejores/v160y2005i3p785-802.html.
[20] S. Konieczny, R. Pino Pérez, Logic based merging, J. Philos. Log. 40 (2011) 239–270, http://www.jstor.org/stable/41487513.
[21] N. Schwind, S. Konieczny, P. Marquis, On belief promotion, in: Proceedings of the Sixteenth International Conference on Principles of Knowledge Representation and Reasoning, 2018, pp. 297–306.
[22] S. Hansson, A survey of non-prioritized belief revision, Erkenntnis 50 (1999) 413–427.
[23] E. Fermé, S. Hansson, Selective revision, Stud. Log. 63 (3) (1999) 331–342.
[24] A. Darwiche, J. Pearl, On the logic of iterated belief revision, Artif. Intell. 89 (1997) 1–29, https://doi.org/10.1016/s0004-3702(96)00038-0.
[25] M.A. Falappa, G. Kern-Isberner, M.D.L. Reis, G.R. Simari, Prioritized and non-prioritized multiple change on belief bases, J. Philos. Log. 41 (2012) 77–113, https://doi.org/10.1007/s10992-011-9200-8.
[26] L. Zhang, On Non-Prioritized Multiple Belief Revision, Ph.D. thesis, KTH Royal Institute of Technology, 2018.