From Classical Description Logic to n-graded Fuzzy Description Logic

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***Abstract*— Description Logics (DLs) are knowledge representation languages built on the basis of classical logic. DLs allow the creation of knowledge bases and provide ways to reason on the contents of these bases. Fuzzy Description Logics (FDLs) are natural extensions of DLs for dealing with vague concepts, commonly present in real applications. Following the ideas of Hájek in [17] and García-Cerdaña et al. in [15] we develop a family of FDLs whose underlying logic is the fuzzy logic of a finite linearly ordered residuated lattice, that is, an n-graded fuzzy logic defined by a divisible finite t-norm over a finite chain. Moreover, the role of the constructor of implication in the languages for FDLs is discussed, and a hierarchy of $\mathcal{AL}$-languages adapted to the behavior of the connectives in the fuzzy logics underlying these description languages is proposed. Finally, we deal with reasoning tasks within the framework of finitely valued DLs.**

I. INTRODUCTION

In the last ten years there has been a large and increasing interest on the attempt to generalize the formalism of Description Logics (DLs) to a multi-valued framework. In the literature there exist several interesting and deep papers on Fuzzy Description Logics (FDLs) dealing with the expressiveness of the languages and reasoning algorithms (see [21] for a survey) rather than with logical foundations.

In this paper, though, we take a metamathematical point of view and we define FDLs on the basis of first order many-valued fuzzy logics in an analogous way as DLs relate with first order classical logic.[1] As a result of the development of Mathematical Fuzzy Logic, we have at our disposal a large family of first order logical systems, the so-called predicate t-norm based fuzzy logics. These systems, presented as well defined Hilbert-style calculi, allow us to interpret the FDLs in them and, therefore, to take advantage of the results and metamathematical tools developed in Fuzzy Logic in the last fifteen years (see http://www.mathfuzzlog.org for an exhaustive list of works and researchers in this area).

This point of view for dealing with FDLs was proposed firstly by Hájek (see [17], [18], [19]) and recently has been also developed in [15] where some lines of research in this direction are proposed. The main idea is to work in a similar way as in classical DLs whose formulas are interpreted into first order classical logic. Consequently, we define FDLs (having the constructors needed to have the expressive capabilities we want to confer to the description language) and we interpret the formulas in the corresponding first order fuzzy logic. In order to go ahead with this agenda, we need to know the fragments of the logics corresponding to our FDLs, their properties, and their reasoning capabilities including complexity and algorithms. The problem is that first order fuzzy logics and their fragments are more complex and less studied than the classical ones. For example, we have that many first order fuzzy logics (like Łukasiewicz or Product logics) are not standard complete while the semantics needed for defining FDLs is the standard one (with respect to a given structure of truth values). In particular, standard tautologies are very complex (for instance, they are not recursively enumerable in the case of Łukasiewicz and not arithmetical in the case of Product). Even though for Łukasiewicz (see [17]) and for Product (see [10]) it is proved that the satisfiability problem for the fragment of the corresponding first order fuzzy logic associated to the $\mathcal{ALC}$-like fuzzy description logic is decidable, many other problems remain open.

In [15] a family of description languages is defined: one language for each continuous t-norm (or divisible finite t-norm) and each countable subalgebra of the corresponding standard algebra on the real interval $[0, 1]$ (or on a finite chain $C_n = \{0 = r_1 < r_2 < \cdots < r_{n-1} < r_n = 1\}$). These languages, denoted by $\mathcal{ALC}^*(S)$, include an involutive negation and constants for representing truth degrees: one for every element of the carrier S of the subalgebra. Taking advantage of the expressive power provided by these truth constants, in the cited paper a graded notion of satisfiability (validity and subsumption) of concepts is defined by means of the satisfiability (validity and subsumption) of certain *evaluated formulas* of the associated first order fuzzy logic denoted by $\tilde{L}^*(S)\forall$.

In the current paper we restrict ourselves to the case that $*$ is a divisible finite t-norm, and we take as algebra of truth values the finite standard algebra C_n^* defined by $*$ in the set C_n. Thus, in Section II, we introduce the logics $\tilde{L}_n^*(C_n)\forall$, which are the logical framework for the n-graded FDLs considered in the paper. These logics, presented by well defined Hilbert-style calculi, are strong canonical complete, that is, they are complete with respect to the so-called canonical $\tilde{L}_n^*(C_n)$-chain, which is the algebra C_n^* expanded with the truth function for the involutive negation and a truth constant for every element in C_n.

In Section III we use the notion of *instance* of a description in order to define the new family of n-graded FDLs as

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[1] Notice that, following this approach, some of the subjects that are studied in papers about FDLs dealing with fuzzy sets notions in a broader sense –such as fuzzy quantifiers [25], fuzzy modifiers or fuzzy data types [29], [7]– are not considered.

fragments of the logics $\tilde{L}^*_n(C_n)\forall$. Therefore, as in the classical case, our n-graded FDLs correspond to fragments of the corresponding Hilbert-style predicate calculi. In the same section we also discuss the role of the constructor for *implication* in FDLs, and we define a hierarchy of description languages from the less expressive $\mathcal{AL}$-like to the more expressive $\mathcal{ALC}$-like adapted to the behavior of the connectives in the fuzzy logics underlying these description languages. Let us remark that the proposed hierarchy is also valid for the infinitely valued FDLs since in this case the relation between the connectives remains basically the same.

The finitely valued FDLs could also be very interesting for applications and a good test for a study of reasoning algorithms and different satisfiability and subsumption definitions. Indeed, we could consider 1-satisfiability, positive satisfiability, and n-graded notions of satisfiability and the same for validity and subsumption. Thus in Section IV we discuss about the reasoning tasks within the framework of finitely valued logics: the different notions of satisfiability and their relationship with the corresponding definitions of validity and subsumption. Let us remark that, as a consequence of a Hájek's result about witnessed models in [17], the satisfiability and validity problems for the n-graded FDLs defined in this paper are decidable.

II. A LOGICAL FRAMEWORK FOR n-GRADED FUZZY DESCRIPTION LOGICS

Finite t-norms are operations analogous to t-norms but defined over finite chains. Given a finite chain of n elements

$$C_n = \{0 = r_1 < r_2 < \ldots < r_{n-1} < r_n = 1\},$$

a finite t-norm $*$ is a binary operation defined on C_n which is associative, commutative, non decreasing in both arguments and having 1 as unit element. This operation has always a residuum, denoted by $\to_*$, since the supremum always exists and it is a maximum. Moreover the negation associated to $*$ is defined as $\neg_* x := x \to_* 0$. Thus the operation $*$ defines the following residuated structure over C_n, called the *canonical chain* defined by $*$, that is,

$$C^*_n = \langle C_n, \max, \min, *, \to_*, 0, 1\rangle.$$

A finite t-norm $*$ is said to be *divisible* if and only if the canonical chain C^*_n is a BL-algebra, that is, if it satisfies the divisibility condition $x * (x \to_* y) = \min(x, y)$ (for the notion of BL-algebra see [16]). It is also well known that any finite BL-chain of n elements is either a finite Łukasiewicz chain (denoted by $Ł_n$), or a finite Gödel chain (denoted by G_n), or a finite ordinal sum of copies of finite Łukasiewicz and Gödel chains (see [22], [23]).

A. The logics L^*_n

The Basic fuzzy Logic (BL) was defined in [16] by means of a Hilbert-style axiomatization, and has the following basic connectives: *multiplicative conjunction* (&), *implication* ($\to$), and *falsity* ($\bar{0}$). The only deduction rule of BL is, as in Classical Propositional Logic, *Modus Ponens*. Further connectives are defined as follows:

$$\varphi \wedge \psi := \varphi \& (\varphi \to \psi), \quad \varphi \leftrightarrow \psi := (\varphi \to \psi) \& (\psi \to \varphi),$$
$$\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi),$$
$$\neg\varphi := \varphi \to \bar{0}, \quad \bar{1} := \neg\bar{0}.$$

Given a divisible finite t-norm $*$, we denote by L^*_n the propositional multi-valued logic whose theorems coincide with the tautologies with respect to interpretations on algebras of the variety generated by C^*_n. The logics L^*_n, studied in [1], [14], [12], have been proved to be finitely axiomatizable as axiomatic extensions of BL, and hence they have only Modus Ponens as inference rule.

Let us stress that one theorem of the logic L^*_n is the formula: $(\varphi_1 \to \varphi_2) \vee (\varphi_2 \to \varphi_3) \vee \ldots \vee (\varphi_n \to \varphi_{n+1})$. Indeed, this formula is a tautology because, being $n + 1$ the number of formulas φ_i, and being n the number of elements in C_n, for every evaluation e we must have, for some j, $e(\varphi_j) \leq e(\varphi_{j+1})$ and hence $e(\varphi_j \to \varphi_{j+1}) = 1$. Notice that this implies that this formula must be a tautology for any chain of the variety generated by C^*_n and so all these chains have no more than n elements.

B. Adding an involutive negation: The logics $\tilde{L}^*_n$

In the case that the negation associated to $*$ is not involutive, that is, when $C^*_n \neq Ł_n$, an interesting expansion is the one obtained by adding an involutive negation as an extra connective. This logic, which we will denote by $\tilde{L}^*_n$, is obtained from L^*_n, as is done in the context of intuitionistic logic (see [24]) or in the context of Gödel logic (cf. [13]), by adding a new unary connective $\sim$ and the axioms:

$$\begin{array}{ll} (\sim 1) & \sim\sim\varphi \leftrightarrow \varphi \\ (\sim 2) & \sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi) \\ (\sim 3) & \neg\varphi \to \sim\varphi \end{array}$$

The truth function corresponding to this involutive negation over C^*_n is the unique involutive negation N that is possible to define over C_n, that is, $N(r_i) = r_{n-i+1}$. Thus the canonical chain associated to this logic is

$$\tilde{C}^*_n = \langle C_n, \min, \max, *, \to_*, N, 0, 1\rangle.$$

Notice that in these logics we can define a connective of *strong disjunction* in this way: $\varphi \veebar \psi =: \sim(\sim\varphi \& \sim\psi)$. Another interesting feature of the logics $\tilde{L}^*_n$ is the definability of the connective Δ (see [3]), whose associated truth function is $\delta(x) = 0$ if $x \neq 1$, and $\delta(1) = 1$. In the logics $\tilde{L}^*_n$, this connective is definable as $\Delta\varphi := (\neg\sim\varphi)^{n-1}$, where ψ^m means $\psi \& \overset{(m)}{\ldots} \& \psi$. Observe that when $C^*_n = Ł_n$, taking $\sim = \neg$, we have $\Delta\varphi = \varphi^{n-1}$; and when $C^*_n = G_n$, we have $\Delta\varphi = \neg\sim\varphi$. As a consequence of [16, Theorem 2.4.14], the logics $\tilde{L}^*_n$ enjoy the Delta Deduction Theorem (DT$_\Delta$ for short), i.e., $\varphi \vdash \psi$ iff $\vdash \Delta\varphi \to \psi$.

C. Adding truth constants: The logics $\tilde{L}^*_n(C_n)$

In FDLs are often used the so-called *graded formulas*, which demand, from the logical side, an explicit representation of the truth values in the underlying logic. The needed logic is obtained by adding to $\tilde{L}_n^*$ a truth constant $\bar{r}$ for each $r \in C_n$ and the following axioms and inference rule,[2]

(1) the Book-keeping axioms:
$\overline{r \diamond s} \leftrightarrow \bar{r} \circ \bar{s}$, for $\circ$ being any binary connective and $\diamond$ its corresponding truth function over C_n, and
$\overline{N(r)} \leftrightarrow \sim \bar{r}$;

(2) the Witnessing axiom,

$$\bigvee_{i=1,\ldots,n} (\varphi \leftrightarrow \bar{r}_i);$$

(3) and the rule: $\bar{r}_{n-1} \vee \varphi \vdash \varphi$.

The resulting logic will be denoted as $\tilde{L}_n^*(C_n)$. The *canonical chain* associated with this logic is

$$\tilde{C}_n^*(C_n) = \langle C_n, \min, \max, *, \to_*, N, r_1, \ldots, r_n \rangle.$$

Remark 2.1: Observe that the algebraic counterpart of each logic $\tilde{L}_n^*(C_n)$ would be a quasivariety and not a variety due to the new inference rule (3), but we can prove that, in fact, it is a variety. Indeed, the connective Δ is definable in the logics $\tilde{L}_n^*$ and, since they enjoy the DT_Δ, we have that the rule (3) is equivalent to the axiom $\Delta(\bar{r}_{n-1} \vee \varphi) \to \varphi$. Moreover, it is easy to see that the unique chain belonging to each of these varieties is the canonical chain.

D. The predicate logics $\tilde{L}_n^*(C_n)\forall$

Now we define the first order versions of the propositional fuzzy logics described above. For the basic notions and results on first order fuzzy logics see [16], [20], [11].

Let us recall that a *predicate language* or *first order signature*[3] is a pair $\Sigma = \langle \mathcal{C}, \mathcal{P} \rangle$, where $\mathcal{C} = \{c, d, \ldots\}$ is a countable set of *object constants*, and $\mathcal{P} = \{P, Q, \ldots\}$ is a a countable set of *predicate symbols*, each one with arity $k \geq 0$. In order to build the set of *predicate formulas*, the *logical symbols* are: a countable set of *object variables* $\{x, y, \ldots\}$, the *connectives* of a propositional language $\mathbb{L}$, and the *quantifiers* $\forall$ and $\exists$. *Terms* are object constants and object variables. An *atomic formula* is an expression of the form P, when P is a predicate symbol of arity 0; or $P(t_1, \ldots, t_k)$, being $t_1, \ldots, t_k$ terms, when P is a predicate symbol of arity $k \geq 1$. The set of $\langle \Sigma, \mathbb{L} \rangle$-formulas is built as it is done in the propositional language $\mathbb{L}$ –but now from atomic formulas instead from propositional letters– and adding the rule stating that if φ is a formula, and x is a variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas. The notions of *free variable*, *open formula* (i.e., with free variables) and *closed formula* or *sentence* (i.e., without free variables) are defined in the usual way.

Given a divisible finite t-norm over a chain of n elements, the first order logic $\tilde{L}_n^*(C_n)\forall$ is presented in the Hilbert-style calculus defined as follows:

- *Axioms:* the ones of the propositional logic $\tilde{L}_n^*(C_n)$, where now the formulas are read as predicate formulas, plus the following axioms on quantifiers (see [16]):

($\forall 1$) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$),
($\exists 1$) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$),
($\forall 2$) $(\forall x)(\varphi \to \psi) \to (\varphi \to (\forall x)\psi)$ (x not free in φ),
($\exists 2$) $(\forall x)(\varphi \to \psi) \to ((\exists x)\varphi \to \psi)$ (x not free in ψ),
($\forall 3$) $(\forall x)(\varphi \vee \psi) \to (\forall x)\varphi \vee \psi$ (x not free in ψ).

- *Deduction rules: Modus Ponens* and *Generalization*.

Now let us recall the definitions corresponding to the *canonical semantics*, i.e., the one defined from valuations over the canonical chain, for the language of the logics $\tilde{L}_n^*(C_n)\forall$ (cf. [15] and references therein).

Definition 2.2 ($*$-Interpretation): Given a divisible finite t-norm, an interpretation over the canonical chain $\tilde{C}_n^*(C_n)$, or $*$-interpretation*, for the predicate language $\Sigma = \langle \mathcal{C}, \mathcal{P} \rangle$ is a tuple $\mathbf{M} = \langle M, \{a^{\mathbf{M}} : a \in \mathcal{C}\}, \{P^{\mathbf{M}} : P \in \mathcal{P}\} \rangle$, where* 1) *$M$ is a non-empty set;* 2) *for each object constant $a \in \mathcal{C}$, $a^{\mathbf{M}}$ is an element of M; and* 3) *for each k-ary predicate symbol P with $k \geq 1$, $P^{\mathbf{M}}$ is an n-graded k-ary relation defined on M, that is, a function $P^{\mathbf{M}} : M^k \to C_n$. For each* 0*-ary predicate symbol P, $P^{\mathbf{M}}$ is an element of C_n.*

Given a $*$-interpretation $\mathbf{M}$, a map v assigning an element $v(x) \in M$ to each variable x is called an *assignation of the variables in* $\mathbf{M}$ (an $\mathbf{M}$-*assignation*). Given $\mathbf{M}$ and v, the *value of a term* t in $\mathbf{M}$, denoted by $\|t\|_{\mathbf{M},v}$, is defined as $v(x)$ when t is a variable x, and as $a^{\mathbf{M}}$ when t is a constant a. In order to emphasize that a formula α has its free variables in $\{x_1, \ldots, x_n\}$, we will denote it by $\alpha(x_1, \ldots, x_n)$. Let v be an $\mathbf{M}$-assignation such that $v(x_1) = b_1, \ldots, v(x_n) = b_n$. Taking $\mathbb{L}$ as the propositional language of $\tilde{L}_n^*(C_n)$, the *truth value in* $\mathbf{M}$ *over the canonical chain* $\tilde{C}_n^*(C_n)$ *of the* $\langle \Sigma, \mathbb{L} \rangle$*-formula* $\varphi(x_1, \ldots, x_n)$ *for the assignation* v, denoted by $\|\varphi\|^*_{\mathbf{M},v}$ or by $\|\varphi(b_1, \ldots, b_n)\|^*_{\mathbf{M}}$, is a value in C_n defined inductively as follows:

$P^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_k\|_{\mathbf{M},v})$, if $\varphi = P(t_1, \ldots, t_k)$;
0, if $\varphi = \bar{0}$;
1, if $\varphi = \bar{1}$;
r, if $\varphi = \bar{r}$;
$N(\|\alpha\|^*_{\mathbf{M},v})$, if $\varphi = \sim \alpha$;
$\|\alpha\|^*_{\mathbf{M},v} * \|\beta\|^*_{\mathbf{M},v}$, if $\varphi = \alpha \& \beta$;
$\|\alpha\|^*_{\mathbf{M},v} \to_* \|\beta\|^*_{\mathbf{M},v}$, if $\varphi = \alpha \to \beta$;
$\inf\{\|\alpha(a, b_1, \ldots, b_n)\|^*_{\mathbf{M}} : a \in M\}$, if $\varphi = (\forall x)\alpha(x, x_1, \ldots, x_n)$;
$\sup\{\|\alpha(a, b_1, \ldots, b_n)\|^*_{\mathbf{M}} : a \in M\}$, if $\varphi = (\exists x)\alpha(x, x_1, \ldots, x_n)$.

The truth value of a formula containing a definable connective is calculated straightforwardly. A $*$-interpretation $\mathbf{M}$ is a $*$-*model* of a set of formulas Γ if, for each $\varphi \in \Gamma$, and each $\mathbf{M}$-assignation v, $\|\varphi\|^*_{\mathbf{M},v} = 1$. If $\Gamma = \{\varphi\}$, we say that $\mathbf{M}$ is a $*$-model of φ. We will say that a formula φ is $*$-*satisfiable* iff there is a $*$-model of φ. We will say that φ is $*$-*valid* iff every $*$-interpretation is a $*$-model of φ.

We have the following result about strong canonical completeness for the logics of the family $\tilde{L}_n^*(C_n)\forall$.

[2] For more information about t-norm based fuzzy logics with truth constants see for instance [12].

[3] In this paper we restrict ourselves to signatures having only functional symbols of arity 0, i.e., object constants.

Theorem 2.3: Given a divisible finite t-norm $*$, the logic $\tilde{L}_n^*(C_n)\forall$ is strongly complete with respect to interpretations over its corresponding canonical chain, that is, for every set of formulas Γ and every formula φ, the following conditions are equivalent:

1) $\Gamma \vdash_{\tilde{L}_n^*(C_n)\forall} \varphi$
2) *Every $*$-model of Γ is also a $*$-model of φ.*

Notice that, being the set C_n of truth values finite, the infimum and supremum used to define the truth values for quantified formulas are in fact minimum and maximum, respectively. This implies that all $*$-interpretations are *witnessed* (cf. [17]), which means that all quantified formulas are witnessed in $\mathbf{M}$ in the sense that for every formula $(\forall x)\varphi(x, y_1, \ldots, y_n)$ (resp. $(\exists x)\varphi(x, y_1, \ldots, y_n)$) and any choice $b_1, \ldots, b_n \in M$ of values of $y_1, \ldots, y_n$, there exists $a \in M$ such that $\|(\forall x)\varphi(x, b_1, \ldots, b_n)\|^*_{\mathbf{M}} = \|\varphi(a, b_1, \ldots, b_n)\|^*_{\mathbf{M}}$ (resp. $\|(\exists x)\varphi(x, b_1, \ldots, b_n)\|^*_{\mathbf{M}} = \|\varphi(a, b_1, \ldots, b_n)\|^*_{\mathbf{M}}$). Therefore, the previous strong canonical completeness theorem expresses in fact completeness with respect to witnessed $*$-interpretations.

Finally we recall a notion which will be used in Section IV-A. Given $r \in C_n$, an *evaluated formula* of a logic $\tilde{L}_n^*(C_n)\forall$ is a formula of one of the forms $\bar{r} \to \varphi$, $\varphi \to \bar{r}$, or $\bar{r} \leftrightarrow \varphi$, where φ does not contain any occurrence of truth constants other than $\bar{0}$ or $\bar{1}$. Notice that the last formula above is, in fact, the conjunction of the other two.

III. Description Logic languages in the n-graded and fuzzy frameworks

The attempts to generalize the formalism of DLs to a multi-valued framework have been focussed on generalizing the semantic interpretations of concept constructors in order to make them work with fuzzy concepts and sets, and have been based on the tacit supposition that the same concept constructors (and, with them, the same formal languages) could be maintained in a multi-valued framework. This supposition worked indeed well when, at the beginning of the research on fuzzy DLs, the logic adopted as underlying formalism was the so called Zadeh's logic (see [28]), whose propositional connectives are a direct generalization of the classical ones. However, due to the absence of a residuated implication, this logic is too weak and it can lead to counter-intuitive consequences (for a discussion see [17], [6]). For this reason, in more recent papers, researchers on FDLs adopted, as underlying logic, a logic between those that provide a residuated implication. However, adopting a multi-valued framework and maintaining the same languages as in the classical case, could produce a slight confusion. This is due to several reasons that are related to differences, arising directly from the underlying logical formalisms, between the classical and the multi-valued framework. Commonly, such differences include the following items:

1) implication is not definable within the language,
2) a residuated negation is definable from (residuated) implication and bottom, but (except for the logic of Ł_n) it is not involutive, which implies:
   a) the quantifiers are not interdefinable,
   b) strong union is not definable from residuated negation and strong intersection.

All these items must be taken into account both when choosing the constructors of our description languages and when building the hierarchy of fuzzy description languages in next sections. As an example remind that, in classical DLs, $\mathcal{ALE}$ is strictly contained in $\mathcal{ALC}$, while within many fuzzy DLs, by item 2a) above, this is not the case.

A. Concept constructors in FDLs

In the tradition of Description Logics, the language $\mathcal{ALC}$ (cf. [2]) is presented using: a) the symbols in $\{\sqcup, \sqcap, \neg, \bot, \top\}$ which, from the first order logic point of view, can be understood as the propositional connectives in $\{\vee, \wedge, \neg, \bar{0}, \bar{1}\}$; and b) the symbols $\forall$ and $\exists$ used in the denotation of the constructors of concepts $\forall R.C$ and $\exists R.C$ (*universal* and *existential quantification* respectively) which can also be read as particular kinds of quantified first order formulas (cf. [26]).

In Fuzzy Description Logics we need a set of constructors that corresponds to the logical connectives and quantifiers existing in our first order setting. We will keep the symbols $\sqcup$, $\sqcap$, $\neg$, $\bot$, and $\top$ to denote the constructors of weak union, weak intersection, weak complementation, empty and universal description, respectively. Moreover, we propose the following symbols for the new *propositional constructors*:

- $\boxplus$ for strong union,
- $\boxtimes$ for strong intersection,
- $\sqsupset$ for residuated implication,
- $\sim$ for strong complementation,
- $\bar{r}$ for constant description.

A *description signature* can be defined as a tuple $\mathcal{D} = \langle N_I, N_T, N_A, N_R\rangle$, where $N_I = \{a, b, \ldots\}$ is a countable set of *individual names*; $N_T = \{\bar{r}, \bar{s}, \ldots\}$ is a countable set of *truth-constant names*; $N_A = \{A, B, \ldots\}$, is a countable set of *atomic concepts* or *concept names*; and $N_R = \{R, S, \ldots\}$ is a countable set of *atomic roles* or *role names*. The logical symbols are: a subset of the propositional constructors considered above, the quantifiers $\forall, \exists$, and the point . as an auxiliary symbol. Given $\bar{r} \in N_T$, $A \in N_A$, and $R \in N_R$, a *description formula* is inductively defined in accordance with the following syntactic rules (we use the symbols C, C_1, C_2 as metavariables for descriptions of concepts):

$C, C_1, C_2 \rightsquigarrow$		
$\bot \mid$	(empty description)	
$\top \mid$	(universal description)	
$\bar{r} \mid$	(constant description)	
$A \mid$	(atomic concept)	
$\sim C \mid$	(strong complementary concept)	$(\mathcal{C})$
$\sim A \mid$	(restricted strong compl. concept)	
$\neg C \mid$	(weak complementary concept)	
$C_1 \boxplus C_2 \mid$	(concept strong union)	$(\mathcal{U})$
$C_1 \boxtimes C_2 \mid$	(concept strong intersection)	

$C_1 \sqcup C_2 \mid$	(concept weak union)	
$C_1 \sqcap C_2 \mid$	(concept weak intersection)	
$C_1 \sqsupset C_2 \mid$	(concept implication)	$(\mathfrak{I})$
$\forall R.C \mid$	(universal quantification)	
$\exists R.C \mid$	(existential quantification)	$(\mathcal{E})$
$\exists R.\top \mid$	(restricted existential quantif.)	
$R \mid$	(atomic role)	

Given a description signature $\mathcal{D} = \langle N_I, N_T, N_A, N_R \rangle$, we define the first order signature $\Sigma_{\mathcal{D}} = \langle \mathcal{C}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}} \rangle$, being $\mathcal{C}_{\mathcal{D}} = N_I$ and $\mathcal{P}_{\mathcal{D}} = N_T \cup N_A \cup N_R$, and where we read

- each individual name in N_I as an object constant,
- each truth constant in N_T as a nullary predicate symbol,
- each atomic concept in N_A as a unary predicate symbol,
- each atomic role in N_R as a binary predicate symbol.

Let $\mathbb{L}$ be the propositional language of a logic $\tilde{L}^*_n(C_n)$. The notion of *instance of a description* allows us to read description formulas of a given description signature $\mathcal{D}$ as $\langle \mathbb{L}, \Sigma_{\mathcal{D}} \rangle$-formulas as it is done in the following definition.

Definition 3.1 (Instance of a description): The instance *of a truth constant is defined as* $\bar{0}$ *for* $\bot$*;* $\bar{1}$ *for* $\top$*; and* $\bar{r}$ *for* $\bar{r}$*. Given a term* t *and a concept* D*, the* instance $D(t)$ *of* D *is defined as*

$A(t)$	if D is an atomic concept A,
$\sim C(t)$	if $D = \sim C$,
$\neg C(t)$	if $D = \neg C$,
$C_1(t) \veebar C_2(t)$	if $D = C_1 \boxplus C_2$,
$C_1(t) \& C_2(t)$	if $D = C_1 \boxtimes C_2$,
$C_1(t) \vee C_2(t)$	if $D = C_1 \sqcup C_2$,
$C_1(t) \wedge C_2(t)$	if $D = C_1 \sqcap C_2$,
$C_1(t) \rightarrow C_2(t)$	if $D = C_1 \sqsupset C_2$,
$(\forall y)(R(t, y) \rightarrow C(y))$	if $D = \forall R.C$,
$(\exists y)(R(t, y) \& C(y))$	if $D = \exists R.C$,
$(\exists y) R(t, y)$	if $D = \exists R.\top$,

where y *is a variable not occurring in* $C(t)$*. Finally, an* instance *of an atomic role* R *is any atomic first order formula* $R(t_1, t_2)$*, where* t_1 *and* t_2 *are terms.*

Given a description signature $\mathcal{D}$, let $\mathbf{M}$ be a $*$-interpretation of the first order signature $\Sigma_{\mathcal{D}} = \langle \mathcal{C}_{\mathcal{D}}, \mathcal{P}_{\mathcal{D}} \rangle$, that is,

$$\mathbf{M} = \langle\, M, \{a^{\mathbf{M}} : a \in N_I\}, \{P^{\mathbf{M}} : P \in N_T \cup N_A \cup N_R\}\,\rangle,$$

where, according to Definition 2.2, M is a non-empty set, for each $a \in N_I$, $a^{\mathbf{M}}$ is an element of M and, for every $P \in \mathcal{P}$, $P^{\mathbf{M}}$ is

- an element of C_n, when $P \in N_T$;
- a function $M \rightarrow C_n$, i.e., an n-graded set on M, when $P \in N_A$; and
- a function $M \times M \rightarrow C_n$, i.e., an n-graded binary relation on M, when $P \in N_R$.

So, in order to calculate the truth value of an instance of a description, we proceed according with the definitions given in Section II-D. Thus, for instance, given two atomic concepts A_1 and A_2, the truth value of the instance $D(x)$ of the concept $D = \forall R.(A_1 \boxplus A_2)$ for an $\mathbf{M}$-assignation v such that $v(x) = a$ is calculated as follows:

$$\|D(x)\|^*_{\mathbf{M},v} = \|(\forall y)(R(x, y) \rightarrow (A_1(y) \veebar A_2(y)))\|^*_{\mathbf{M},v} = \\ = \inf_{b \in M}\{R^{\mathbf{M}}(a, b) \rightarrow_* (A_1^{\mathbf{M}}(b) \oplus A_2^{\mathbf{M}}(b))\},$$

where $\oplus$ is the truth function for the connective $\veebar$, i.e., the t-conorm defined by $x \oplus y := N(N(x) * N(y))$, being N the involutive negation.

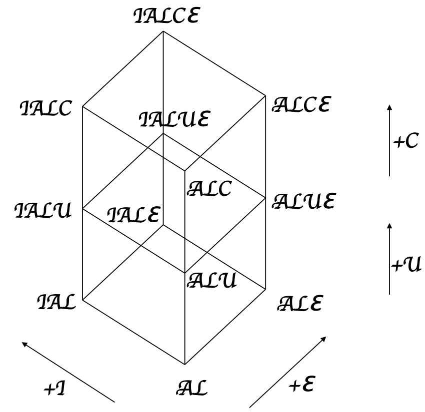

Fig. 1. Hierarchy of languages

B. The family of fuzzy attributive languages

Before moving to basic languages like those already existing in classical DLs, we find worth discussing the case of implication. In classical DL, no language has an explicit concept constructor for implication (even if implication is often implicitly used), because the implication is definable from conjunction and negation. Nevertheless, in the logic BL and many of its extensions, implication is in general not definable from other connectives and, moreover, from implication it is possible to define some other connectives, including residuated negation (see Section II). It is not the first time that a concept constructor for the implication is included in the definition of the language as a primitive connective (see [17], [8]). This allows, on the one hand, to utilize, in a finitely valued framework, a concept constructor that is not otherwise definable, even if quite necessary and, on the other hand, to define other concept constructors like those for weak intersection (whose semantics and symbol are the minimum and $\sqcap$ respectively), weak union (whose semantics and symbol are the maximum and $\sqcup$ respectively) and weak complementation (whose definition is $C \sqsupset \bot$). So, we introduce the symbol $\sqsupset$ to denote concept implication and the letter $\mathfrak{I}$ (for *implicative*) before the name of the language to extend, in order to denote its presence in this language.[4]

[4]We have choosen $\mathfrak{I}$ to denote the presence in the language of the constructor *concept implication* because the calligraphic $\mathcal{I}$ is extensively used in the literature of DLs to denote the constructor for inverse roles.

In accordance with classical DLs tradition, the constructors present in our basic fuzzy description language $\mathcal{AL}$ are *empty concept, universal concept, restricted strong complementary concept, concept strong intersection, universal quantification* and *restricted existential quantification*. Adding the constructor for implication we obtain the language $\mathfrak{I}\mathcal{AL}$. On the other side, allowing the unrestricted existential quantification, we obtain the language $\mathcal{ALE}$. As a third possibility we can add the concept constructor for the concept strong union and obtain the language $\mathcal{ALU}$. When we add an unrestricted strong negation to $\mathcal{AL}$ (so, obtaining $\mathcal{ALC}$) the landscape is slightly different than in classical DL: in our framework the languages $\mathcal{ALE}$ and $\mathfrak{I}\mathcal{AL}$ are not strictly contained in $\mathcal{ALC}$. Since here we do not have the same possibility of reducing languages between each other as in the classical case, the hierarchy of basic languages obtained is more cumbersome. Figure 1 shows the inclusion lattice of the languages obtained by successively adding a basic operator or another. As we can see, since strong union is definable from strong intersection and strong negation (see Section II), $\mathcal{ALU}$ is strictly contained in $\mathcal{ALC}$.

As we have remarked in Section II, the constructors in $\{\sqcup, \sqcap, \neg\}$ are definable from implication, strong intersection and bottom. Hence, these operators are implicitly present as definable constructors in $\mathfrak{I}\mathcal{ALC}$ and in every language extending it. Notice that the language $\mathcal{S}$ (obtained in the classical case from $\mathcal{ALC}$ by adding role transitivity) will be in our framework an extension of $\mathfrak{I}\mathcal{ALCE}$.

Let $\mathcal{X}$ be a language from the hierarchy of Fig. 1. Fixed a divisible finite t-norm $*$ over a chain of n elements, following a similar notation as in [15], we denote by $\mathcal{X}^*(C_n)$ the language obtained by adding a truth constant for each element of the carrier C_n of the corresponding canonical chain. In our framework we are considering languages with constants, but this does not modify the hierarchy in Fig. 1.

Definition 3.2 (The description logics $\mathcal{X}^*(C_n)$): Let $\Gamma \cup \{\varphi\}$ be a finite set of instances of $\mathcal{X}^*(C_n)$-descriptions. We define the $\mathcal{X}^*(C_n)$-logic in the following way:
$\Gamma \models_{\mathcal{X}^*(C_n)} \varphi$ iff *every $*$-model of Γ is also a $*$-model of φ.*

Notice that by Theorem 2.3 (canonical completeness), we have the following result.

Corollary 3.3: Let $\mathcal{X}$ be a language of the hierarchy of Fig. 1. For every language $\mathcal{X}^*(C_n)$, and every set $\Gamma \cup \{\varphi\}$ of instances of $\mathcal{X}^*(C_n)$-descriptions,

$$\Gamma \models_{\mathcal{X}^*(C_n)} \varphi \text{ iff } \Gamma \vdash_{\tilde{L}^*_n(C_n)\forall} \varphi.$$

Therefore, every $\mathcal{X}^*(C_n)$-logic coincides with a fragment of the Hilbert-style calculus defining the logic $\tilde{L}^*_n(C_n)\forall$.

Remark 3.4: Notice that the general hierarchy in Fig. 1 can be simplified when we deal with a concrete standard algebra C^*_n. For instance, when $n = 2$ (the classical case) or when $C^*_n = Ł_n$, the fact that the residuated negation is involutive implies that $\mathcal{E}$ and $\mathcal{U}$ are definable by duality from the universal quantifier and the strong intersection, respectively. Thus, in these cases, the logics $\mathcal{ALCE}$, $\mathfrak{I}\mathcal{ALC}$, $\mathfrak{I}\mathcal{ALCE}$ coincide with $\mathcal{ALC}$.

IV. Reasoning with n-graded DLs

In this section we define the notions concerning knowledge bases for n-graded DLs. In particular, since we are interested in reasoning on partial truth of formulas, we will restrict ourselves to using evaluated formulas for representing the knowledge contained in knowledge bases. With truth constants in the language we can handle graded inclusion axioms in addition to graded assertional axioms (see [15]), as usually done in FDLs (see [28], [29]).

A. Knowledge bases for n-graded DLs

Let C, D be concepts without occurrences of any truth constant other than $\perp$ or $\top$, R be an atomic role and a, b be constant objects. Finally let $r \in C_n$. A *graded concept inclusion formula* is an expression of one of the forms:

$$\langle C \sqsubseteq D, \succcurlyeq \bar{r}\rangle, \quad \langle C \sqsubseteq D, \preccurlyeq \bar{r}\rangle, \quad \langle C \sqsubseteq D, \approx \bar{r}\rangle,$$

whose corresponding evaluated first order sentences are

- $\bar{r} \to (\forall x)(C(x) \to D(x))$
- $(\forall x)(C(x) \to D(x)) \to \bar{r}$
- $\bar{r} \leftrightarrow (\forall x)(C(x) \to D(x))$

A *graded concept assertion formula* is an expression of one of the forms:

$$\langle C(a), \succcurlyeq \bar{r}\rangle, \quad \langle C(a), \preccurlyeq \bar{r}\rangle, \quad \langle C(a), \approx \bar{r}\rangle,$$

whose corresponding evaluated first order sentences are

$$\bar{r} \to C(a), \quad C(a) \to \bar{r}, \quad \bar{r} \leftrightarrow C(a).$$

A *graded role assertion formula* is an expression of the form $\langle R(a, b), \succcurlyeq \bar{r}\rangle$. Its corresponding evaluated first order sentence is $\bar{r} \to R(a, b)$.

As in the classical case, a KB for the languages that fall within the scope of this paper has two components: $TBox$ and $ABox$. A $TBox$ for a graded DL language is a finite set of graded concept inclusion formulas. An $ABox$ is a finite set of graded concept and role assertion formulas. A KB is a pair $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}\rangle$, where the first component is a $TBox$ and the second one is an $ABox$. The formulas in the $ABox$ are called *assertion axioms* and the formulas in the $TBox$ are called *concept inclusion axioms*. Notice that the graded notation of these evaluated formulas is similar to the notation used in some papers on FDLs (see for instance [28], [29]); however, in our framework these expressions correspond to sentences of our first order fuzzy logics. Therefore, the fact that an interpretation satisfies an axiom of a knowledge base is equivalent to saying that the associated $*$-interpretation satisfies the corresponding first order sentence.

Remark 4.1 (Two observations about the fuzzy axioms):
1) Notice that we do not allow sentences of the form $\langle R(a, b), \preccurlyeq \bar{r}\rangle$ in the $ABox$. This choice is made in order to define the fuzzy KB associated to an n-graded DL language as a generalization of the KB associated to the corresponding classic DL language. Allowing sentences such as $\langle R(a, b), \preccurlyeq \bar{r}\rangle$ implies the possibility of allowing negation of atomic roles in the $ABox$, which is not allowed

in classical $\mathcal{ALC}$. But the negation is allowed for concepts (formulas of type $\langle C(a), \preccurlyeq \bar{0}\rangle$) as in the classical case.
2) The presence in the $TBox$ of graded inclusion axioms of the form $\langle C \sqsubseteq D, \preccurlyeq \bar{r}\rangle$ can be surprising from a practical standpoint. However, this kind of axioms can express inclusions between concepts in a range of values. For example, given $s < r$, it can be useful to say that C is included in D to a degree greater or equal than s but less or equal than r. We can express this fact including the axioms $\langle C \sqsubseteq D, \preccurlyeq \bar{r}\rangle$ and $\langle C \sqsubseteq D, \succcurlyeq \bar{s}\rangle$ in the $TBox$.

B. Reasoning tasks

Reasoning in n-graded DLs involves the same kind of tasks as in the classical case but their results depend on the chosen divisible finite t-norm. As it has been proposed in [15], in what follows we will consider graded notions of reasoning tasks. Given concepts C and D, a divisible finite t-norm $*$, the corresponding canonical algebra C_n^*, and a truth value $r \in C_n$,

1) C is *$*$-satisfiable to a degree greater (resp. lower) or equal than* r iff the first order sentence $\bar{r} \to C(x)$ (resp. $C(x) \to \bar{r}$) is $*$-satisfiable.
2) C is *$*$-valid to a degree greater (resp. lower) or equal than* r iff the first order sentence $\bar{r} \to (\forall x)C(x)$ (resp. $(\forall x)C(x) \to \bar{r}$) is $*$-valid.
3) C is *$*$-subsumed* by D *to a degree greater (resp. lower) or equal than* r iff the first order sentence $\bar{r} \to (\forall x)(C(x) \to D(x))$ (resp. $(\forall x)(C(x) \to D(x)) \to \bar{r}$) is $*$-valid.

Notice that the usual notions of 1-*satisfiability* and *positive satisfiability* are reducible to the above notion (1). Indeed a concept C is 1-satisfiable iff it is $*$-satisfiable to a degree greater or equal to 1, and it is positively satisfiable iff it is satisfiable to a degree greater or equal to r_2, where r_2 is the least positive truth value in C_n. Since we have defined satisfiability (validity, subsumption) to a degree both greater and lower or equal than a certain value, it is also possible to define the notions of satisfiability (validity, subsumption) to a degree belonging to an interval of truth values $r, s \in C_n$, being $r \leq s$. For instance, a concept C is *$*$-satisfiable to a degree in the interval* $[r, s]$ iff $(\bar{r} \to C(x)) \,\&\, (C(x) \to \bar{s})$ is $*$-satisfiable. In particular, when $r = s$ we will say that C is *satisfiable to a degree equal to* r.

Next we show that the problems of $*$-validity and $*$-subsumption can be related with $*$-satisfiability.

Proposition 4.2 (Reduction to Satisfiability): Let C and D be concepts and $r_m \in C_n$, with $m > 1$. Then, the following statements hold:

1) *C is $*$-valid to a degree greater or equal than r_m if and only if C is not $*$-satisfiable to a degree lower or equal than r_{m-1},*
2) *C is $*$-subsumed by D to a degree greater or equal than r_m if and only if the concept $C \sqsupset D$ is not $*$-satisfiable to a degree lower or equal than r_{m-1}.*

All these reasoning tasks are decidable. Being the chains C_n^* finite, all $*$-models are witnessed. Thus Hájek's algorithm in [17], that he has also proved to be applicable when adding truth constants (see Section 4 of [18]), also applies to this case with the obvious changes. As a consequence, the satisfiability (resp. validity) problem in the $\mathcal{ALC}$ description language over our logics can be equivalently transformed in a satisfiability (resp. deduction) problem in the corresponding propositional logic that is a decidable problem.

Now we move to reasoning with respect to a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}\rangle$. We can also define the notions of $*$-satisfiability, $*$-validity and $*$-subsumption (and their corresponding versions with degrees) with respect to $\mathcal{K}$. The definitions are the obvious generalizations of the previous ones when restricting to models that satisfy $\mathcal{K}$. Moreover we define:

1) A knowledge base $\mathcal{K}$ is *$*$-consistent* iff there exists an interpretation which $*$-satisfies each axiom in $\mathcal{K}$.
2) a graded assertion formula (resp. a graded concept inclusion formula) is *$*$-entailed* by a knowledge base $\mathcal{K}$ if every $*$-model $\mathbf{M}$ of $\mathcal{K}$ is also a $*$-model of the formula.

In our n-graded FDLs, the notion of *$*$-entailement* defined above can be reduced to a $*$-validity problem. First, a knowledge base $\mathcal{K}$ corresponds to a finite set of first order sentences and thus equivalent to a unique formula φ_k, which is the conjunction of all formulas in $\mathcal{K}$. Second, our logics satisfy the DT_Δ (see Section II-B). Therefore, we have that the $*$-entailment of a formula α from $\mathcal{K}$ can be reduced to the problem of validity of the sentence $\Delta\varphi_k \to \alpha$. Is easy to see that the considered reasoning tasks are also decidable problems with respect to an $ABox$ since they are reducible to $*$-satisfiability of concepts. However, the same problem with respect to a general $TBox$ remains open.

Finally, within the n-graded framework it is also possible to define, as in papers on FDLs as [28], [29], the following reasoning tasks, not used in classical DLs. Given a concept C, a divisible finite t-norm $*$, and the corresponding canonical algebra, the *greatest lower bound* of C, denoted by $glb(C)$, is the maximum degree $r \in C_n$, such that C is $*$-satisfiable to a degree greater or equal than r. In a similar way we define the *least upper bound* of C, denoted by $lub(C)$. Then,

$$glb(C) := \max\{r \in C_n : \bar{r} \to C(x) \text{ is } *\text{-satisfiable}\},$$
$$lub(C) := \min\{r \in C_n : C(x) \to \bar{r} \text{ is } *\text{-satisfiable}\}.$$

V. Conclusions

In this paper we have given an account of the logical results on finitely valued t-norm based logics that we need in order to provide a well-established logical framework to finitely valued FDLs. Subsequently, we have defined n-graded FDLs as the description logics associated to the logics of a finite BL-chain. Taking into account the relationships between the logical connectives and quantifiers in our logical setting, we have provided a new hierarchy of the associated basic description languages. Moreover, we have dealt with the relationships between the reasoning tasks in our framework, and we have shown their decidability by adapting the Hájek's methods presented in [17], [18].

As future work we plan: a) to develop examples of conceptualization of real domains using our FDLs in order to explore their capabilities; b) to investigate the adaptability to our finitely valued framework of some satisfiability algorithms and results existing for infinitely graded FDLs (for instance, those in [6], [7], [27], [29]); and c) to extend the present approach to more expressive languages.

Finally, there are two related topics that could be interesting to explore:

On the one hand, a finite and complete system of natural deduction for the first order many-valued logic of any finite residuated chain defined by truth value functions is given in [5]. Moreover, an automated method to build a finite and complete Gentzen System for these logics, called MUltlog, is provided in [4]. It seems interesting to explore the applicability of these results to our framework.

On the other hand, results on modal many-valued logics, specially results of [9] dealing with modal logics evaluated over a finite residuated lattice, could be interesting in order to study the translation of FDLs to modal systems, in a similar way as it is done in the classical DLs (see [26] and references therein).

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