On strong standard completeness of $MTL_*^{\mathbb{Q}}$ expansions

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Within the mathematical logic field, much effort has been devoted to prove completeness of different axiomatizations with respect to classes of algebras defined on the real unit interval $[0,1]$ (see for instance [1] and [2]), but in general, what has been mainly achieved are axiomatizations and results concerning *finitary* completeness, that is, for deductions from *a finite number of premises.*

In this work we are concerned with the problem of *strong* completeness, i.e., completeness for deductions from an arbitrary number of premises. In particular, we will focus on showing strong completeness for logics of a left-continuous t-norm. These will be extensions of the monoidal t-norm based logic, MTL, the logic of prelinear, bounded, commutative and integral residuated lattices [3], expanded with rational truth-constants and with an arbitrary set of connectives under some constraints.

It is known that MTL is strongly complete with respect to the class of all standard algebras based of left-continuous t-norms [2]. Some particular extensions of MTL enjoy more concrete completeness results: BL, Gödel, Product or Łukasiewicz logics are complete wrt single particular standard algebras. However, these completeness results are for finitary deductions, only in a few cases (e.g. in Gödel logic) they hold in general.

Regarding the issue of enforcing strong completeness in MTL logics expanded with rational constants, the main references are [4] and [5]. While the former focuses on the strong standard completeness for Product logic extended with rational constants following the usual algebraic approach, the latter is framed in the context of Pavelka-style completeness, a different (infinitary) notion of completeness originally introduced by Pavelka in the context of Łukasiewicz logic [6]. We will not deal here with this kind of completeness, we only notice that it is a weaker notion than that of strong standard completeness.

The paper by Cintula [5] explores different notions of rational expansions of *MTL*, and shows that adding a pair of infinitary deduction rules for each discontinuity point in the connectives' truth-functions on the unit real interval $[0,1]$ makes these logics Pavelka-style complete. Cintula also makes an observation that will partially orient our work: for a rational standard algebra (i.e., over $[0,1]$ with rational constants) with a non-continuous operation, there is no finitary axiomatic system that is strongly complete with respect to it.

In this abstract we present an alternative way (with respect to the Pavelka-style approach) to enforce strong standard completeness of rational expansions of *MTL*. The approach is based on the idea that the problem of devising an axiomatization that is strongly standard complete is not exactly linked to the discontinuity points of the connectives but rather to changes in some regularity conditions of the corresponding operations, like monotonicity and continuity. Our main result is, given any arbitrary left-continuous t-norm $*$, to present an axiomatic system $MTL_*^{\mathbb{Q}}$ strongly complete with respect to the rational standard algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$. It is defined as the extension of the MTL with the usual book-keeping axioms for $\&, \rightarrow$ and $\wedge$ connectives, the rule $\overline{c} \vee \varphi \vdash \varphi$ for each rational $c < 1$, and the following adaptation of the density rule of some first order logics:

$$(\vee\mathsf{D}^\infty)\ \frac{\{\gamma \vee (\varphi \rightarrow \overline{c}) \vee (\overline{c} \rightarrow \psi)\}_{c\in[0,1]_{\mathbb{Q}}}}{\gamma \vee (\varphi \rightarrow \psi)}$$

Note that the definition of proof when infinitary rules are present is a tree (rather than a sequence), where the root is the consequence of the deduction, the leaves are either axioms or formulas from the premise set, and each branch represents the application of a deduction rule. The finite depth of the tree maintains the correction for what respects reasoning by induction on proofs.

The rule $(\vee\mathsf{D}^\infty)$ is strong enough for our purposes over the MTL language, but if we want to expand the logic with more general connectives (with corresponding "regular enough" operations), particular rules for each one of them are needed. Indeed, the density rule allows us to determine the values of the new operations from the ones given by the rational constants, under certain regularity conditions referring to both monotonicity and continuity. We call these well behaved operations *representable*, and we show that for them it is possible to specify rules on a language expanding $MTL_*^{\mathbb{Q}}$ modeling those regularity conditions. We also show that for a set OP of representable operations, the system $MTL_*^{\mathbb{Q}}(OP)$ resulting from adding to $MTL_*^{\mathbb{Q}}$ the regularity rules for the operations in OP, (and the book-keeping axioms and congruence rules of OP), is strongly complete with respect to extended standard algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$.

We note that our proposed axiomatizations are in many cases finitely presented -except for the set of book-keeping axioms. It is also remarkable that, if the Monteiro-Baaz operator Δ belongs to OP, the axiomatic system $MTL_*^{\mathbb{Q}}(OP)$ can be simplified a lot.

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