Almost (MP)-based substructural logics

Petr Cintula, Institute of Computer Science, Czech Academy of Sciences, cintula@cs.cas.cz
Carles Noguera, Artificial Intelligence Research Institute, CSIC, cnoguera@iiia.csic.es

This paper is a contribution to the theory of substructural logics. We introduce the notions of (MP)-*based* and *almost* (MP)-*based* logics (w.r.t. a special set of formulae D), which leads to an alternative proof of the well-known forms of the local deduction theorems for prominent substructural logics (FL, FL_e, FL_{ew}, etc.). Roughly speaking, we decompose the proof of the local deduction theorem into the trivial part, which works almost classically, and the non-trivial part of determining with respect to which set (if any) the logic is almost (MP)-based. We can also show connection of (almost) (MP)-based condition and the proof by cases properties for generalized disjunctions and the description of (deductive) filters generated by some elements of a given algebra.

In order to provide as general theory as possible, i.e., to cover more logics than the usual Ono's definition of substructural logics [6] (i.e. axiomatic extensions of the logic of pointed residuated lattices) we propose a more general notion of substructural logic based on a very weak system lacking not only structural rules, but also associativity of multiplicative conjunction, and consider all its (even non-axiomatic) extensions, expansions (by new connectives), and well-behaved fragments thereof. This defines a wide family of logical systems containing pretty much all prominent substructural logics.

Our basic logic will be the non-associative variant for the Full Lambek Calculus [6, 7], here denoted as SL. Its language, $\mathcal{L}_{\mathrm{SL}}$, consists of residuated conjunction &, right $\searrow$ and left $\swarrow$ residual implications,[1] lattice conjunction $\wedge$ and disjunction $\vee$, and truth constants $\overline{0}, \overline{1}$. The logic SL is given by the following axiomatic system:

$$\vdash \varphi \searrow \varphi \qquad \varphi, \varphi \searrow \psi \vdash \psi \qquad \varphi \vdash (\varphi \searrow \psi) \searrow \psi \qquad \varphi \searrow \psi \vdash (\psi \searrow \chi) \searrow (\varphi \searrow \chi) \qquad \psi \searrow \chi \vdash (\varphi \searrow \psi) \searrow (\varphi \searrow \chi)$$

$$\vdash \varphi \searrow ((\psi \swarrow \varphi) \searrow \psi) \qquad \varphi \searrow (\psi \searrow \chi) \vdash \psi \searrow (\chi \swarrow \varphi) \qquad \psi \swarrow \varphi \vdash \varphi \searrow \psi$$

$$\vdash \varphi \wedge \psi \searrow \varphi \qquad \vdash \varphi \wedge \psi \searrow \psi \qquad \varphi, \psi \vdash \varphi \wedge \psi \qquad \vdash (\chi \searrow \varphi) \wedge (\chi \searrow \psi) \searrow (\chi \searrow \varphi \wedge \psi)$$

$$\vdash \varphi \searrow \varphi \vee \psi \qquad \vdash \psi \searrow \varphi \vee \psi \qquad \vdash (\varphi \searrow \chi) \wedge (\psi \searrow \chi) \searrow (\varphi \vee \psi \searrow \chi) \qquad \vdash (\chi \swarrow \varphi) \wedge (\chi \swarrow \psi) \searrow (\chi \swarrow \varphi \vee \psi)$$

$$\vdash \psi \searrow (\varphi \searrow \varphi \,\&\, \psi) \qquad \psi \searrow (\varphi \searrow \chi) \vdash \varphi \,\&\, \psi \searrow \chi$$

$$\vdash \overline{1} \qquad \vdash \overline{1} \searrow (\varphi \searrow \varphi) \qquad \vdash \varphi \searrow (\overline{1} \searrow \varphi)$$

Definition 1 *A logic* L *in a language* $\mathcal{L}$ *containing* $\searrow$ *is a* substructural logic *if*

- L *is the expansion of the* $\mathcal{L} \cap \mathcal{L}_{\mathrm{SL}}$*-fragment of* SL.
- *for each* n*,* $i < n$*, and each* n*-ary connective* $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$ *holds:*

$$\varphi \searrow \psi, p \searrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots \chi_i, \varphi, \ldots, \chi_n) \searrow c(\chi_1, \ldots \chi_i, \psi, , \ldots, \chi_n)$$

Note that the first condition implies that the second condition holds for connectives from $\mathcal{L}_{\mathrm{SL}}$. Any substructural logic is finitely equivalential [5], order algebraizable [9], weakly implicative [2, 3], and algebraizable in the sense of Blok and Pigozzi [1] in the presence of either $\vee$ or $\wedge$ in its language. The class of substructural logics as just defined contains:

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[0] P. Cintula was supported by project 1M0545 of the Ministry of Education, Youth, and Sports of CR and by Institutional Research Plan AV0Z10300504. C. Noguera was supported by the research contract "Juan de la Cierva" JCI-2009-05453 and the projects TASSAT TIN2010-20967-C04-01 and ESF Eurocores-LogICCC/MICINN FFI2008-03126-E/FILO.

[1] In the literature on substructural logics, the implications are usually denoted by /, \, whereas in the literature on non-commutative fuzzy logic are used the symbols $\to$ (with swapped arguments) and $\rightsquigarrow$. Here we use the signs $\swarrow, \searrow$ suggested by L.N. Stout, since besides indicating the side of conjoining the antecedent in the residuation law they also mark the direction of the implication from the antecedent to the succedent. In logics satisfying the exchange rule $\varphi \searrow (\psi \searrow \chi) \vdash \psi \searrow (\varphi \searrow \chi)$ both implications coincide and then we denote them by the usual symbol $\to$.

- substructural logics Ono's sense, including e.g. monoidal logic, uninorm logic, psBL, GBL, BL, Intuitionistic logic, (variants of) relevance logics, Łukasiewicz logic;
- expansion of the mentioned logics by additional connectives, e.g. (classical) modalities, exponentials in (variants of) Linear Logic and Baaz delta in fuzzy logics;
- fragments of the mentioned logics to languages containing implication, e.g. BCK, BCI, psBCK, BCC, hoop logics, etc.;
- non-associative logics recently developed by Buszkowski, Farulewski, Galatos, Ono, Halaš, Botur, etc.

What seems to be left aside is e.g. the logic $\mathrm{BCK}_\wedge$ of BCK-semilattices [8] (because it does not satisfy one of our axioms, namely: $(\chi \searrow \varphi) \wedge (\chi \searrow \psi) \searrow (\chi \searrow \varphi \wedge \psi))$. This observation calls for a comment on the postulative nature of our definition: when we claim that some logic is substructural and it has a connective $c \in \mathcal{L}_{\mathrm{SL}}$ we *postulate* how this connective should behave. Thus $\mathrm{BCK}_\wedge$ in the language $\{\searrow, \wedge\}$ is not a substructural logic ($\wedge$ does not behave as it should) but (!) if we would formulate $\mathrm{BCK}_\wedge$ in the language $\{\searrow, \overline{\wedge}\}$ it would indeed be a substructural logic (because then the only SL connective present in its language, implication, behaves as it should).

Definition 2 ((MP)-based substructural logic) *A substructural logic is* (MP)-based *if it has a presentation where* (MP) *is the only deduction rule.*

In substructural logics *with* & *and* $\overline{1}$ *in the language* we can introduce the following notation: $\varphi^0 = \{\overline{1}\}$, $\varphi^1 = \{\varphi\}$, $\varphi^{n+1} = \{\varphi \,\&\, \psi, \psi \,\&\, \varphi \mid \psi \in \varphi^n\}$ for every $n \geq 1$; note that if the logic is *associative*, we can identify φ^n just with any of its elements. The presence of $\overline{1}$ (or &) could be avoided at the price of more cumbersome formulations of the theorems (in case of & also we would also need to assume, implicationally expressed, associativity). The proof of the next theorem is almost trivial:

Theorem 3 (Implicational deduction theorem) *Let* L *be a substructural logics with* & *and* $\overline{1}$ *in the language. Then:* L *is* (MP)*-based iff* L *is finitary and for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \chi \searrow \psi \textit{ for some } n \geq 0 \textit{ and } \chi \in \varphi^n.$$

Clearly FL_{ew} is an example of (MP)-based logic, thus we have just shown that it enjoys this form of deduction theorem (and obviously the same holds for its axiomatic extensions). On the other hand, we can use the previous theorem to show that FL_e is not (MP)-based: indeed, $\varphi \vdash \varphi \wedge \overline{1}$ would entail provability of the theorem $\varphi^n \searrow \varphi \wedge \overline{1}$ for some n which can be refuted by a simple semantical counterexample.

Our next aim is to obtain some form of deduction theorem for FL_e, FL, and other substructural logics. To this end, we need to introduce three auxiliary notions. First, given a set S of formulae, we denote by $\prod S$ the smallest set of formulae containing $S \cup \{\overline{1}\}$ and closed under & (it can be seen as the free groupoid with unit generated by S). Second, we introduce a notion of (MP)-based companion for a given logic:

Definition 4 (Logic $\mathrm{L}^{(\mathrm{MP})}$) *Let* L *be a substructural logic. By* $\mathrm{L}^{(\mathrm{MP})}$ *we denote the logic axiomatized by all theorems of* L *and* modus ponens *as the only inference rule.*[2]

Note that L is (MP)-*based* iff $\mathrm{L} = \mathrm{L}^{(\mathrm{MP})}$ and that $\mathrm{L}^{(\mathrm{MP})}$ need not be a substructural logic in the sense of Definition 1. Not withstanding this, we are able to easily prove a deduction theorem for $\mathrm{L}^{(\mathrm{MP})}$ (we formulate already it in a stronger form needed for the next corollary), which in turn will allow to obtain a deduction theorem for L.

[2]Example: if L is the global variant of a normal modal logic, then $\mathrm{L}^{(\mathrm{MP})}$ is its local variant.

Lemma 5 *Let* L *be a substructural logic with* & *and* $\overline{1}$ *in its language. Then:*

$$\Gamma, S \vdash_{\mathrm{L(MP)}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L(MP)}} \hat{\varphi} \searrow \psi \textit{ for some } \hat{\varphi} \in \prod S.$$

The third auxiliary, but crucial, notion is that of *almost* (MP)*-based substructural logic*:

Definition 6 (Almost (MP)**-based substructural logic)** *A substructural logic* L *is* almost (MP)-based *if there is a subset* $D(v, \vec{p}) \subseteq \mathrm{Th_L}(v)$ *(v is a variable and* $\vec{p}$ *are possibly present parameters) such that*

$$\Gamma \vdash_{\mathrm{L}} \varphi \qquad \textit{iff} \qquad \bigcup\{D(\psi, \vec{p}) \mid \psi \in \Gamma\} \vdash_{\mathrm{L(MP)}} \varphi.$$

Notice that each (MP)-based logic is almost (MP)-based (with $D = \{v\}$) and that without loss of generality we can assume that $v \in D$ (unless explicitly said otherwise). Also notice that any axiomatic extension of an almost (MP)-based logic is almost (MP)-based too. Finally, note that each almost (MP)-based logic can be axiomatized with (MP) as the only *non-unary* rule, the question whether the converse is true as well seems to be open. The previous lemma allows us to straightforwardly extend the scope of the implicational deduction theorem to *almost* MP-based logics.

Corollary 7 (Deduction theorem and almost (MP)**-based substructural logics)** *Let* L *be a substructural logic with* & *and* $\overline{1}$ *in the language and* $D(v, \vec{p}) \subseteq \mathrm{Th_L}(v)$*. Then:* L *is* almost (MP)-*based w.r.t. the set* $D(v, \vec{p})$ *if, and only if, for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \hat{\varphi} \searrow \psi \textit{ for some } \hat{\varphi} \in \prod D(\varphi, \vec{p}).$$

Thus the proof of the deduction theorem in a given logic was rather effortlessly reduced to the proof that the logic is almost (MP)-based. Recall the footnote 2 and observe that the well-known connection of global and local variants of modal logic K can, in our terminology, be formulated as 'global K is almost (MP)-based with the $D(v) = \{v, \Box v, \Box\Box v, \dots\}$', which immediately give us the deduction theorem of K. We show that the logics FL_e and FL (and so all their axiomatic extensions) are almost (MP)-based and determine the corresponding set D. As we can see, even the proof of the most complicated case is rather simple:

Theorem 8

- *The logic* FL_e *is almost* (MP)*-based with the set* $D_{\mathrm{FL}_e} = \{v \wedge \overline{1}\}$.
- *The logic* FL *is almost* (MP)*-based with the following set:*

$$D_{\mathrm{FL}} = \{\gamma(v) \mid \gamma(v) \textit{ an iterated conjugate}\}.$$

Proof: We know that FL can be axiomatized by (MP) and rules (con_l) and (con_r): $\varphi \vdash \lambda_\alpha(\varphi)$ and $\varphi \vdash \rho_\alpha(\varphi)$.

Let us denote the set $\bigcup\{D(\chi, \vec{p}) \mid \chi \in \Gamma\}$ as $\hat{\Gamma}$. We show that for each ψ in the proof of $\Gamma \vdash_{\mathrm{FL}} \varphi$ we have $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\psi)$, for each iterated conjugate γ. The claim then follow from taking $\gamma = \lambda_{\overline{1}}$ and the trivial fact that $\vdash_{\mathrm{FL}} \varphi \wedge \overline{1} \searrow \varphi$.

If ψ is an axiom or an element of Γ, the claim is trivial. Assume that ψ was proved using the rule (con_l). Then $\psi = \lambda_\alpha(\chi)$ for some formula χ appearing the proof before ψ. The induction assumption will give us $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\chi)$ for each iterated conjugate γ. Thus, in particular, for each iterated conjugate γ' we have $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma'(\lambda_\alpha(\chi))$. The proof for ($\mathrm{con}_r$) is completely analogous.

Finally, assume that $\Gamma \vdash_{\mathrm{FL}} \chi$ and $\Gamma \vdash_{\mathrm{FL}} \chi \searrow \psi$. Thus, by the induction assumption, for each iterated conjugate γ: $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\chi)$ and $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\chi \searrow \psi)$. The fact that $\vdash_{\mathrm{FL}} \gamma(\varphi \searrow \psi) \searrow (\gamma(\varphi) \searrow \gamma(\psi))$ and *modus ponens* complete the proof.

Interestingly enough, these deductions theorems yield a connection with a variant of the classical proof by cases property. Recall that classical logic enjoys the following metarule:

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

We will see now how a similar property can be obtained for associative substructural logics with a more complex form of disjunction built from the sets $D(v, \vec{p})$ used to show that these logics are almost (MP)-based.

Theorem 9 *Let* L *be an* associative *substructural logic with* & *and* $\overline{1}$ *in the language such that* L *is* almost (MP)*-based w.r.t. the set* $D(v, \vec{p})$. *Then each set* $\Gamma \cup \{\varphi, \psi, \chi\}$ *of formulae holds:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \cup \{\alpha \vee \beta \mid \alpha \in D(\varphi, \vec{p}), \beta \in D(\psi, \vec{p})\} \vdash \chi}$$

Corollary 10 *The following meta-rule holds in* FL*:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \cup \{\gamma_1(\varphi) \vee \gamma_2(\psi) \mid \gamma_1(v), \gamma_1(v) \textit{ iterated conjugates}\} \vdash \chi}$$

The following meta-rule holds in FL_e*:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1}) \vdash \chi}$$

The following meta-rule holds in FL_{ew}*:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

There is also a clear relation of almost (MP)-basedness and the description of the (deductive) filters generated by a set (which is exactly what the deduction theorem says for filters in the Lindebaum algebra (theories), taking in account that implication defined the order). However due to the lack of space we will not go into details here.

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