Characterizing Fuzzy Modal Semantics by fuzzy multimodal systems with crisp accessibility relations

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Abstract— *In [1] the authors considered finitely-valued modal logics with Kripke style semantics where both propositions and the accessibility relation are valued over a finite residuated lattice. Unfortunately, the necessity operator does not satisfy in general the normality axiom (K). In this paper we focus on the case of finite chains, and we consider a different approach based on introducing a multimodal logic where the previous necessity operator is replaced with a family, parametrized by truth values different from zero, of necessity operators each one semantically defined using the crisp accessibility relation given by the corresponding cut of the finitely-valued original accessibility relation. This multimodal logic is somehow more appealing than the original modal one because axiom (K) holds for each necessity operator. In this paper we axiomatize this multimodal logic and we prove that, in the case the starting residuated lattice is a finite BL chain, the modal and the multimodal languages have the same expressive power iff this algebra is an MV chain.*

Keywords— many-valued modal logic, fuzzy modal logic, Łukasiewicz modal logic, fuzzy logic.

1 Introduction

Fuzzy modal logics is a subfield of mathematical fuzzy logic with growing interest. The interested reader is referred to [1] and the references therein for recent developments. Indeed, in [1] the same authors proposed a mathematical definition of what a many-valued (uni)modal logic is. According to this definition *many-valued modal logics* are sets[1] of modal formulas, denoted by $\Lambda(\mathsf{K}, \mathbf{A})$, arising naturally from the semantics given by a complete residuated lattice $\mathbf{A}$, whose support A corresponds to the intended set of truth values, and a class K of Kripke frames valuated over the set A.

In [1], due to simplicity reasons, the authors only considered *modal formulas* obtained enriching the propositional language of residuated lattices with a necessity operator $\Box$ (i.e., without a primitive possibility operator $\Diamond$). These modal formulas may include or not canonical constants to talk about the truth values. By adding *canonical constants* we mean to add one constant $\overline{a}$ for every element a in the residuated lattice $\mathbf{A}$ in such a way that each one of these constants is semantically interpreted by its canonical interpretation (i.e., the very element a). In the case that we allow canonical constants in the modal language we use the notation $\Lambda(\mathsf{K}, \mathbf{A^c})$ to stress their presence. In the present paper we will always assume that the language has canonical constants.

An assumption that we consider throughout the present paper is that $\mathbf{A}$ is a chain (to keep us inside the fuzzy realm) and finite, i.e., $\mathbf{A}$ is a finite MTL chain. We remind the reader that finiteness is crucial in the results obtained in [1], but linearity is not required there. In the next paragraphs we point out some of the results in [1] for the particular case of finite chains.

One of the main results in [1] is the presentation of a complete calculus for the many-valued modal logic $\Lambda(\mathsf{K}, \mathbf{A^c})$ when K is the class Fr of all frames valued over $\mathbf{A}$, assuming a complete calculus is already known for the non-modal logic without canonical constants (denoted by $\mathbf{\Lambda(A)}$). This axiomatization is shown in Table 1. It is worth pointing out that in the case that $\mathbf{A}$ is a finite BL chain, it is known [1, Propositions 2.5 and 2.7] that there are axiomatizations of $\mathbf{\Lambda(A)}$ based on only one rule, the Modus Ponens rule; but there are examples of finite MTL chains where it is strictly necessary to add more rules besides Modus Ponens.

Among the difficulties to find such an axiomatization is that while the meet distributivity axiom

$$(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi) \qquad \text{(MD)}$$

is valid (as in the classical modal case), in general this is not the case in the many-valued modal setting for the normality axiom

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) \qquad \text{(K)}$$

The normality axiom is indeed valid in $\Lambda(\mathsf{Fr}, \mathbf{A^c})$ iff the residuated lattice is a Heyting algebra (i.e., the interpretation of the strong conjunction coincides with the meet).

The situation is much more difficult in case that there are no canonical constants in the language, and as far as the authors are aware the only known axiomatizations for many-valued modal logics of the form $\Lambda(\mathsf{Fr}, \mathbf{A})$, where $\mathbf{A}$ is not a Heyting algebra, are the ones given in [1] for the case that $\mathbf{A}$ is a finite MV chain. The main drawback of the axiomatization there given is that it is rather artificial, and hence it is not absolutely clear what are the basic principles of these many-valued modal logics.

On the other hand, if one considers K as the class CFr of *crisp Kripke frames* (i.e., those Kripke frames valuating the accessibility relation over the set[2] $\{0, 1\}$) then the problem

[1]Besides these sets in the same paper it is also considered the consequence relations $\mathbf{\Lambda}(l, \mathsf{K}, \mathbf{A})$ and $\mathbf{\Lambda}(g, \mathsf{K}, \mathbf{A})$ corresponding to the local and global many-valued modal consequence relations.

[2]It is worth pointing out that since $\mathbf{A}$ is a chain it holds that the set $\{0, 1\}$ coincides with the set $\{a \in A : 1 = a \vee \neg a\}$ of Boolean elements of $\mathbf{A}$.

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Table 1: Axiomatization of the set $\Lambda(\mathsf{Fr}, \mathbf{A^c})$ when $\mathbf{A}$ is a finite chain

- the set of axioms is the smallest set closed under substitutions containing
  - the axiomatic basis for $\mathbf{\Lambda(A)}$,
  - the witnessing axiom $\bigvee_{a\in A}(\varphi \leftrightarrow \overline{a})$
  - the bookkeeping axioms $(\overline{a_1} * \overline{a_2}) \leftrightarrow \overline{a_1 * a_2}$ (for every $a_1, a_2 \in A$ and every $* \in \{\wedge, \vee, \odot, \to\}$),
  - $\Box 1$, $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ and $\Box(\overline{a} \to \varphi) \leftrightarrow (\overline{a} \to \Box\varphi)$ (for every $a \in A$),
- the rules of the basis for $\mathbf{\Lambda(A)}$, the rule $\overline{k} \vee \varphi \vdash \varphi$ (where k is the coatom of $\mathbf{A}$) and the Monotonicity rule $\varphi \to \psi \vdash \Box\varphi \to \Box\psi$.

resembles much more the one in the classical modal setting since now the normality axiom is valid. It is known [1] that $\Lambda(\mathsf{CFr}, \mathbf{A^c})$ is the set of modal formulas derivable in the calculus given in Table 2. This table is, roughly speaking, saying that we only need to add to $\Lambda(\mathsf{Fr}, \mathbf{A^c})$ the normality axiom plus the axiom $\Box(\overline{k}\vee\varphi) \to (\overline{k}\vee\Box\varphi)$ (where k is the coatom of $\mathbf{A}$) in order to capture the logic of crisp frames. We point out it is known that in general it is not enough to add the normality axiom[3] (cf. Remark 3.2); and we notice that the last axiom is a particular case of the formulas $\Box(\overline{a} \vee \varphi) \to (\overline{a} \vee \Box\varphi)$ with $a \in A$, all these formulas being valid in $\Lambda(\mathsf{CFr}, \mathbf{A^c})$. Notice that in Table 2 we have replaced the Monotonicity rule by the Necessity rule, but this is not important due to the presence of the normality axiom.

The aim of this paper is to apply to the realm of many-valued modal logics the well known [2] correspondence between

- a fuzzy binary relation over A (i.e., a function $R : W \times W \longrightarrow A$), and
- an $A \setminus \{0\}$-indexed and decreasing family of crisp binary relations (i.e., a family $\{R_a : a \in A, a \neq 0\}$ such that if $a \leqslant b$ then $R_b \subseteq R_a \subseteq W \times W$).

This correspondence is given by the following identities

$$R_a = \{(w_1, w_2) \in W \times W : R(w_1, w_2) \geqslant a\} \quad (1)$$

$$R(w_1, w_2) = \sup\{a \in A : (w_1, w_2) \in R_a\} \quad (2)$$

The fact that $\mathbf{A}$ is a finite chain (in particular a complete lattice) is crucial in order to see that the previous identities induce a bijective correspondence.

In other words, the previous correspondence transforms a Kripke frame into a family of crisp Kripke frames. This transformation suggests to use a multimodal language (with one modality $\Box_a$ for every element $a \in A \setminus \{0\}$) to describe properties of Kripke frames. The advantage of this method is that, since the modalities $\Box_a$'s are induced by crisp Kripke frames, the normality axioms

$$\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$$

are valid.

[3] In [1] it is proved that if we only add the normality axiom to $\Lambda(\mathsf{Fr}, \mathbf{A^c})$ then we get an axiomatization for the modal logic given by the class of idempotent frames (i.e., those Kripke frames valuating the accessibility relation over the set $\{a \in A : a = a \odot a\}$).

The aim of the present paper is to pursue this research line. To this purpose in Section 2 these multimodal logics are introduced, in Section 3 a complete calculus is given for them, and in Section 4 we compare the expressive power of the crisp multimodal approach with the unimodal one from [1]. Finally, in Section 5 the authors analyze what it is known about these results when there are no canonical constants in the language.

2 Defining the multimodal systems

Throughout this paper we assume that $\mathbf{A}$ is a fixed *finite residuated lattice* whose underlying order is a *chain*. We remind the reader that a residuated lattice is an algebra $\mathbf{A}$ such that:

- $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with a linear associated order $\leqslant$,
- $\langle A, \odot, 1\rangle$ is a commutative monoid with the unit 1,
- $x \odot z \leqslant y \;\Leftrightarrow\; z \leqslant x \to y$ (the law of *residuation*).

In the literature these algebras are well known under different names: residuated lattices, integral, commutative residuated monoids, $\mathsf{FL_{ew}}$-algebras, etc. [3, 4, 5]. We stress that finite MTL-algebras [6], finite BL-algebras [7] and finite MV-algebras [8] are particular cases satisfying our assumption.

We stress that the algebraic language of $\mathbf{A}$ is given by $\langle \wedge, \vee, \odot, \to, 1, 0\rangle$ (with arities $\langle 2, 2, 2, 2, 0, 0\rangle$). The *multimodal language* is the one obtained by enriching the previous one with canonical constants and a unary operator $\Box_a$ for every $a \in A \setminus \{0\}$. We remind the reader that in this paper (see p. 1) with the term *modal language* we denote the language obtained enriching $\langle \wedge, \vee, \odot, \to, 1, 0\rangle$ with canonical constants and one unary operator $\Box$. In case we are interested in the expansion having simultaneously, besides canonical constants, the operators $\Box_a$'s and $\Box$ we will use the expression *full modal language*. Of course we adopt the analogous convention to talk about *multimodal formulas*, *modal formulas* and *full modal formulas*.

A *Kripke frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ where W is a non empty set (whose elements are called *worlds*) and R is a binary relation valued in A (i.e., $R : W \times W \longrightarrow A$) called *accessibility relation*. $\mathfrak{F}$ is said to be *crisp* in case that the range of R is included in $\{0, 1\}$. The classes of Kripke frames and crisp Kripke frames will be denoted, respectively, by Fr and CFr.

A *Kripke model* is a 3-tuple $\mathfrak{M} = \langle W, R, V\rangle$ where $\langle W, R\rangle$ is a Kripke frame and V is a map, called *valuation*, assigning

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Table 2: Axiomatization of the set $\Lambda(\mathsf{CFr}, \mathbf{A^c})$ when $\mathbf{A}$ is a finite chain

- the set of axioms is the smallest set closed under substitutions containing
  - the axiomatic basis for $\mathbf{\Lambda(A)}$,
  - the witnessing axiom $\bigvee_{a\in A}(\varphi \leftrightarrow \overline{a})$
  - the bookkeeping axioms $(\overline{a_1} * \overline{a_2}) \leftrightarrow \overline{a_1 * a_2}$ (for every $a_1, a_2 \in A$ and every $* \in \{\wedge, \vee, \odot, \to\}$),
  - $\Box 1$, $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ and $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  - $\Box(\overline{a} \to \varphi) \leftrightarrow (\overline{a} \to \Box\varphi)$ (for every $a \in A$) and $\Box(\overline{k} \vee \varphi) \to (\overline{k} \vee \Box\varphi)$ (where k is the coatom of $\mathbf{A}$),
- the rules of the basis for $\mathbf{\Lambda(A)}$, the rule $\overline{k} \vee \varphi \vdash \varphi$ (where k is the coatom of $\mathbf{A}$) and the Necessity rule $\varphi \vdash \Box\varphi$.

to each propositional variable and each world in W an element of A (i.e., $V : \mathsf{Var} \times W \longrightarrow A$ where Var is the set of propositional variables). In such a case we say that $\mathfrak{M}$ is based on the Kripke frame $\langle W, R\rangle$.

If $\mathfrak{M} = \langle W, R, V\rangle$ is a Kripke model, then the map V can be uniquely extended to a map V' assigning to each full modal formula (in particular also multimodal and modal formulas) and each world in W an element of A (i.e., $V' : Fm \times W \longrightarrow A$) satisfying that:[4]

- V' is an algebraic homomorphism, in its first component, for the connectives $\wedge, \vee, \odot, \to, 1$ and 0,
- $V'(\overline{a}, w) = a$ for every $a \in A$.
- $V'(\Box\varphi, w) = \bigwedge\{R(w, w') \to V'(\varphi, w') : w' \in W\}$.
- $V'(\Box_a\varphi, w) = \bigwedge\{R_a(w, w') \to V'(\varphi, w') : w' \in W\} = \bigwedge\{V'(\varphi, w') : w' \in W, R(w, w') \geqslant a\}$.

Although V and V' are different mappings there is no problem, since one is an extension of the other, to use the same notation V for both. As usual we say that two formulas φ and ψ are *equivalent* (in symbols $\varphi \equiv \psi$) iff for every Kripke model $\mathfrak{M}$ it holds that $V(\varphi) = V(\psi)$.

Definition 2.1. The *local (many-valued) full modal logic* $\mathsf{F}\mathbf{\Lambda}(l, \mathsf{Fr}, \mathbf{A^c})$ is the consequence relation obtained by defining, for all sets $\Gamma \cup \{\varphi\}$ of full modal formulas,

- $\Gamma \vdash_{\mathsf{F}\mathbf{\Lambda}(l,\mathsf{Fr},\mathbf{A^c})} \varphi$, iff
- For every Kripke model $\langle W, R, V\rangle$ and $w \in W$, if $V(\gamma, w) = 1$ for every $\gamma \in \Gamma$, then $V(\varphi, w) = 1$.

And the *global (many-valued) full modal logic* $\mathsf{F}\mathbf{\Lambda}(g, \mathsf{Fr}, \mathbf{A^c})$ is the one given by defining

- $\Gamma \vdash_{\mathsf{F}\mathbf{\Lambda}(g,\mathsf{Fr},\mathbf{A^c})} \varphi$, iff
- For every Kripke model $\langle W, R, V\rangle$, if $V(\gamma, w) = 1$ for every $\gamma \in \Gamma$ and every $w \in W$, then $V(\varphi, w) = 1$ for every $w \in W$.

In the case that we restrict our attention to multimodal formulas or modal ones we will analogously use the notations $\mathsf{M}\mathbf{\Lambda}(l, \mathsf{Fr}, \mathbf{A^c})$, $\mathsf{M}\mathbf{\Lambda}(g, \mathsf{Fr}, \mathbf{A^c})$, $\mathbf{\Lambda}(l, \mathsf{Fr}, \mathbf{A^c})$ and $\mathbf{\Lambda}(g, \mathsf{Fr}, \mathbf{A^c})$. A formula is *valid* iff it is a theorem of the local (or the global) consequence relation. We will write $\mathsf{M}\Lambda(\mathsf{Fr}, \mathbf{A^c})$, $\mathsf{F}\Lambda(\mathsf{Fr}, \mathbf{A^c})$ and $\Lambda(\mathsf{Fr}, \mathbf{A^c})$ to denote the set of valid formulas in the corresponding language.

From this definition it is obvious that all these local and global consequence relations are conservative expansions of the non-modal consequence relations with and without canonical constants, denoted respectively by $\mathbf{\Lambda(A^c)}$ and $\mathbf{\Lambda(A)}$.

3 Completeness of the multimodal systems

In this section we give an sketch of the proof of the following completeness theorem.

Theorem 3.1 (Completeness)**.** *Let* $\mathbf{A}$ *be a finite* MTL *chain.*

1. *The consequence relation* $\mathsf{M}\mathbf{\Lambda}(g, \mathsf{Fr}, \mathbf{A^c})$ *is the consequence relation axiomatized by the axioms and rules given in Table 3.*
2. *The set* $\mathsf{M}\Lambda(\mathsf{Fr}, \mathbf{A^c})$ *is the set of modal formulas derivable in the calculus given in Table 3.*
3. *The consequence relation* $\mathsf{M}\mathbf{\Lambda}(l, \mathsf{Fr}, \mathbf{A^c})$ *is axiomatized by (i)* $\mathsf{M}\Lambda(\mathsf{Fr}, \mathbf{A^c})$ *as the set of axioms, and (ii) the rules of the basis for* $\mathbf{\Lambda(A)}$ *together with the rule* $\overline{k} \vee \varphi \vdash \varphi$ *(where* k *is the coatom of* $\mathbf{A}$*).*

We notice that the behaviour of the $\Box_b$ operators given in this axiomatization (Table 3) is the same one that was given for $\Box$ in the case of crisp Kripke frames (cf. Table 2), plus the addition of some axioms telling that the $\Box_b$'s are somehow nested.

Remark 3.2. Before explaining the proof it is worth pointing out that if we delete the axioms $\Box_b(\overline{k} \vee \varphi) \to (\overline{k} \vee \Box_b\varphi)$ from Table 3 then we get an incomplete system. This fact can be checked using an standard matrix argument. Let us consider the Gödel algebra $\mathbf{G_3}$ with three elements $\{0, 0.5, 1\}$, and expand it with canonical constants (i.e., $\overline{0.5} = 0.5$) and with the connectives $\Box_{0.5}$ and $\Box_1$ being interpreted by the unary function $f : \mathbf{G_3} \longrightarrow \mathbf{G_3}$ defined by $f(x) := 0.5 \to x$. Then, this algebra enriched with the set $\{1\}$ of designated elements is a matrix that is a model of all axioms and rules given in Table 3 except for the ones of the form $\Box_b(\overline{k} \vee \varphi) \to (\overline{k} \vee \Box_b\varphi)$. It is not a model of these last axioms because for example $\Box_1(\overline{0.5} \vee 0) \to (\overline{0.5} \vee \Box_1 0) = 0.5$.

[4] The infimun of the empty set is taken, as usual, equal to 1.

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Table 3: Axiomatization of the set $\mathsf{M}\Lambda(\mathsf{Fr}, \mathbf{A^c})$ when $\mathbf{A}$ is a finite chain

- the set of axioms is the smallest set closed under substitutions containing
  - the axiomatic basis for $\mathbf{\Lambda(A)}$,
  - the witnessing axiom $\bigvee_{a \in A}(\varphi \leftrightarrow \overline{a})$
  - the bookkeeping axioms $(\overline{a_1} * \overline{a_2}) \leftrightarrow \overline{a_1 * a_2}$ (for every $a_1, a_2 \in A$ and every $* \in \{\wedge, \vee, \odot, \rightarrow\}$),
  - $\Box_b 1$, $(\Box_b\varphi \wedge \Box_b\psi) \rightarrow \Box_b(\varphi \wedge \psi)$ and $\Box_b(\overline{a} \rightarrow \varphi) \leftrightarrow (\overline{a} \rightarrow \Box_b\varphi)$ (for every $a \in A$ and $b \in A \setminus \{0\}$),
  - $\Box_b(\varphi \rightarrow \psi) \rightarrow (\Box_b\varphi \rightarrow \Box_b\psi)$ and $\Box_b(\overline{k} \vee \varphi) \rightarrow (\overline{k} \vee \Box_b\varphi)$ (for every $b \in A \setminus \{0\}$) (where k is the coatom of $\mathbf{A}$),
  - $\Box_{b_1}\varphi \rightarrow \Box_{b_2}\varphi$ (for every $b_1, b_2 \in A \setminus \{0\}$ such that $b_1 \leqslant b_2$),
- the rules of the basis for $\mathbf{\Lambda(A)}$, the rule $\overline{k} \vee \varphi \vdash \varphi$ (where k is the coatom of $\mathbf{A}$) and the Necessity rules $\varphi \vdash \Box_b\varphi$ (for every $b \in A \setminus \{0\}$).

Next we give some hints on the proof of Theorem 3.1. To this purpose for the rest of this section we will use the symbol L to denote the set of modal formulas derivable from the calculus given in Table 3, and the symbol $\vdash_{\mathsf{L}}$ to denote the consequence relation axiomatized by L as set of axioms and the rules of the basis for $\mathbf{\Lambda(A)}$ together with the rule $\overline{k} \vee \varphi \vdash \varphi$ (where k is the coatom of $\mathbf{A}$). The completeness theorem can be proved using two steps: (i) a reduction of the multimodal completeness problem to the already known strong completeness of the non-modal logic $\mathbf{\Lambda(A^c)}$, and (ii) a Truth Lemma based on a canonical Kripke model construction. Next we briefly sketch the proofs of each one of these steps in the completeness proof.

The first step is based on the fact that $\vdash_{\mathsf{L}}$ is strongly complete by definition with respect to $\mathbf{\Lambda(A)}$, and hence by [1, Corollary 2.16] we know that it is also strongly complete with respect to $\mathbf{\Lambda(A^c)}$. We point out that the rule $\overline{k} \vee \varphi \vdash \varphi$ plays a remarkable role in the proof of [1, Corollary 2.16]. Therefore, we get the following trivial consequence.

Lemma 3.3 (Non-Modal Reduction)**.** *Let $\Gamma \cup \{\varphi\}$ be a set of multimodal formulas. Then*

- *$\Gamma \vdash_{\mathsf{L}} \varphi$, iff*
- *$\Gamma \vdash_{\mathbf{\Lambda(A^c)}} \varphi$, i.e., for every homomorphism h from the algebra of multimodal formulas*[5] *into the algebra $\mathbf{A^c}$, if $h[\Gamma] \subseteq \{1\}$ then $h(\varphi) = 1$.*

On the other hand, the second step consists on a canonical Kripke model construction. The definition of this construction is the following one.

Definition 3.4. The *multi canonical Kripke model* $\mathfrak{M}_{\mathfrak{mcan}}$ is the Kripke model $\langle W_{mcan}, R_{mcan}, V_{mcan}\rangle$ where

- the set W_{mcan} is the set of non-modal homomorphisms $v : \mathbf{Fm} \longrightarrow \mathbf{A^c}$ (we point out that the algebra $\mathbf{Fm}$ is the one given by multimodal formulas) such that $v[\mathsf{L}] = \{1\}$.
- the accessibility relation R_{mcan} is defined by[6]

$$R_{mcan}(v_1, v_2) := \bigvee\{b \in A\backslash\{0\} : \forall\varphi(v_1(\Box_b\varphi) \leqslant v_2(\varphi))\}.$$

- the valuation map is defined by $V_{mcan}(p, v) := v(p)$ for every variable p.

It is obvious that for every $b \in A \setminus \{0\}$, it holds that $b \leqslant R_{mcan}(v_1, v_2)$ iff $v_1(\Box_b\varphi) \leqslant v_2(\varphi)$ for every multimodal formula φ.

Lemma 3.5 (Truth Lemma)**.** *The multi canonical Kripke model $\mathfrak{M}_{\mathfrak{mcan}}$ satisfies that $V_{mcan}(\varphi, v) = v(\varphi)$ for every multimodal formula φ and world v.*

Proof. The proof is done by induction on the multimodal formula. The only non trivial case is when this formula starts with a necessity operator $\Box_b$. By the inductive hypothesis it is clear that it is enough to prove that

$$\bigwedge\{v'(\varphi) : v' \in W_{mcan}, b \leqslant R_{mcan}(v, v')\} = v(\Box_b\varphi)$$

where v is a world. And indeed the only non trivial inequality is $\leqslant$. Hence, let us consider $a := \bigwedge\{v'(\varphi) : v' \in W_{mcan}, b \leqslant R_{mcan}(v, v')\}$ and try to prove that $a \leqslant v(\Box_b\varphi)$.

First of all we claim that

$$\mathsf{L} \cup \{\overline{d} \rightarrow \psi : \psi \in Fm, d = v(\Box_b\psi) \in A\} \vdash_{\mathbf{\Lambda(A^c)}} \overline{a} \rightarrow \varphi.$$

Why? Let us consider a homomorphism h from the algebra of multimodal formulas into the algebra $\mathbf{A^c}$ such that $h[\mathsf{L} \cup \{\overline{d} \rightarrow \psi : \psi \in Fm, d = v(\Box_b\psi) \in A\}] = \{1\}$. We have to prove that $h(\overline{a} \rightarrow \varphi) = 1$, i.e., $a \leqslant h(\varphi)$. The assumptions on h imply that $h \in W_{mcan}$, and that $v(\Box_b\psi) \leqslant h(\psi)$ for every ψ. Hence, $b \leqslant R_{mcan}(v, h)$. Using the definition of a we get that $a \leqslant h(\varphi)$. This finishes the proof of this first claim.

Hence, using that $\mathbf{\Lambda(A^c)}$ is a finitary logic (because A is finite) we get from the previous claim that there is some $m \in \omega$, some multimodal formulas $\psi_1, \ldots, \psi_m$ and some elements $d_1, \ldots, d_m \in A$ such that $d_i = v(\Box_b\psi_i)$ for every $i \in \{1, \ldots, m\}$ and

$$\mathsf{L} \cup \{\overline{d_1} \rightarrow \psi_1, \ldots, \overline{d_m} \rightarrow \psi_m\} \vdash_{\mathbf{\Lambda(A^c)}} \overline{a} \rightarrow \varphi.$$

From here it is clear[7] that $\mathsf{L} \vdash_{\mathbf{\Lambda(A^c)}} ((\overline{d_1} \rightarrow \psi_1) \wedge \ldots \wedge (\overline{d_m} \rightarrow \psi_m)) \rightarrow (\overline{k} \vee (\overline{a} \rightarrow \varphi))$. Using Lemma 3.3 we get from the

[5]When we look at the multimodal formulas as non-modal ones we are thinking that $\{\Box_a\varphi : a \in A \setminus \{0\}, \varphi$ multimodal formula$\}$ are the variables of this non-modal language.

[6]The supremum of the empty set is taken, as usual, equal to 0.

[7]The trick used here is a particular case of the following more general statement:

$$\Gamma, \gamma \vdash_{\mathbf{\Lambda(A^c)}} \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathbf{\Lambda(A^c)}} \gamma \rightarrow (\overline{k} \vee \varphi).$$

This statement is a consequence of the fact that k is the coatom of $\mathbf{A}$.

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previous claim that $\mathsf{L} \vdash_{\mathsf{L}} ((\overline{d_1} \to \psi_1) \wedge \ldots \wedge (\overline{d_m} \to \psi_m)) \to (\overline{k} \vee (\overline{a} \to \varphi))$. Thus, $((\overline{d_1} \to \psi_1) \wedge \ldots \wedge (\overline{d_m} \to \psi_m)) \to (\overline{k} \vee (\overline{a} \to \varphi)) \in \mathsf{L}$. Therefore, $\Box_b((\overline{d_1} \to \psi_1) \wedge \ldots \wedge (\overline{d_m} \to \psi_m)) \to \Box_b(\overline{k} \vee (\overline{a} \to \varphi)) \in \mathsf{L}$ by the Monotonicity Rule. Thus, $((\overline{d_1} \to \Box_b\psi_1) \wedge \ldots \wedge (\overline{d_m} \to \Box_b\psi_m)) \to (\overline{k} \vee (\overline{a} \to \Box_b\varphi)) \in \mathsf{L}$ using some of the axioms in Table 3. Finally, using that $v(\overline{d_1} \to \Box_b\psi_1) = \ldots = v(\overline{d_m} \to \Box_b\psi_m) = 1$ and that $k \neq 1$, we obtain that $v(\overline{a} \to \Box_b\varphi) = 1$, i.e., $a \leqslant v(\Box_b\varphi)$. This finishes the proof. □

Remark 3.6. In the calculus given in Table 3 we have included for each one of the $\Box_b$ modalities the meet distributivity axiom, the normality axiom, the Monotonicity rule and the Necessity rule. Checking the details of the proof of Lemma 3.5 the reader can realize that among the previous four axioms/rules it is enough to include the meet distributivity axiom and the Monotonicity rule in order to get completeness. And the same is also true if we only include the normality axiom and the Necessity rule, but this time the argument is slightly different (just replace $\wedge$ with $\odot$ in the proof of Lemma 3.5).

4 Comparing Expressive Power

The aim of this section is to compare the expressive power of the multimodal language with the one of the modal language. To this purpose we introduce the following concepts.

Definition 4.1. Given two pointed[8] Kripke models $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{M}', w'\rangle$, we will say that they are *modally equivalent*, in symbols $\langle \mathfrak{M}, w\rangle \equiv \langle \mathfrak{M}', w'\rangle$, in the case that $V(\varphi, w) = V'(\varphi, w')$ for every modal formula φ. Analogously we will talk about *multimodally equivalent* and *full modally equivalent*, in symbols $\equiv_{\mathsf{M}}$ and $\equiv_{\mathsf{F}}$, in the case we focus, respectively, on multimodal formulas or full modal formulas.

The first remark comparing expressive powers is that in every finite residuated lattice $\mathbf{A}$ (here it is not needed the chain assumption) it holds that

$$\Box\varphi \;\equiv\; \bigwedge\{\overline{a} \to \Box_a\varphi : a \in A \setminus \{0\}\} \tag{3}$$

Therefore, the modality $\Box$ is explicitly definable using the modalities $\Box_a$'s (and these last ones have the advantage that satisfy the normality axiom). In other words, the expressive power of the modal language is smaller than the one of the multimodal one. Thus, $\Lambda(\mathsf{Fr}, \mathbf{A^c})$ can be seen as a fragment of $\mathsf{M}\Lambda(\mathsf{Fr}, \mathbf{A^c})$. It is obvious that if two pointed Kripke models are multimodally equivalent then they are also modally equivalent.

What about the converse direction in the statements from last paragraph? That is, (i) is it possible to explicitly define the modalities $\Box_a$'s in the modal language?, and (ii) are modally equivalent pointed Kripke models also multimodally equivalent? In the rest of the section we will discuss these two questions. It is worth pointing out that as far as the authors are aware there is no general result in the many-valued modal setting relating these last two questions: an study of Beth definability in this setting has not been undertaken (cf. [9, p. 277]). As a matter of fact it is necessary to distinguish between having an empty set Var of propositional variables or not.

[8] By a *pointed Kripke model* we mean a Kripke model together with a distinguished point.

Proposition 4.2 (Case $\mathsf{Var} = \emptyset$)**.** *Let $\mathbf{A}$ be a finite MTL chain. Then, if two pointed Kripke models are modally equivalent then they are also multimodally equivalent.*

Proof. First of all we point out that it is enough to prove that if $\langle \mathfrak{M}, w\rangle \equiv \langle \mathfrak{M}', w'\rangle$, $w_0 \in W$ and $R(w, w_0) \geqslant a \neq 0$, then there is some $w_0' \in W'$ such that $R'(w', w_0') \geqslant a$ and $\langle \mathfrak{M}, w_0\rangle \equiv \langle \mathfrak{M}', w_0'\rangle$. The proof finishes by realizing that the previous statement is true because for every $a \in A \setminus \{0\}$, it holds that $V(\Box\overline{\mathsf{pred}(a)}, w) \neq 1$ iff there is some w_0 such that $R(w, w_0) \geqslant a$. The notation $\mathsf{pred}(a)$ refers to the predecessor of element a. □

Therefore, it is obvious that if $\mathsf{Var} = \emptyset$ then two pointed Kripke models are modally equivalent iff they are multimodally equivalent. It is still open whether the modalities $\Box_a$'s are explicitly definable or not when there are no propositional variables.

In the case that there is some propositional variable then the situation is quite different as next proposition shows. We will see later in Theorem 4.5 that the assumption in this proposition is also a necessary condition for the case of finite BL chains.

Proposition 4.3 (Case $\mathsf{Var} \neq \emptyset$)**.** *Let $\mathbf{A}$ be the ordinal sum $\mathbf{A_1} \oplus \mathbf{A_2}$ of two finite MTL chains such that $\mathbf{A_1}$ and $\mathbf{A_2}$ are non trivial (i.e., $\min\{|A_1|, |A_2|\} \geqslant 2$). Then, there are two pointed Kripke models that are modally equivalent but not multimodally equivalent.*

Proof. Let us define $a \in A$ as the minimum element of $\mathbf{A_2}$ (i.e., a is the idempotent element separating both components). Since $\mathbf{A_i}$'s are non trivial it is obvious that $a \notin \{0^{\mathbf{A}}, 1^{\mathbf{A}}\}$, and also that for every $b \in A$, $a \odot b = a \wedge b$. Next we consider the two Kripke models given[9] in Fig. 1. It is obvious that for every modal formula φ it holds that $V(\varphi, w_1) = V'(\varphi, w_1')$. On the other hand, by induction on the length of formulas the reader can easily prove[10] that for every modal formula φ (we remind that canonical constants are allowed), it holds both that

- $a \leqslant V'(\varphi, w_1')$ iff $a \leqslant V'(\varphi, w_2')$,
- if $a \nleqslant V'(\varphi, w_1')$ then $V'(\varphi, w_1') = V'(\varphi, w_2')$.

These last two properties guarantees that for every modal formula φ, it holds that $V'(\varphi, w_1') \leqslant a \to V'(\varphi, w_2')$; and so $V'(\Box\varphi, w_0') = V'(\varphi, w_1')$. Once we know this last fact, it becomes easy to show by an straightforward induction that for every modal formula φ, it holds that $V(\varphi, w_0) = V'(\varphi, w_0')$. Thus, $\langle \mathfrak{M}, w_0\rangle$ and $\langle \mathfrak{M}', w_0'\rangle$ are modally equivalent. Finally, using that $V(\Box_a p, w_0) = 1 \neq a = V'(\Box_a p, w_0')$ we get that $\langle \mathfrak{M}, w_0\rangle$ and $\langle \mathfrak{M}', w_0'\rangle$ are not multimodally equivalent. □

Lemma 4.4. *Let $\mathbf{A}$ be the finite MV chain of cardinal n. Then, for every $a \in A \setminus \{0\}$ it holds that*[11]

$$\Box_a\varphi \;\equiv\; \bigwedge\left\{\left(\overline{a} \to \neg\Box\neg((\varphi \leftrightarrow \overline{b})^{n-1})\right)^{n-1} \to \overline{b} : b \in A\right\}$$

[9] The convention here adopted about presenting Kripke models as diagrams is the same one stated in [1, Convention 3.6].

[10] In this inductive proof it plays a crucial role the definition of ordinal sum.

[11] Notice that φ^{n-1} only takes crisp values (i.e., in $\{0, 1\}$). Indeed, φ^{n-1} takes value 1 when φ takes value 1, and φ^{n-1} takes value 0 elsewhere. Hence, φ^{n-1} takes the same value than $\Delta\varphi$ where Δ is the well-known operator used in fuzzy logic (see [7]).

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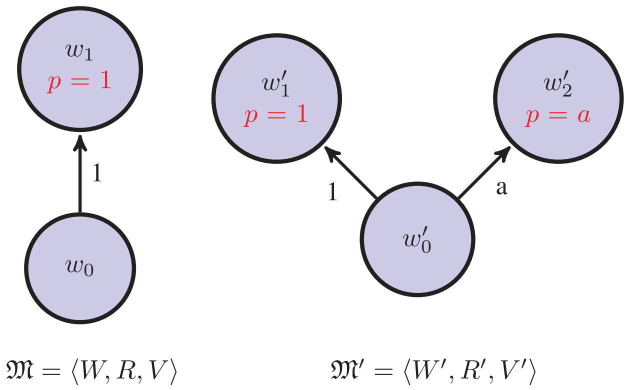

Figure 1: Two interesting Kripke models

Proof. Let us introduce the abbreviations $\Diamond\varphi := \neg\Box\neg\varphi$ and $\Delta\varphi := \varphi^{n-1}$. Then, the statement says that

$$\Box_a\varphi \;\equiv\; \bigwedge\{\Delta(\bar{a} \to \Diamond\Delta(\varphi \leftrightarrow \bar{b})) \to \bar{b} : b \in A\}$$

In this proof we will use the notation

$$f(w,\varphi,b) := \bigvee\{R(w,w') : w' \in W, V(\varphi,w') = b\}.$$

The reader can easily check the following steps.

$$V(\Diamond\Delta(\varphi \leftrightarrow \bar{b}), w) = f(w,\varphi,b)$$

$$V(\Delta(\bar{a} \to \Diamond\Delta(\varphi \leftrightarrow \bar{b})), w) = \begin{cases} 1, & \text{if } a \leqslant f(w,\varphi,b) \\ 0, & \text{if not} \end{cases}$$

$$V(\Delta(\bar{a} \to \Diamond\Delta(\varphi \leftrightarrow \bar{b})) \to \bar{b}, w) = \begin{cases} b, & \text{if } a \leqslant f(w,\varphi,b) \\ 1, & \text{if not} \end{cases}$$

Then, it easily follows that $V(\bigwedge\{\Delta(\bar{a} \to \Diamond\Delta(\varphi \leftrightarrow \bar{b})) \to \bar{b} : b \in A\}, w) = \bigwedge\{V(\varphi, w') : w' \in W, R(w,w') \geqslant a\} = V(\Box_a\varphi, w)$. This finishes the proof. $\Box$

Theorem 4.5 (Case $\mathsf{Var} \neq \emptyset$)**.** *Let* $\mathbf{A}$ *be a finite* BL *chain. Then, the following statements are equivalent.*

1. $\mathbf{A}$ *is a finite* MV *chain (i.e., the only idempotent elements of* A *are* 0 *and* 1*),*
2. *The modalities* $\Box_a$*'s are explicitly definable*[12] *in the modal language.*
3. *Two pointed Kripke models are modally equivalent iff they are multimodally equivalent,*

Proof. For the case that $|A_1| \leqslant 2$ it is trivial that these statements are equivalent. Hence, we only have to deal with the case that $|A_1| \geqslant 3$; and this case is a consequence of Proposition 4.3 and Lemma 4.4 together with the decomposition of BL chains as ordinal sums. $\Box$

5 Concluding Remarks

As it has been stressed in this paper, and also in [1], the role played by canonical constants is crucial in several of the proofs given above. For example, it is unknown to the authors whether there is a general method for converting an axiomatization of $\Lambda(\mathbf{A})$ (and so without canonical constants) into one of $\Lambda(\mathsf{CFr}, \mathbf{A})$. One of the few statements that we know that remains true when we remove canonical constants from the language is Theorem 4.5. That is, for finite MV chains it is possible to find a modal formula (and so using $\Box$) without canonical constants that is equivalent to $\Box_a p$. The proof of this stronger version of Theorem 4.5 is completely different and is based on a combinatorial analysis of all possibilities; it is a rather involved proof and due to space limitations has not been included in this paper.

[12] That is, there is some modal formula φ such that $\varphi \equiv \Box_a p$. We remind that p refers to a propositional variable.

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