Logical Reasoning and Computation: Essays dedicated to Luis Fariñas del Cerro

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Preface

Near the end of 2015, Luis Fariñas del Cerro officially retired as directeur de recherche in the CNRS and became an Emeritus researcher of the CNRS. The present volume is a Festschrift in his honour to celebrate Luis's achievements in science, both as an outstanding scholar as well as a remarkable and highly successful organiser, administrator and leader in science and technology policy and management.

The volume contains 15 scientific contributions by 21 authors, among them Luis's colleagues, former students and friends. They will be presented at an international workshop, Logical Reasoning and Computation, to be held at IRIT, Université Paul Sabatier, Toulouse, on March 3-4, 2016. The volume includes a short scientific biography, written by Philippe Balbiani and Andreas Herzig, that describes the many different areas of logic and computation where Luis has made significant advances to the field.

Despite setting a tight deadline for contributions, we received a fantastic response from all the scholars we contacted. It became clear that Luis is held in great affection and esteem by his students, co-authors and close collaborators. This is also witnessed by the breadth of Luis's geographical reach: this volume alone includes scholars from 10 different countries and 4 continents. Besides scientific papers, we also received contributions in the form of personal reminiscences, poems and even a song, that will be presented and performed at the celebratory workshop.

Since Luis has been slowly winding down his administrative responsibilities, he has recently been able to dedicate a greater effort to research once again, entering with great enthusiasm new and exciting fields such as computational biology. Luis, we surely speak on behalf of all the contributors here to wish you enormous success and enjoyment in your new role and we look forward to many more years of inspiring cooperation with you in the future.

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Table of Contents

On the Relation between Possibilistic Logic and Modal Logics of Belief

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Abstract. Possibility theory and modal logic are two knowledge representation frameworks that share some common features, such as the duality between possibility and necessity, as well as some obvious differences since possibility theory is graded but is not primarily a logical setting. In the last thirty years there have been a series of attempts, reviewed in this paper, for bridging the two frameworks in one way or another. Possibility theory relies on possibility distributions and modal logic on accessibility relations, at the semantic level. Beyond the observation that many properties of possibility theory have qualitative counterparts in terms of axioms of well-known modal logic systems, the first works have looked for (graded) accessibility relations that can account for the behavior of possibility and necessity measures. More recently, another view has emerged from the study of logics of incomplete information, which is no longer based on Kripke-like models. On the one hand, possibilistic logic, closely related to possibility theory, mainly handles beliefs having various strength. On the other hand, in the so-called meta-epistemic logic (MEL) an agent can express both beliefs and explicitly ignored facts (both without strength), by only using modal formulas of depth 1, and no objective ones; its semantics is based on epistemic states. The system MEL$^+$ is an extension of MEL having the syntax of S5. Generalized possibilistic logic (GPL) extends both possibilistic logic and MEL, and has a semantics in terms of sets of possibility distributions. After a survey of these different attempts, the paper presents GPL$^+$, a graded counterpart of MEL$^+$ that extends MEL by allowing objective (sub)formulas. The axioms of GPL$^+$ are graded counterparts of those of S5 modal system, the semantics being based on pairs made of an interpretation (representing the real state of facts) and a possibility distribution (representing an epistemic state). Soundness and completeness are established. The paper also discusses the difference with S5 used as a logic for rough sets that accounts for indiscernibility rather than incomplete information, using also the square of opposition as a common structure underlying modal logic, possibility theory, and rough set theory.

Keywords: Modal logic, possibility theory, epistemic logic, rough sets

1 Introduction

Possibility theory has been introduced by Zadeh [54] as a framework for representing the uncertainty conveyed by linguistic statements. It is based on the notion of possibility distribution π, from which a maxitive possibility measure $\Pi(A)$ is defined as a consistency degree between this distribution representing the available information and the considered event A. This proposal is formally similar to, although fully independent of the one previously developed in economics by Shackle [50] based on the notion of degree of surprise (which corresponds to impossibility).

Although possibility theory has been the basis of an original approximate reasoning theory [56], this setting is not a logical setting strictly speaking. It is only later, in the 1980's, that possibilistic logic, a logic of classical logic formulas associated with certainty levels (thought as lower bounds of a necessity measure) has emerged (see [15,18] for introductions and overviews). Still, in the setting of his representation language PRUF [55] Zadeh discusses the representation of statements of the form "X is A" (meaning that the possible values of the single-valued variable X are fuzzily restricted by fuzzy set A) linguistically qualified in terms of truth, probability, or possibility. Interestingly enough, the representation of possibility-qualified statements led to possibility distributions over possibility distributions, but certainty-qualified statements were not considered at all, just because necessity measures as dual of possibility measures were playing almost no role in Zadeh's view (with the exception of half a page in [57]). Certainty-qualified statements were first considered in [45], and rediscussed in [14] in relation with two resolution principles (respectively involving two certainty-qualified propositions, and one certainty-qualified proposition together with a possibility qualified proposition), whose formal analogy with the inference rules existing in modal logic was stressed.

Such an analogy between possibility theory calculus (including necessity measures) and modal logic was not coming as a surprise since the parallels between $N(A) = 1 - \Pi(A)$ and $\Box p \leftrightarrow \neg \Diamond \neg p$ (duality between necessity and possibility), between $N(A) \leq \Pi(A)$ and $\Diamond p \rightarrow \Box p$ (axiom D in modal logic systems), or between the characteristic axiom of necessity measures $N(A \cap B) = \min(N(A), N(B))$ and $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ (a theorem valid in modal system K) had been already noticed. Nevertheless, no formal connection between modal logic and possibility theory existed in those days, even if the idea of graded accessibility relations had been already proposed independently [32] [49] some years before.

The striking parallel between possibility theory and modal logic eventually led to proposals for a modal analysis and encoding of possibility theory, one of which by L. Fariñas and A. Herzig [25], later by Boutilier [5], then extended to multiple-valued propositions [29]. Another more semantically-oriented trend was to build particular accessibility relations [22][31] agreeing with possibility theory. The work in [36,37,38,35] is also worth-mentioning in that respect.

Rather than putting possibility theory under the umbrella of (graded) modal logics, a quite different view has finally emerged by designing a logical system closer to classical logic capable of handling simple certainty- or possibility-qualified statements. This epistemic logic is a two-tiered propositional logic (an idea that first appears in [16]) where propositional combinations of modal formulas of depth 1 can be handled. The resulting logic, called meta-epistemic logic (MEL), when necessity and possibility are binary-valued, proved to be equivalent to a fragment of the normal modal logic system KD [1,3]. MEL can be extended to graded modalities, thus extending possibilistic logic [33,11] (where only conjunctions of certainty- or possibility-qualified statements are allowed) to a generalized possibilistic logic (GPL) [20], where negation and disjunctions of weighted formulas are allowed . The semantics of MEL (resp. GPL) is no longer expressed by means of an accessibility relation, but in terms of a set of sets of models (resp. a set of possibility distributions), which agrees with Zadeh's original semantical view of possibility-qualified statements (applied in his case to linguistic degrees of possibility and thus leading to a fuzzy set of possibility distributions).

MEL has been more recently extended to MEL$^+$ [2] where propositional combinations of objective formulas and modal formulas of depth 1 are allowed. These formulas are semantically evaluated by pairs made of one interpretation (representing the real state of facts) and a non-empty set of interpretations (representing an epistemic state). The axioms of MEL$^+$ are those of propositional logic, modal axioms K (distributivity), and D, plus $\Box p$ if p is a tautology, while MEL^{++} also includes axiom T ($\Box p \rightarrow p$). MEL$^+$ and MEL^{++} are respectively equivalent to modal systems KD45 and S5. The purpose of this paper is to extend such a construct to GPL.

The paper is structured as follows. The next two sections provides a detailed background organized in several subsections. Section 2 first covers a square of opposition-based view of modal logic, possibility theory, and rough sets whose logic obey the axioms of modal system S5. Then Section 2 surveys early attempts at bridging possibility theory and modal logics. Section 3 offers overviews of MEL, MEL$^+$ and generalized possibilistic logic. Section 4 is dedicated to the joint extension of MEL$^+$ and GPL in GPL$^+$, and then to the joint extension of MEL^{++} and GPL$^+$ in GPL^{++}; soundness and completeness results are established.

2 Background

This background section is organized into two pieces. First, we indicate how the square of opposition captures and exhibits the roots of the formal similarities underlying modal logic, possibility theory, and rough sets. Then different early attempts at bridging possibility theory and modal logic are reviewed.

2.1 Possibility theory, rough sets and modal logics: a square of opposition viewpoint

Recent studies [19] have pointed out that many artificial intelligence knowledge representation settings are sharing the same structures of opposition that extend or generalize the traditional square of opposition which dates back to Aristotle, and whose logical interest has been rediscovered more than one decade ago [4]. The traditional square involves four logically related statements exhibiting universal or existential quantifications: a statement **A** of the form "every x is p" is negated by the statement **O** "some x is not p", while a statement like **E** "no x is p" is clearly in even stronger opposition to the first statement (**A**). These three statements, together with the negation of the last one, namely **I** "some x is p", give birth to the Aristotelian square of opposition in terms of quantifiers **A** : $\forall x \ p(x)$, **E** : $\forall x \ \neg p(x)$, **I** : $\exists x \ p(x)$, **O** : $\exists x \ \neg p(x)$. This square, pictured in Fig. 1.1, is usually

denoted by the letters **A**, **I** (affirmative half) and **E**, **O** (negative half). The names of the vertices come from a traditional Latin reading: **A**ff**I**rmo, n**E**g**O**).
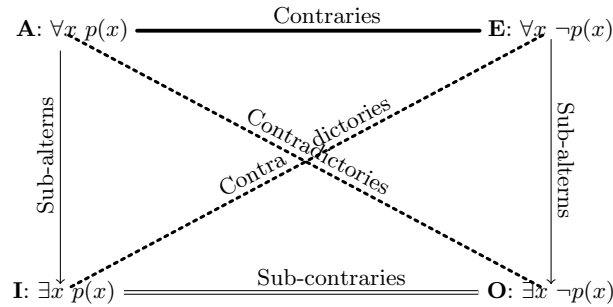

Fig. 1.1. Square of opposition

Note that we assume that some x do exist, thus avoiding existential import problems in Fig. 1.1. The different edges and diagonals of the square exhibits simple logical relations: i) **A** and **O**, as well as **E** and **I** are contraries; ii) **A** entails **I**, and **E** entails **0**; iii) **A** and **E** cannot be true together, while iv) **I** and **O** cannot be false together.

Another well-known instance of this square is in terms of the *necessary* ($\square$) and *possible* ($\diamond$) modalities, with the following reading **A** : $\square p$, **E** : $\square\neg p$, **I** : $\diamond p$, **O** : $\diamond\neg p$, where $\diamond p =_{def} \neg\square\neg p$ (with $p \neq \perp, \top$). Then the entailment from **A** to **I** is nothing but the axiom (D) in modal logic, namely $\square p \to \diamond p$. This reading has an easy counterpart in terms of binary-valued possibility theory replacing $\square p$ by $N([p])$ and $\diamond p$ by $\Pi([p])$ where $[p]$ is the set of models of proposition p [17]. This framework can be extended to graded possibility theory using a graded extension of the square of opposition [8].

A relation-based reading of the square of opposition has been proposed in [7,8]. Let us now consider a binary relation R on a Cartesian product $X \times Y$ (one may have $Y = X$). We assume $R \neq \emptyset$. Let xR denote the set $\{y \in Y \mid (x,y) \in R\}$. We write xRy when $(x,y) \in R$ holds, and $\neg(xRy)$ when $(x,y) \notin R$. Moreover, we assume that $\forall x,\ xR \neq \emptyset$, which means that the relation R is *serial*, namely $\forall x, \exists y$ such that xRy. We further assume that the complementary relation $\overline{R}$ ($x\overline{R}y$ iff $\neg(xRy)$), and its transpose are also serial, i.e. $\forall x,\ xR \neq Y$ and $\forall y,\ Ry \neq X$. These conditions enforce a non trivial relation between X and Y. In the following, set complementations are denoted by means of overbars.

Let S be a subset of Y. We assume $S \neq \emptyset$ and $S \neq Y$. The relation R and the subset S, also considering its complement $\overline{S}$, give birth to the two following subsets of X, namely the (left) images of S and $\overline{S}$ by R

$$R(S) = \{x \in X \mid \exists s \in S, xRs\} = \{x \in X \mid S \cap xR \neq \emptyset\} = \bigcup_{s \in S} Rs \tag{1.1}$$

$$R(\overline{S}) = \{x \in X \mid \exists s \in \overline{S}, xRs\} = \bigcup_{s \in \overline{S}} Rs$$

and their complements

$$\overline{R(S)} = \{x \in X \mid \forall s \in S, \neg(xRs)\} = \overline{\bigcup_{s \in S} Rs} = \bigcap_{s \in S} \overline{Rs} = \bigcap_{s \in S} \overline{R}s$$

$$\overline{R(\overline{S})} = \{x \in X \mid \forall s \in \overline{S}, \neg(xRs)\} = \{x \in X \mid xR \subseteq S\} = \overline{\bigcup_{s \in \overline{S}} Rs} = \bigcap_{s \in \overline{S}} \overline{R}s \tag{1.2}$$

The four subsets thus defined can be nicely organized into a square of opposition, see Fig. 1.2. Indeed, it can be checked that the set counterparts of the logical relations existing between the logical statements of the traditional square of opposition still hold here. Namely,

- $\overline{R(\overline{S})}$ and $R(\overline{S})$ are complements of each other, as are $\overline{R(S)}$ and $R(S)$;
  they correspond to the diagonals of the square;
- $\overline{R(\overline{S})} \subseteq R(S)$, and $\overline{R(S)} \subseteq R(\overline{S})$,
  thanks to condition $\forall x,\ xR \neq \emptyset$. These inclusions are represented by vertical arrows in Fig. 1.2;
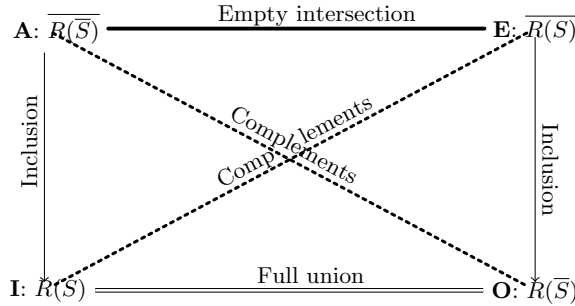
Fig. 1.2. Square of opposition induced by a relation R and a subset S

- $\overline{R(\overline{S})} \cap \overline{R(S)} = \emptyset$;
  this empty intersection corresponds to the thick line in Fig. 1.2,
  and one may have $\overline{R(\overline{S})} \cup \overline{R(S)} \neq Y$;
- $R(S) \cup R(\overline{S}) = X$;
  this full union corresponds to the double thin line in Fig. 1.2,
  and one may have $R(S) \cap R(\overline{S}) \neq \emptyset$.

Conditions (c)-(d) hold also thanks to the X-normalization of R.

Note that this fits with a modal logic reading of this square where R is viewed as an accessibility relation defined on $X \times X$, and S as the set of models of a proposition p. Indeed, $\Box p$ (resp. $\Diamond p$) is true in world x means that p is true at *every* (resp. at *some*) possible world accessible from x; this corresponds to $\overline{R(\overline{S})}$ (resp. $R(S)$) which is the set of worlds where $\Box p$ (resp. $\Diamond p$) is true. Moreover, the entailment from **A** to **I** is the axiom (D) of modal logic which is known to require serial accessibility relations [6].

Note that the relation R is serial if and only if $\overline{R(\overline{S})} \subseteq R(S)$. An interesting instantiation is in terms of rough sets [7], where in the classical case R is an equivalence relation. Then given the above definitions, we recognize that

- $R(S)$ is the upper approximation of S wrt the relation R;
- $\overline{R(\overline{S})}$ is the lower approximation of S wrt the relation R;
- $\overline{R(S)}$ is the exterior region of S;
- $R(\overline{S})$ is the complement of the lower approximation of S.

At this point one may observe that these relationships hold as well for fuzzy rough sets [13], if we replace the approximation operators by fuzzy ones – consider fuzzy box and diamond operators on crisp or fuzzy sets, also studied by Helmut Thiele [51]. A study of fuzzy rough sets in relation to the square of opposition appears in Ciucci et al. [9].

2.2 Early attempts at bridging possibility theory and modal logics

The first attempt at bridging possibility theory with modal logic can be found in a paper co-authored by L. Fariñas [27]. This paper establishes a formal parallel between rough sets and twofold fuzzy sets [12], namely a pair of fuzzy sets of elements that respectively certainly and possibly belong to an ill-known set. Then, taking advantage of the existence of the modal logic DAL for rough sets [26] and of a modal logic view of incomplete information databases [41], the paper discusses some possible options for a modal logic agreeing with possibility theory and with the issue of dealing with incomplete information rather than indiscernibility as in the case of rough sets.

A couple of years later, the idea of building a modal logic from a graded accessibility relation between different incomplete states of knowledge was investigated in detail in the case of binary-valued possibility theory and suggested for the graded case [22]. Then a state of knowledge s_2 is accessible from a state s_1 if and only if the information in state s_1 is consistent with the information in state s_2, but more incomplete (which was formalized as a set inclusion in the binary-valued case). In the general case, the inclusion becomes a matter of degree and the accessibility relation becomes graded. But the underlying axiom system remained an open issue.

Another attempt at the semantical level at bridging uncertainty theories with modal logic can be found in [46,48,47]; it includes the cases of possibility theory [31] and Shafer theory of evidence [30].

In the case of possibility theory, the authors use an accessibility relation assumed to be transitive and complete (connected), which corresponds to modal system `S43`. Necessity and possibility are built as ratios of the number of worlds in which the corresponding propositions are true.

3 From MEL to GPL

This section completes the background by providing a brief introduction to the meta-epistemic logic MEL, and to MEL$^+$ and then to generalized possibilistic logic GPL.

3.1 MEL and MEL$^+$, two simple epistemic logics

The usual truth values *true* (1) and *false* (0) assigned to propositions are of ontological nature (which means that they are part of the definition of what we call *proposition*), whereas assigning to a proposition a value whose meaning is expressed by the word *unknown* sounds like having an epistemic nature: it reveals a knowledge state according to which the truth value of a proposition (in the usual Boolean sense) in a given situation is out of reach (for instance one cannot compute it, either by lack of computing power, or due to a sheer lack of information). It corresponds to an epistemic state for an agent that can neither assert the truth of a Boolean proposition nor its falsity.

Admitting that the concept of "unknown" refers to a knowledge state rather than to an ontic truth value, we may keep the logic Boolean and add to its syntax the capability of stating that we ignore the truth value (1 or 0) of propositions. The natural framework to syntactically encode statements about knowledge states of classical propositional logic (CPL) statements is modal logic, and in particular, the logic `KD`. Nevertheless, if one only wants to reason about e.g. the beliefs of another agent, a very limited fragment of this language is needed. The logic MEL [1,3] was defined for that purpose.

Let us consider $\mathcal{L}$ to be a standard propositional language built up from a finite set of propositional variables $\mathcal{V} = \{p_1, \ldots, p_k\}$ along with the Boolean connectives of conjunction and negation $\neg$. As usual, a disjunction $\varphi \vee \psi$ stands for $\neg(\neg\varphi \wedge \neg\psi)$ and an implication $\varphi \to \psi$ stands for $\neg\varphi \vee \psi$. Further we use $\top$ to denote $\varphi \vee \neg\varphi$, and $\bot$ to denote $\neg\top$. Let us consider another propositional language $\mathcal{L}_\square$ whose set of propositional variables is of the form $\mathcal{V}_\square = \{\square\varphi \mid \varphi \in \mathcal{L}\}$ to which the classical connectives can be applied. It is endowed with a modality operator expressing certainty, that encapsulates formulas in $\mathcal{L}$. In other words $\mathcal{L}_\square = \{\square\alpha : \alpha \in \mathcal{L}\} \mid \neg\Phi \mid \Phi \wedge \Psi$.

MEL is a propositional logic on the language $\mathcal{L}_\square$ and with the following semantics. Let Ω be the set of classical interpretations for the propositional language $\mathcal{L}$, i.e. Ω consists of the set of mappings $w : \mathcal{L} \to \{0, 1\}$ conforming to the rules of classical propositional logic. For a propositional formula $\varphi \in \mathcal{L}$ we will denote by $Mod(\varphi)$ the set of $w \in \Omega$ such that $w(\varphi) = 1$. Models (or interpretations) for MEL correspond to epistemic states, which are simply subsets $\emptyset \neq E \subseteq \Omega$. The truth-evaluation rules of formulas of $\mathcal{L}_\square$ in a given epistemic model E are defined as follows:

- $E \models \square\varphi$ if $E \subseteq Mod(\varphi)$
- $E \models \neg\Phi$ if $E \not\models \Phi$
- $E \models \Phi \wedge \Psi$ if $E \models \Phi$ and $E \models \Psi$

Note that contrary to what is usual in modal logic, modal formulas are not evaluated on particular interpretations of the langage $\mathcal{L}$ because modal formulas in MEL do not refer to the actual world.

The notion of logical consequence is defined as usual $\Gamma \models \Phi$ if, for every epistemic model E, $E \models \Phi$ whenever $E \models \Psi$ for all $\Psi \in \Gamma$.

MEL can be axiomatized in a rather simple way (see [3]). The following are a possible set of axioms for MEL in the language of $\mathcal{L}_\square$:

(CPL) Axioms of CPL for $\mathcal{L}_\square$-formulas
  (K) $\square(\varphi \to \psi) \to (\square\varphi \to \square\psi)$
  (D) $\square\varphi \to \Diamond\varphi$
 (Nec) $\square\varphi$, for each $\varphi \in \mathcal{L}$ that is a CPL tautology, i.e. if $Mod(\varphi) = \Omega$.
The only inference rule is modus ponens. The corresponding notion of proof, denoted by $\vdash$, is defined as usual from the above set of axioms and modus ponens.

This set of axioms provides a sound and complete axiomatization of MEL, that is, it holds that, for any set of MEL formulas $\Gamma \cup \{\varphi\}$, $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$. This is not surprizing: MEL is just a standard propositional logic with additional axioms, whose propositional variables are the formulas of another propositional logic, and whose interpretations are subsets of interpretations of the latter.

MEL has been extended in [2] to allow dealing with not only subjective formulas that express an agent's beliefs, but also objective formulas (i.e. non-modal formulas) that express propositions that hold true in the actual world (whatever it might be). The extended language will be denoted by $\mathcal{L}_\square^+$, and it thus contains both propositional and modal formulas. It exactly corresponds to the non-nested fragment of the language of usual modal logic.

More precisely, the language $\mathcal{L}_\square^+$ of MEL$^+$ extends $\mathcal{L}_\square$ and is defined by the following formation rules:

- If $\varphi \in \mathcal{L}$ then $\varphi, \square\varphi \in \mathcal{L}_\square^+$
- If $\Phi, \Psi \in \mathcal{L}_\square^+$ then $\neg\Phi, \Phi \wedge \Psi \in \mathcal{L}_\square^+$

$\Diamond\varphi$ is defined as an abbreviation of $\neg\square\neg\varphi$. Note that $\mathcal{L} \subseteq \mathcal{L}_\square^+$ and that in $\mathcal{L}_\square^+$ there are no formulas with nested modalities.

Semantics for MEL$^+$ are given now by "pointed" MEL epistemic models, i.e. by structures (w, E), where $w \in \Omega$ and $\emptyset \neq E \subseteq \Omega$. The truth-evaluation rules of formulas of $\mathcal{L}_\square^+$ in a given structure (w, E) are defined as follows:

- $(w, E) \models \varphi$ if $w \in Mod(\varphi)$, in case $\varphi \in \mathcal{L}$
- $(w, E) \models \square\varphi$ if $E \subseteq Mod(\varphi)$
- usual rules for $\neg$ and $\wedge$

Logical consequence, as usual: $\Gamma \models \Phi$ if, for every structure (w, E), $(w, E) \models \Phi$ whenever $(w, E) \models \Psi$ for all $\Psi \in \Gamma$. The following are the axioms for MEL$^+$ in the language of $\mathcal{L}_\square^+$:

(CPL) Axioms of propositional logic
   (K) $\square(\varphi \to \psi) \to (\square\varphi \to \square\psi)$
   (D) $\square\varphi \to \Diamond\varphi$
(Nec) $\square\varphi$, for each $\varphi \in \mathcal{L}$ that is a CPL tautology, i.e. if $Mod(\varphi) = \Omega$.
   The only inference rule is *modus ponens*.[4]

It can be proven that the above axiomatization of MEL$^+$ is sound and complete with respect to the intended semantics, as defined above. Moreover, as it could be expected, if we call MEL^{++} the extension of MEL$^+$ with the axiom:

   (T) $\square\varphi \to \varphi$

then it can be shown that MEL^{++} is complete with respect to the class of *reflexive* pointed epistemic models (w, E), i.e. where $w \in E$.

Actually, MEL, MEL$^+$ and MEL^{++} capture different *non-nested* fragments of the normal modal logics of belief KD, KD4, KD45 and S5 (see e.g. [6] for details). In [2] the following relationships are shown:

- Let φ a formula from $\mathcal{L}_\square$. Then MEL $\vdash \varphi$ iff $L \vdash \varphi$,
  for $L \in \{$KD,KD4,KD45,S5$\}$.
- Let φ a formula from $\mathcal{L}_\square^+$. Then MEL$^+ \vdash \varphi$ iff $L \vdash \varphi$,
  for $L \in \{$KD,KD4,KD45$\}$.
- Let φ a formula from $\mathcal{L}_\square^+$. Then, MEL$^{++} \vdash \varphi$ iff S5 $\vdash \varphi$.

Moreover, by recalling the well-known result that any formula of KD45 and S5 is logically equivalent to another formula without nested modalities, the following stronger relationships hold:

- For any arbitrary modal formula φ, there is a formula $\varphi' \in \mathcal{L}_\square^+$ such that KD45 $\vdash \varphi$ iff MEL$^+ \vdash \varphi'$.
- For any arbitrary modal formula φ, there is a formula $\varphi' \in \mathcal{L}_\square^+$ such that S5 $\vdash \varphi$ iff MEL$^{++} \vdash \varphi'$.

3.2 About generalized possibilistic logic

A natural generalization of MEL is to extend epistemic states $E \subseteq \Omega$ to rankings of possible worlds in terms of plausibility. This can be done by means of a mapping $\pi : \Omega \to U$ that assigns to each possible world w a value $\pi(w)$ from a totally ordered uncertainty scale $\langle U, \leq, 0, 1 \rangle$ (which we will assume furthermore to be such that $\{0, 1\} \subseteq U \subseteq [0, 1]$ and closed by $n(x) = 1 - x$), with the following conventions:

- $\pi(w) = 1$ if w is fully plausible

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[4] An equivalent presentation could be to replace (Nec) by the usual Necessitation rule in modal logics, but restricted to tautologies of propositional logic: if $\varphi \in \mathcal{L}$ is a theorem, derive $\square\varphi$.

- $\pi(w) = 0$ if w is rejected as a possible world
- $\pi(w) \leq \pi(w')$ if w' is at least as plausible as w.

Such a mapping is called possibility distribution. A possibility distribution $\pi : \Omega \to U$ induces a pair of dual possibility and necessity measures on propositions, defined respectively as:

$$\Pi(\varphi) := \sup\{\pi(w) \mid w \in \Omega, w(\varphi) = 1\}$$
$$N(\varphi) := \inf\{1 - \pi(w) \mid w \in \Omega, w(\varphi) = 0\} .$$

They are dual in the sense that $\Pi(\varphi) = 1 - N(\neg\varphi)$ for every proposition φ.

Actually, *possibilistic logic* (see e.g. [11,15,18]), nowadays a well-known uncertainty logic, was initially devised to reason with graded beliefs on classical propositions by means of necessity and possiblity measures. For instance, the necessity fragment of possibilistic logic deals with weighted formulas (φ, r), where φ is a classical proposition and $r \in U$ is a weight, interpreted as a lower bound for the necessity degree of φ. It has a very simple axiomatization:

(CPL) $(\varphi, 1)$, for φ being a tautology of CPL
(GMP) from (φ, r) and $(\varphi \to \psi, s)$ derive $(\psi, \min(r, s))$
(Nes) from (φ, r) derive (ψ, s), if $s \leq r$

A graded extension of MEL capturing possibilistic logic has been proposed under the name *Generalized Possibilistic Logic*, GPL for short, in [20]. To deal with graded possibility and necessity they fix a *finite* scale of uncertainty values $\Lambda = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$ and for each value $a \in \Lambda \setminus \{0\}$ introduce a pair of modal operators $\square_a$ and $\diamondsuit_a$. In this case models (epistemic states) are possibility distributions $\pi : \Omega \to \Lambda$ on the set Ω of classical interpretations for the language L_1 with values in Λ, and the evaluation of the modal formulas is as follows:

$$\pi \models \square_a \varphi \qquad \text{if} \qquad N_\pi(\varphi) = \min\{1 - \pi(w) \mid w(\varphi) = 0\} \geq a.$$

The dual possibility operators are defined as $\diamondsuit_a \varphi := \neg\square_{s(1-a)}\neg\varphi$, where the superscript $s(a)$ refers to the successor of a in Λ. The semantics of $\diamondsuit_a \varphi$ is the natural one, i.e. $\pi \models \diamondsuit_a \varphi$ whenever the possibility degree of φ induced by π, $\Pi(\varphi) = \max\{\pi(w) \mid w(\varphi) = 1\}$, is at least a. A complete axiomatization of GPL is given in [20], an equivalent and shorter axiomatization is given by the following additional set of axioms and rules to those of CPL[21]:

(K) $\square_a(\varphi \to \psi) \to (\square_a\varphi \to \square_a\psi)$
(D) $\diamondsuit_1\top$
(Nes) $\square_{a_1}\varphi \to \square_{a_2}\varphi$, if $a_1 \geq a_2$
(Nec) $\square_1\varphi$, for each $\varphi \in \mathcal{L}$ that is a CPL tautology.

4 GPL$^+$: extending generalized possibilistic logic with objective formulas

Let again $\Lambda = \{0, \frac{1}{k}, \frac{2}{k}, \ldots 1\}$ where $k \in \mathbb{N} \setminus \{0\}$ be the finite uncertainty scale we will assume. Moreover we let $\Lambda^+ = \Lambda \setminus \{0\}$, and if $a \in \Lambda^+$, we denote by $p(a)$ the value in the scale that preceeds a.

In this section we extend the language of generalized possibilistic logic (GPL) to allow dealing with not only subjective formulas that express an agent's beliefs, but also objective formulas (i.e. non-modal formulas) that express propositions that hold true in the actual world (whatever it might be). The extended language will be denoted by $\mathcal{L}_\square^{k+}$, and it thus contains both propositional and modal formulas. It exactly corresponds to the non-nested fragment of the language of usual modal logic.

More precisely, the language $\mathcal{L}_\square^{k+}$ of GPL$^+$ extends the one of GPL, $\mathcal{L}_\square^+$, and is defined by the following formation rules:

- If $\varphi \in \mathcal{L}$ and $a \in \Lambda^+$ then $\varphi, \square_a\varphi \in \mathcal{L}_\square^{k+}$
- If $\Phi, \Psi \in \mathcal{L}_\square^{k+}$ then $\neg\Phi, \Phi \wedge \Psi \in \mathcal{L}_\square^{k+}$

$\diamondsuit_b\varphi$ is defined as an abbreviation of $\neg\square_a\neg\varphi$, with $b = 1 - p(a)$. Note that $\mathcal{L} \subseteq \mathcal{L}_\square^+$ and that in $\mathcal{L}_\square^+$ there are no formulas with nested modalities.

Semantics for GPL$^+$ are given now by "pointed" possibilistic models, i.e. by structures (w, π), where $w \in \Omega$ and $\pi : \Omega \to \Lambda$ such that there is at least one $w \in \Omega$ with $\pi(w) = 1$. For each proposition $\varphi \in \mathcal{L}$, let $N_\pi(\varphi) = \inf_{w \notin Mod(\varphi)} \pi(w)$. The truth-evaluation rules of formulas of $\mathcal{L}_\square^{k+}$ in a given structure (w, π) is defined as follows:

- $(w, \pi) \models \varphi$ if $w(\varphi) = 1$, in case $\varphi \in \mathcal{L}$

- $(w, \pi) \models \Box_a \varphi$ if $N_\pi(\varphi) \geq a$
- usual rules for $\neg$ and $\wedge$

If we let $\pi_a = \{w \in \Omega \mid \pi(w) \geq a\}$, note that $(w, \pi) \models \Box_a \varphi$ whenever $\pi_{1-p(a)} \subseteq Mod(\varphi)$. Therefore, it becomes clear that each $\Box_a$ operator is a MEL^+ modality.

The corresponding logical consequence is defined as usual: $\Gamma \models \Phi$ if, for every structure (w, π), $(w, \pi) \models \Phi$ whenever $(w, \pi) \models \Psi$ for all $\Psi \in \Gamma$.

The following are the axioms for GPL^+ in the language of $\mathcal{L}_\Box^{k+}$:

(CPL) Axioms of propositional logic

(K$_a$) $\Box_a(\varphi \to \psi) \to (\Box_a \varphi \to \Box_a \psi)$, for every $a \in \Lambda^+$

(D$_a$) $\Box_a \varphi \to \Diamond_1 \varphi$, for every $a \in \Lambda^+$

(Nes) $\Box_a \varphi \to \Box_b \varphi$, where $b \leq a$

(Nec) $\Box_1 \varphi$, for each $\varphi \in \mathcal{L}$ that is a CPL tautology

The only inference rule is *modus ponens*. We will write $\Gamma \vdash \Phi$ to denote that φ can be derived from a set of formulas Γ using the above axioms and modus ponens. Also, in what follows, we will denote by $\vdash_{CPL}$ the notion of proof of classical propositional language on the language $\mathcal{L}_\Box^{k+}$ taking all $\Box$-formulas as new propositional variables.

To prove completeness, we first recall the following useful lemma that allows to express deductions in GPL^+ as deductions in CPL.

Lemma 1. *Let $\Gamma \cup \{\Phi\}$ be a set of $\mathcal{L}_\Box^{k+}$-formulas. Then it holds that $\Gamma \vdash \Phi$ iff*
$\Gamma \cup \{\Box_1 \varphi \mid \vdash_{CPL} \varphi\} \cup \{instances\ of\ axioms\ (K_a),\ (D_a),\ (Nes)\ and\ (Nec)\} \vdash_{CPL} \Phi$.

Theorem 1 (Completeness). *For any set of $\mathcal{L}_\Box^{k+}$-formulas $\Gamma \cup \{\Phi\}$, it holds that $\Gamma \vdash \Phi$ iff $\Gamma \models \Phi$.*

Proof. From left to right is easy, as usual. For the converse direction, assume $\Gamma \nvdash \Phi$. By the preceding lemma and the completeness of PL, there exists a propositional evaluation v on the whole language $\mathcal{L}_\Box^{k+}$ (taking $\Box$-formulas as genuine propositional variables) such that $v(\Psi) = 1$ for all $\Psi \in \Gamma \cup \{\Box_1 \varphi \mid \vdash_{PL} \varphi\} \cup \{instances\ of\ axioms\ (K),\ (D)\ and\ (Nes)\}$ but $v(\Phi) = 0$. We have to build a structure (w, π) that it is a model of Γ but not of Φ. So, we take (w, π) as follows:

- w is defined as the restriction of v to $\mathcal{L}$, i.e. $w(\varphi) = v(\varphi)$ for all $\varphi \in \mathcal{L}$.
- For each $a \in \Lambda^+$, let us first define $E_{1-p(a)} = \bigcap \{Mod(\varphi) \mid v(\Box_a \varphi) = 1\}$. Then define $\pi : \Omega \to \Lambda$ as follows: $\pi(w) = \max\{a \in \Lambda^+ \mid w \in E_a\}$, where we adopt the usual convention of taking $\max \emptyset = 0$. In other words, we define π in such a way that each a-cut π_a coincides with E_a.

Note that, since by axioms (D) and (Nec) we have $v(\Diamond_1 \top) = 1$, $E_1 \neq \emptyset$. Then the last step is to show that, for every $\Psi \in \mathcal{L}_\Box^{k+}$, $v(\Psi) = 1$ iff $(w, \pi) \models \Psi$.

We prove this by induction. The case Ψ being a non-modal formula from $\mathcal{L}$ is clear, since in that case $w(\Psi) = v(\Psi)$. The interesting case is when $\Psi = \Box_a \psi$. Then we have:

(i) If $v(\Box_a \psi) = 1$ then, by definition of $E_{1-p(a)}$, $E_{1-p(a)} \subseteq Mod(\psi)$, and hence $(w, \pi) \models \Box_a \psi$.

(ii) Conversely, if $E_{1-p(a)} \subseteq Mod(\psi)$, then there must exist γ such that $v(\Box_a \gamma) = 1$ and $Mod(\gamma) \subseteq Mod(\psi)$. Hence this means that $\gamma \to \psi$ is a PL theorem, and hence we have first, by the necessitation axiom, that $v(\Box_a(\gamma \to \psi)) = 1$, and thus $v(\Box_a \gamma) \leq v(\Box_a \psi)$ holds as well by axiom (K), and therefore $v(\Box_a \psi) = 1$ holds as well.

As a consequence, we have that $(w, \pi) \models \Psi$ for all $\Psi \in \Gamma$ but $(w, \pi) \not\models \Phi$.

Similar to the non graded case of MEL^+, we may consider an S5-like extension of GPL^+, capturing the pointed possibilistic epistemic models (w, π), where the 'actual world' w is one of the non-discarded possible worlds by π. In this case, the higher $\pi(w)$ is, the more the actual world w belongs to the set of plausible worlds, and hence we can speak of a notion of graded reflexive pointed possibilistic epistemic models (w, π).

Definition 1. *Let (w, π) be a pointed possibilistic structure and let $a \in \Lambda^+$. We call (w, π) to be a-reflexive when $\pi(w) \geq a$.*

Let us define GPL_a^{++} to be the axiomatic extension of GPL^+ with the following generalized (T) axiom:

(T$_a$) $\Box_a \varphi \to \varphi$

One can check that (T$_a$) is valid in all b-reflexive pointed possibilistic structures, with $b = 1 - p(a)$. Indeed, if $(w, \pi) \models \Box_a \varphi$ then $N_\pi(\varphi) \geq a$, and thus $\pi_{1-p(a)} \subseteq Mod(\varphi)$. But if (w, π) is b-reflexive, we have $\pi(w) \geq 1 - p(a)$, and hence $w \in \pi_{1-p(a)} \subseteq Mod(\varphi)$. Therefore $(w, \pi) \models \varphi$ as well.

Theorem 2. *GPL_a^{++} is complete with respect to the class of $(1-p(a))$-reflexive pointed possibilistic structures.*

Proof. The proof is analogous to that of Theorem 1.

It is interesting to point out that Liau and Lin [36,37] propose a language similar to GPL^+, albeit using $[0,1]$ as a possibility scale (which forces them to introduce additional multimodal formulas to deal with strict inequalities) and graded accessibility relations. Their tableau-based proof methods could be of interest to develop inference techiques for GPL.

5 Concluding remarks

In this paper, following the fact that the fragment MEL^+ (resp. MEL^{++}) of the KD45 (resp. S5) logic, the richest of doxastic (resp. epistemic) logics, involving modal formulas of depth 0 or 1 can have simplified semantics, we show that this state of facts extends to graded modalities with the extensions GPL^+ and GPL^{++} of the generalized possibilistic logic GPL.

Besides, it has been recently shown that the graded notion of guaranteed possibility can be expressed in GPL enabling us to express "all I know" statements [21] (see also [3] for the crisp case). This result calls for for a deeper comparison with the modal logic presented in [10] that involves the classical modalities of the possible and the necessary together with the nonstandard modalities that are the guaranteed possibility and its dual, having also in mind that these four modalities and their negations makes a cube of opposition [8] that generalizes the square of opposition.

Dedication

This article is particularly dedicated to Luis Fariñas del Cerro. It perfectly illustrates one of the topics at the junction of our respective subjects of interest, namely modal logic and possibility theory. Discussions along 35 years of friendship have repeatedly triggered two of the authors to dig more and more about the relations between these two knowledge representation frameworks, thanks also to the help of the two other authors of this note. Interestingly enough, while gaining mutual understanding of our respective reference theories, each of us has remained a supporter of one's own theory. Let us hope that in the long range, the now obvious bridge between the two formalisms will become routine knowledge so that both can be used appropriately by the same people according to the particulars of the applications at hand.

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