ON PRODUCT FUZZY LOGIC WITH TRUTH CONSTANTS *

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ABSTRACT: In this paper we investigate expansions of Product logic by adding into the language a countable set of truth constants and by adding the corresponding book-keeping axioms for the truth constants. In particular we consider expansions with sets of truth constants defined by the natural and rational powers of an arbitrary real $a \in [0, 1]$, for which we prove standard completeness. Finite strong completeness results for these logics are studied when we restrict ourselves to formulas of the kind $\overline{r} \rightarrow \varphi$, where $\overline{r}$ is a truth constant denoting the truth degree r and φ is a formula without truth constants.

Keywords: Product logic, truth constants, standard completeness, fuzzy logic.

1 INTRODUCTION

In the context of fuzzy logical systems, introducing truth constants in the language is an elegant means to be able to explicitely reasoning with partial degrees of truth. This goes back to Pavelka [13] who built a propositional many-valued logical system over Lukasiewicz logic by adding into the language a truth constant $\overline{r}$ for each real $r \in [0, 1]$, together with a number of additional axioms. Although the resulting logic is not strong complete (like Lukasiewicz logic), Pavelka proved that his logic, call it PL, is complete in a weaker sense. Namely, by defining the truth degree of a formula φ in a theory T as

$$\| \varphi \|_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

and the degree of provability of φ in T as

$$| \varphi |_T = \sup\{r \mid T \vdash_{PL} \overline{r} \rightarrow \varphi\},$$

Pavelka proved that these degrees coincide. This kind of completeness, is usually known as Pavelka-style completeness, and strongly relies in the continuity of Lukasiewicz truth functions. Novák extended Pavelka approach to Lukasiewicz first order logic.

Later, Hájek [10] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results, indeed it is enough to extend the language only by a countable number of truth constants, one per each *rational* in $[0, 1]$, and by two additional axiom schemata, called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \rightarrow \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ are Lukasiewicz t-norm and its residuum respectively. He denoted this new system Rational Pavelka Logic, RPL for short. Moreover he proved that RPL is strong complete for finite theories.

Similar *rational* extensions for other popular fuzzy logics can be obviously defined, but remark that Pavelka-style completeness cannot be obtained since Lukasiewicz is the only fuzzy logic with continuous truth functions in the real unit interval [0, 1]. Among different works in this direction we may cite [10] where an extension of G_Δ (the extension of Gödel logic with Baaz's Delta operator) with a finite number of rational truth constants, and [7] where the authors define logical systems obtained by adding (rational) truth constants to $G_\sim$ (Gödel logic with an involutive negation) and to Π (Product logic) and $\Pi_\sim$ (Product logic with an involutive negation). More recently, in [8] the authors consider the extension of Gödel and Weak Nilpotent Minimum logics (and some of its extensions) with rational truth constants. Standard (weak) completeness is shown for those logics as well finite strong completeness when restricted to formulas of the kind $\overline{r} \rightarrow \varphi$, where $\overline{r}$ is a truth constant denoting the truth degree r and φ is a formula without truth constants. Actually, this kind of formulas have been extensively considered in other frameworks for reasoning with partial degrees of truth. Indeed, these formulas correspond to a particular class of Novák's *evaluated* formulas in the setting of graded formal logical systems [12]. Evaluated formulas are expressions a/A where a is a truth value (from a given algebra) and A is a formula of a language built using truth constants. Our formula $\overline{r} \rightarrow \varphi$ would be expressed as r/φ in Novák's syntax. They also appear in the framework of abstract fuzzy logics developed by Gerla [9] based on the notion of fuzzy consequence or deduction operators over fuzzy sets of formulas, where the membership degree of formulas are interpreted as lower bounds on their truth degrees.

In this paper we present preliminar results on expansions à la Pavelka of another popular fuzzy logic, the Product fuzzy logic Π [11, 10]. In particular we consider expansions with sets of truth constants defined by the natural and rational powers of an arbitrary real $a \in [0, 1]$, for which we prove standard completeness. After some preliminaries in Section 2, this is done in Section 3. In Section 4, strong completeness of these logics are studied when we restrict ourselves to formulas of the kind $\overline{r} \rightarrow \varphi$. We conclude with some final remarks on open problems and future research. Notice that, in contrast to the above other logics, to prove standard completeness of

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the expansion of Π with a truth constant for each rational in $[0, 1]$ remains as an open problem.

2 PRELIMINARIES

Our general logical framework is the Product fuzzy logic Π defined in [11] and further studied in e.g. [10, 1, 3, 5] as a propositional logic in the language $\mathcal{L} = \{\&, \rightarrow, \overline{0}\}$. We will denote by $Fm_{\mathcal{L}}$ the set of well-formed formulas built over the language $\mathcal{L}$ and a countable set of propositional variables. Other connectives are defined as follows: $\overline{1}$ is $\varphi \rightarrow \varphi$, $\neg\varphi$ is $\varphi \rightarrow \overline{0}$, $\varphi \wedge \psi$ is $(\varphi\&(\varphi \rightarrow \psi))$, $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, and $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Axioms of Π are:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi\&\psi) \rightarrow \varphi$
(A3) $(\varphi\&\psi) \rightarrow (\psi\&\varphi)$
(A4) $(\varphi\&(\varphi \rightarrow \psi)) \rightarrow (\psi\&(\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi\&\psi) \rightarrow \chi)$
(A5b) $((\varphi\&\psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $\overline{0} \rightarrow \varphi$
(A8) $\varphi \wedge \neg\varphi \rightarrow \overline{0}$
(A9) $\neg\neg\varphi \rightarrow [((\varphi\&\psi) \rightarrow (\varphi\&\chi)) \rightarrow (\psi \rightarrow \chi)]$

The rule of inference of Π is *modus ponens.*

The notion of (finitary) proof is as usual from the above axioms and inference rule. If T is an arbitray theory we shall write $T \vdash_\Pi \varphi$ to denote that there exists a proof of φ from T.

The algebraic counterpart of Π logic are the so-called *Product algebras.* A product algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \Rightarrow, 0, 1)$ is a BL-algebra satisfying two further conditions:

- $x \wedge \neg x = 0$
- if $x \neq 0$ then $x * y = x * z$ implies $y = z$

Product algebras (or Π-algebras) form a variety, which is a subvariety of BL-algebras. The so-called *standard Π-algebra* is the product algebra on the unit real interval defined by the algebraic product and its residuum, $[0, 1]_\Pi = ([0, 1], \min, \max, \cdot, \Rightarrow_\Pi, 0, 1)$, where $\cdot$ denotes the algebraic product and $x \Rightarrow_\Pi y = 1$ if $x \leq y$ and $x \Rightarrow_\Pi y = y/x$ otherwise.

Given a product algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \Rightarrow, 0, 1)$ an $\mathbf{A}$-evaluation e is a mapping $e : Var \rightarrow A$ which extends to arbitrary formulas by means of the algebra operations:

$$e(\overline{0}) = 0$$

$$e(\varphi\&\psi) = e(\varphi) \odot e(\psi)$$

$$e(\varphi \rightarrow \psi) = e(\varphi) \Rightarrow e(\psi)$$

An $\mathbf{A}$-evaluation e is an $\mathbf{A}$-model of a formula φ if $e(\varphi) = 1$. φ is an $\mathbf{A}$-tautology if $e(\varphi) = 1$ for all $\mathbf{A}$-evaluation e. If T is a theory, we write $T \models_{\mathbf{A}} \varphi$ when $e(\varphi) = 1$ for all $\mathbf{A}$-evaluation e which is model of all formulas in T.

Completeness results for Π logic [11, 10] read as follows. For any finite theory T and formula φ the following conditions are equivalent:

(i) $T \vdash_\Pi \varphi$

(ii) $T \models_{\mathbf{A}} \varphi$ for all product algebra $\mathbf{A}$

(iii) $T \models_{\mathbf{A}} \varphi$ for all linearly ordered product algebra $\mathbf{A}$

(iv) $T \models_{[0,1]_\Pi} \varphi$

3 EXPANDING PRODUCT LOGIC WITH TRUTH CONSTANTS

Let C be a countable subset of [0, 1] such that $\mathbf{C} = (C, \min, \max, \cdot, \Rightarrow_\Pi, 0, 1)$ is a (product) subalgebra of $[0, 1]_\Pi$. Relevant examples of sets C are:

(i) the set $\mathbb{Q} \cap [0, 1]$ of rational numbers in [0, 1],

(ii) the sets $NP[a_1, \dots, a_m] = \{a_1^{n_1} \cdot \ldots \cdot a_m^{n_m} \mid n_1, \dots, n_m \in \mathbb{N}\} \cup \{0, 1\}$ for any reals $0 < a_1 < \ldots < a_m < 1$,

(iii) the sets $RP[a_1, \dots, a_m] = \{a_1^{r_1} \cdot \ldots \cdot a_m^{r_m} \mid r_1, \dots, r_m \in \mathbb{Q}^+\} \cup \{0, 1\}$ for any reals $0 < a_1 < \ldots < a_m < 1$,

To simplify notation, we will simply write N_a for $NP[a]$ and R_a for $RP[a]$.

Given such a C, one can define the logic $\Pi(C)$ as the expansion of Product logic Π with the countable set $\overline{C} = \{\overline{c} \mid c \in C\}$ of truth constants and by adding the corresponding book-keeping axioms, i.e., for all $r, s \in C$, the axioms

$$\overline{r}\&\overline{s} \equiv \overline{r \cdot s}$$
$$\overline{r} \rightarrow \overline{s} \equiv \overline{r \Rightarrow_\Pi s}$$

The only inference rule is always modus ponens. The notion of proof is as in Π logic. We will use the notation $\vdash_{\Pi(C)}$ to refer to proofs in $\Pi(C)$.

A $\Pi(C)$-algebra is a structure $\mathcal{A} = (A, \wedge, \vee, \odot, \Rightarrow, \{\overline{r}^A\}_{r\in C})$, where $\mathbf{A} = (A, \wedge, \vee, \odot, \Rightarrow, \overline{0}^A, \overline{1}^A)$ is a product algebra, satisfying the following book-keeping equations:

$$\overline{r}^A \odot \overline{s}^A = \overline{r \cdot s}^A$$
$$\overline{r}^A \rightarrow \overline{s}^A = \overline{r \Rightarrow_\Pi s}^A$$

for any $r, s \in C$.

Given a $\Pi(C)$-algebra $\mathcal{A}$, an $\mathcal{A}$-evaluation e is just an $\mathbf{A}$-evaluation which further satisfies $e(\overline{r}) = \overline{r}^A$ for all $r \in C$. The notions of $\mathcal{A}$-model, $\mathcal{A}$-tautology and logical consequence $\models_{\mathcal{A}}$ are then as in the case of Π logic.

One can check that the logic $\Pi(C)$ is algebraizable in the sense of [2] (cf. [8]) and its equivalent algebraic semantics is given by the variety of $\Pi(C)$-algebras. Moreover, as in the case of Π-algebras, $\Pi(C)$-algebras also decompose as subdirect products of linearly ordered ones. As a consequence we have the following general completeness results.

Theorem 1 (General Completeness). *Let φ be a formula of $\Pi(C)$. Then the following conditions are equivalent:*

- $\vdash_{\Pi(C)} \varphi$
- $\models_{\mathcal{A}} \varphi$ *for all $\Pi(C)$-algebra $\mathcal{A}$*

- $\models_{\mathcal{A}} \varphi$ *for all linearly ordered* $\Pi(C)$*-algebra* $\mathcal{A}$

The *standard* $\Pi(C)$*-algebra* is the algebra $[0,1]_{\Pi(C)}$ over the unit real interval [0, 1] where the truth constants are interpreted by their own values, i.e.

$$[0,1]_{\Pi(C)} = ([0,1], \min, \max, \cdot, \Rightarrow_\Pi, \{r\}_{r\in C})$$

The following are some general results about the structure of linearly ordered $\Pi(C)$-algebras.

Lemma 1. *Let* $\mathcal{A} = (A, \wedge, \vee, \odot, \Rightarrow, \{\overline{r}^A\}_{r\in C})$ *be a* $\Pi(C)$*-chain. Then:*

- *Either* $\overline{r}^A < \overline{s}^A$ *for any* $r, s \in C$ *such that* $r < s$*, or* $\overline{r}^A = \overline{1}^A$ *for all* $r > 0$*. Algebras of the first kind will be called* type I *algebras, while algebras of the second kind will be called of* type II.
- A *is finite iff* $A = \{\overline{0}^A, \overline{1}^A\}$ *and* $\overline{r}^A = \overline{1}^A$ *for all* $r > 0$.

The standard $\Pi(C)$-algebra $[0,1]_{\Pi(C)}$ is of type I. The $\Pi(C)$-chain of type II over the unit interval [0, 1] will be denoted as $[0,1]^*_{\Pi(C)}$.

Lemma 2. *Let* $\mathcal{A}$ *be a* $\Pi(C)$*-algebra over* $[0,1]$ *of type I, where* $C = N_a$ *or* $C = R_a$*. Then* $\mathcal{A}$ *is isomorphic to the standard algebra* $[0,1]_{\Pi(C)}$.

Proof. Let $\alpha = \overline{a}^A$. There exists a real β such that $\alpha^\beta = a$. Then the mapping $f : [0,1] \to [0,1]$ defined by $f(x) = x^\beta$ is a $\Pi(C)$-product algebra isomorphism such that $f(\overline{r}^A) = r$ for all $r \in C$. □

Let us consider an interesting linearly ordered $\Pi(C)$-algebra, which is not Archimedean, that we will use later. Let $\mathbf{R}^+ \times_{\mathbf{lex}} \mathbf{R}^+ = (\mathbb{R}^+ \times \mathbb{R}^+, (1,1), \cdot, \leq_{lex})$ denote the l.o. abelian group defined by the lexicographic product of two copies of the multiplicative group of positive reals. Then, let $[0,1]^{lex}_\Pi$ be the product algebra defined as the negative cone of $\mathbf{R}^+ \times_{\mathbf{lex}} \mathbf{R}^+$ adding a bottom element $(0,1)$. Namely,

$$[0,1]^{lex}_\Pi = (Cone^-(\mathbf{R}^+ \times_{\mathbf{lex}} \mathbf{R}^+) \cup \{(0,1)\}, \star, \Rightarrow_\star, (0,1), (1,1))$$

where $Cone^-(\mathbf{R}^+ \times_{\mathbf{lex}} \mathbf{R}^+) = \{(1,y) \mid y \in (0,1]\} \cup \{(x,y) \mid x \in (0,1), y \in \mathbb{R}^+\}$, and the operations are defined for all $(x,y),(x',y') \in Cone^-(\mathbf{R}^+ \times_{\mathbf{lex}} \mathbf{R}^+)$ by:

$$(x,y) \star (x',y') = (x \cdot x', y \cdot y')$$

$$(x,y) \Rightarrow_\star (x',y') = \begin{cases} (1,1), & \text{if } (x,y) \leq_{lex} (x',y') \\ (x \Rightarrow_\Pi x', y'/y), & \text{otherwise} \end{cases}$$

together with $(0,1) \star (x,y) = (x,y) \star (0,1) = (0,1)$, and $(x,y) \Rightarrow_\star (0,1) = (0,1)$, $(0,1) \Rightarrow_\star (x,y) = (1,1)$ for all $(x,y) \in Cone^-(\mathbf{R}^+ \times_{\mathbf{lex}} \mathbf{R}^+)$ and $(0,1) \Rightarrow_\star (0,1) = (1,1)$. Clearly, $[0,1]^{lex}_\Pi$ is a non-Archimedean product algebra, indeed, it has two Archimedean components, $F = \{(1,y) \mid y \in (0,1]\}$ and the rest $Cone^-(\mathbf{R}^+ \times_{\mathbf{lex}} \mathbf{R}^+) \setminus F$. Moreover F is in fact a filter.

It is clear that the mapping $f : x \mapsto (x,1)$ embedds the standard product algebra $[0,1]_\Pi$ into $[0,1]^{lex}_\Pi$.

For a given C, we can define several $\Pi(C)$-algebras over the product algebra $[0,1]^{lex}_\Pi$, depending on how truth constants are interpreted. Let us consider these ones (we omit superscripts in the constants for a ligther notation):

(i) The algebra $[0,1]^{lex}_{C,1}$ where $\overline{c} = (1,c)$, for all $c \in C$, $c \neq 0$, and $\overline{0} = (0,1)$.

(ii) The algebra $[0,1]^{lex}_{C,2}$ where $\overline{1} = (1,1)$ and $\overline{c} = (c,1)$ for all $c \in C$, $c \neq 1$.

(iii) The algebra $[0,1]^{lex}_{C,*}$where $\overline{c} = (1,1)$ for all $c \in C$, $c \neq 0$, and $\overline{0} = (0,1)$.

Lemma 3. *The following statements hold:*

(i) The standard algebra $[0,1]_{\Pi(C)}$ *belongs to both to* $\mathbb{V}([0,1]^{lex}_{C,1})$ *and to* $\mathbb{V}([0,1]^{lex}_{C,2})$*, the varieties generated by* $[0,1]^{lex}_{C,1}$ *and by* $[0,1]^{lex}_{C,2}$ *respectively.*

(ii) The algebra $[0,1]^*_{\Pi(C)}$ *belongs to* $\mathbb{V}([0,1]^{lex}_{C,1})$*, the variety generated by* $[0,1]^{lex}_{C,1}$

Proof. (i) On the one hand, $[0,1]_{\Pi(C)}$ is isomorphic to the subalgebra of $[0,1]^{lex}_{C,1}$ consisting of $F \cup \{(0,1)\}$. On the other hand, $[0,1]_{\Pi(C)}$ is also an homomorphic image of $[0,1]^{lex}_{C,2}$, namely it is isomorphic to the quotient algebra $[0,1]^{lex}_{C,2}/F$.

(ii) In a similar way, $[0,1]^*_{\Pi(C)}$ is isomorphic to the quotient algebra $[0,1]^{lex}_{C,1}/F$. □

In the sequel we investigate the issue of standard completeness of the logics $\Pi(C)$ for different sets of truth constants C.

Theorem 2 (Standard Completeness of $\Pi(R_a)$). *For any* $0 < a < 1$*, the logic* $\Pi(R_a)$ *is weak standard complete, that is, for any formula* φ, $\vdash_{\Pi(R_a)} \varphi$ *if and only if* $\models_{[0,1]_{\Pi(R_a)}} \varphi$.

Proof. Soundness is obvious. Suppose $\not\vdash_{\Pi(R_a)} \varphi$. Then, by general completeness, there is a linearly ordered $\Pi(R_a)$-algebra $\mathcal{A}$ and an $\mathcal{A}$-interpretation e such that $e_A(\varphi) < 1$. We have to prove that there exists a $[0,1]_{\Pi(R_a)}$-evaluation e' such that $e'(\varphi) < 1$ as well. We will distinguish two cases:

(1) Suppose $\mathcal{A}$ is of type I, that is, the interpretation of the truth constant $\overline{a}$ in $\mathcal{A}$, $\overline{a}^A$, is different from $\overline{1}^A$.

Take $X = \{e(\psi) \mid \psi \text{ subformula of } \varphi\} \cup \{\overline{0}^A, \overline{1}^A, \overline{a}^A\}$. As the set X is finite, let $k = l.c.m.\{q \in \mathbb{N} \mid$ for all irreducible fraction $r = p/q$ such that $\overline{a^r}^A \in X\}$, and let $k' \in \mathbb{N}$ the minimum number such that $k'/k \geq r$ for all r such that $\overline{a^r}^A \in X$. Take now $Y = X \cup \{\overline{a^{n/k}}^A \mid n \leq k'\}$.

Considering $\mathbf{A}$ as the Π-algebra reduct of $\mathcal{A}$, by Gurevich-Kokorin's theorem, there is a partial embedding h from Y into the standard Π-algebra $[0,1]_\Pi$. Note that, if $h(\overline{a}^A) = c$, then $h(\overline{a^{p/k}}^A) = c^{p/k}$. In fact, letting $h(\overline{a^{1/k}}^A) = d$, we have $h(\overline{a^{p/k}}^A) = h(\overline{a^{1/k}}^A \odot \overset{p}{\ldots} \odot \overline{a^{1/k}}^A) = d \cdot \overset{p}{\ldots} \cdot d = d^p$. In particular, $h(\overline{a}^A) = d^k = c$, hence $d = c^{1/k}$ and thus $h(\overline{a^{p/k}}^A) = c^{p/k}$. Moreover, $h(e(\varphi)) < 1$.

Finally we can consider an isomorphism of Π-algebras $f : [0,1] \to [0,1]$ defined by $f(x) = x^{\beta}$, where β is such that $c^{\beta} = a$. Note that $f \circ h$ is a partial embedding from $Y \subset \mathcal{A}$ to $[0,1]_{\Pi}$ such that $(f \circ h)(\overline{r}^{A}) = r$ for all $\overline{r}^{A} \in X$ and $(f \circ h)(e(\varphi)) < 1$. Hence $f \circ h \circ e$ can be easily extended to a full $[0,1]_{\Pi(R_a)}$-evaluation e' and still $e'(\varphi) < 1$.

(2) Suppose $\mathcal{A}$ is a $\Pi(R_a)$-algebra of type II, i.e. $\overline{a}^{A} = \overline{1}^{A}$. Take again $X = \{e_A(\psi) \mid \psi \text{ is a subformula of } \varphi\} \cup \{\overline{0}, \overline{1}\}$. Then there is a partial isomorphism (as Product algebras) h of X into the standard Π-algebra $[0,1]_{\Pi}$, and since $h(\overline{1}^{A}) = 1$, $h \circ e$ can be easily extended to a full $[0,1]^{*}_{\Pi(R_a)}$-evaluation e' such that $e'(\varphi) < 1$. This means that φ is not valid in $[0,1]^{*}_{\Pi(R_a)}$. Since, by (ii) of Lemma 3, $[0,1]^{*}_{\Pi(R_a)} \in \mathbb{V}([0,1]^{lex}_{R_a,1})$, φ is not valid in $[0,1]^{lex}_{R_a,1}$ either.

Now we reason as follows. Notice that in the previous part (1) we have actually proved that any equation which is not valid in a $\Pi(R_a)$-algebra of type I, hence in particular in the algebra $[0,1]^{lex}_{R_a,1}$, it is also not valid in the standard algebra $[0,1]_{\Pi(R_a)}$. In other words $[0,1]^{lex}_{R_a,1} \in \mathbb{V}([0,1]_{\Pi(R_a)})$, which together with (i) of Lemma 3, it amounts to establish that $\mathbb{V}([0,1]_{\Pi(R_a)}) = \mathbb{V}([0,1]^{lex}_{R_a,1})$. Therefore, φ is not valid either in the standard algebra $[0,1]_{\Pi(R_a)}$. This ends the proof. $\square$

Corollary 4. $\mathbb{V}([0,1]_{\Pi(R_a)}) = \mathbb{V}([0,1]^{lex}_{R_a,1}) = \mathbb{V}([0,1]^{lex}_{R_a,2})$.

Theorem 3 (Standard Completeness of $\Pi(N_a)$). *For any $0 < a < 1$, the logic $\Pi(N_a)$ is weak standard complete, that is, for any formula φ, $\vdash_{\Pi(R_a)} \varphi$ if and only if $\models_{[0,1]_{\Pi(N_a)}} \varphi$.*

Proof. The proof is analogous (in fact simpler) to the previous case, hence it is ommitted. $\square$

The question whether $\Pi(\mathbb{Q})$ is standard complete remains open.

4 FINITE STRONG COMPLETENESS RESULTS

Similarly to the case of Lukasiewicz, Gödel or Nilpotent Minimum logics, the expansion of Product logic with truth constants is not strong complete, the same counter-examples apply ($p \vee \overline{r} \models_{[0,1]_{\Pi(C)}} p$ while $p \vee \overline{r} \not\vdash_{\Pi(C)} p$, for any propositional variable p).

However, we want to show that, analogously again to the above mentioned cases, $\Pi(C)$ is strongly standard complete if we restrict ourselves to formulas of the type $\overline{r} \to \varphi$, where φ is a formula without rational truth constants (a formula of Π), and expressing that φ is true at least to the degree r. This type of formulas, that we will call *graded* formulas, are also commonly expressed in different fuzzy logic settings as pairs (φ, r). Making use of this notation, the finite strong completeness for $\Pi(C)$ we want to show reads as follows:

$$\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \vdash_{\Pi(C)} (\varphi, s)$$
$$\text{if and only if}$$
$$\{(\psi_i, r_i) \mid i = 1, 2, .., n\} \models_{[0,1]_{\Pi(C)}} (\varphi, s)$$

where $\psi_1, \dots, \psi_n, \varphi$ are formulas without truth constants and $\Pi(C)$ is any expansion of Π which is (weak) standard complete, for instance for $C = R_a$ and $C = N_a$.

Actually, as always, one direction is easy:

Lemma 5. *If $\{(\varphi_i, r_i) \mid i = 1, 2, .., n\} \vdash_{\Pi(C)} (\psi, s)$ then $\{(\varphi_i, r_i) \mid i = 1, 2, .., n\} \models_{[0,1]_{\Pi(C)}} (\psi, s)$.*

The rest of the section is devoted to prove the converse direction:

$$\text{If } \{(\varphi_i, r_i) \mid i = 1, 2, .., n\} \models_{[0,1]_{\Pi(C)}} (\psi, s) \text{ then } \{(\varphi_i, r_i) \mid i = 1, 2, .., n\} \vdash_{\Pi(C)} (\psi, s) \qquad \text{(FSC)}$$

Lemma 6. *Let $\alpha \in \mathbb{R}^{+}$ and define a mapping $f_w : [0,1] \to [0,1]$ as follows:*

$$f_{\alpha}(x) = x^{\alpha}$$

Then f_{α} is a morphism with respect to the standard Product truth functions. Therefore, if e is a Π-evaluation of formulas, then $e_{\alpha} = f_{\alpha} \circ e$ is another Π-evaluation.

Lemma 7. *If $\{(\varphi_1, r_1), \dots, (\varphi_n, r_n)\} \models_{[0,1]_{\Pi(C)}} (\psi, s)$ with $s > 0$ then $\{\varphi_1, \dots, \varphi_n\} \models_{[0,1]_{\Pi}} \psi$, and hence $\{\varphi_1, \dots, \varphi_n\} \vdash_{\Pi} \psi$ as well.*

Proof. Asumme $\{\varphi_1, \dots, \varphi_n\} \not\models_{[0,1]_{\Pi}} \psi$. Then there exists an evaluation e such that $e(\varphi_1) = \ldots = e(\varphi_n) = 1$ and $e(\psi) < 1$. If $e(\psi) = 0$ the result is obvious. Assume $e(\psi) > 0$. Let $\alpha \in \mathbb{R}^{+}$ such that $(e(\psi))^{\alpha} < s$. Then $e' = f_{\alpha} \circ e$ is such that $e'((\varphi_i, r_i)) = 1$ for all i but $e'((\psi, s)) < 1$. $\square$

Lemma 8. *If $\{\varphi_1, \dots, \varphi_n\} \vdash_{\Pi} \psi$ iff there exists k such that $\{p_1 \to \varphi_1, \dots, p_n \to \varphi_n\} \vdash_{\Pi} (p_1 \& \ldots \& p_n)^k \to \psi$.*

Proof. The following are theorems of Π logic:
- $((p_1 \to \varphi_1) \& \ldots \& (p_n \to \varphi_n)) \to ((p_1 \& \ldots \& p_n) \to (\varphi_1 \& \ldots \& \varphi_n))$
- $((p_1 \to \varphi_1) \& \ldots \& (p_n \to \varphi_n))^k \to ((p_1 \& \ldots \& p_n) \to (\varphi_1 \& \ldots \& \varphi_n))^k$
- $((p_1 \& \ldots \& p_n) \to (\varphi_1 \& \ldots \& \varphi_n))^k \to ((p_1 \& \ldots \& p_n)^k \to (\varphi_1 \& \ldots \& \varphi_n)^k)$

Now, if $\{\varphi_1, \dots, \varphi_n\} \vdash_{\Pi} \psi$ then, by the deduction theorem, there exists k such that $\vdash_{\Pi} (\varphi_1 \& \ldots \& \varphi_n)^k \to \psi$. Combining this with the above theorems, we have that

$$\vdash_{\Pi} ((p_1 \to \varphi_1) \& \ldots \& (p_n \to \varphi_n))^k \& (p_1 \& \ldots \& p_n)^k \to \psi$$

from where it easily follows that

$$(p_1 \to \varphi_1), \dots, (p_n \to \varphi_n) \vdash_{\Pi} (p_1 \& \ldots \& p_n)^k \to \psi$$

Conversely, taking $p_i = \varphi_i$, it follows that $\vdash_{\Pi} (\varphi_1 \& \ldots \& \varphi_n)^k \to \psi$, and hence $\{\varphi_1, \dots, \varphi_n\} \vdash_{\Pi} \psi$. $\square$

Lemma 9. *If $\{\varphi_1, \dots, \varphi_n\} \vdash_{\Pi} \psi$ then $\{(\varphi_1, r_1), \dots, (\varphi_n, r_n)\} \vdash_{R\Pi} (\psi, (r_1 \cdot \ldots \cdot r_n)^{k_0})$, where $k_0 = \min\{k \in \mathbb{N} \mid \vdash_{\Pi} (\varphi_1 \& \ldots \& \varphi_n)^k \to \psi\}$.*

Proof: By Lemma 8, it is easy to see that taking $p_i = \overline{r_i}$ we can still prove over Π that

$$\{\overline{r_1} \to \varphi_1, \dots, \overline{r_n} \to \varphi_n\} \vdash_{\Pi} (\overline{r_1} \& \ldots \& \overline{r_n})^{k_0} \to \psi,$$

and over RΠ (hence using the book-keeping axioms) we have

$$\{\overline{r_1} \to \varphi_1, \ldots, \overline{r_n} \to \varphi_n\} \vdash_\Pi \overline{(r_1 \cdot \ldots \cdot r_n)^{k_0}} \to \psi.$$

□

Now, using Lemma 7 and reformulating the above two lemmas by using the following version of the deduction theorem

$$\{\varphi_1, \ldots, \varphi_n\} \vdash_\Pi \psi \text{ iff there exist } k_1, \ldots, k_n \text{ such that } \vdash_\Pi \varphi^{k_1} \& \ldots \& \varphi^{k_n} \to \psi$$

we can formulate the following result.

Theorem 10. *Assume* $\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\} \models_{[0,1]_{R\Pi}} (\psi, s)$ *with* $s > 0$*, and hence* $\{\varphi_1, \ldots, \varphi_n\} \vdash_\Pi \psi$*. Let* $I = \{(i_1, \ldots, i_n) \mid \vdash_\Pi \varphi^{i_1} \& \ldots \& \varphi^{i_n} \to \psi\}$*. Then* $\{(\varphi_1, \alpha_1), \ldots, (\varphi_n, \alpha_n)\} \vdash_{R\Pi} (\psi, \beta)$*, where* $\beta = \max\{\alpha_1^{i_1} \cdot \ldots \cdot \alpha_n^{i_n} \mid (i_1, \ldots, i_n) \in I\}$*.*

5 EXPANDING Π_Δ WITH TRUTH CONSTANTS

A natural extension of the considered logical framework is to introduce the well-known Baaz's Δ connective into the logic. In such a case, instead of the Product logic Π, we take now as starting point the logic Π_Δ, the extension of Π with the Δ connective as done in [10].

As before, given a countable subset C of [0, 1] such that $\mathbf{C} = (C, \min, \max, \cdot, \Rightarrow_\Pi, 0, 1)$ is a (product) subalgebra of $[0,1]_\Pi$, then we define a $\Pi_\Delta(C)$-algebra as a structure $\mathcal{A} = (A, \wedge, \vee, \odot, \Rightarrow, \Delta, \{\overline{r}^A\}_{r \in C})$, where $(A, \wedge, \vee, \odot, \Rightarrow, \{\overline{r}^A\}_{r \in C})$ is a $\Pi(C)$-algebra and $(A, \wedge, \vee, \odot, \Rightarrow, \Delta, \overline{0}^A, \overline{1}^A)$ is a Π_Δ algebra, satisfying further these additional book-keeping axioms:

$$\Delta\overline{r} = \overline{\Delta r}$$

where the second Δ is meant as the standard truth function in [0, 1], i.e. $\Delta x = 1$ if $x = 1$ and $\Delta x = 0$ otherwise.

It is clear then that if $\mathcal{A}$ is a linearly ordered $\Pi_\Delta(C)$-algebra, $\Delta\overline{r} = \overline{0}$ for all $1 > r \in C$.

One can check again that the logic $\Pi_\Delta(C)$ is also algebraizable with equivalent algebraic semantics given by the variety of $\Pi_\Delta(C)$-algebras and that $\Pi_\Delta(C)$-algebras still decompose as subdirect product of linearly ordered ones.

Theorem 4 (General Completeness). *Let* φ *be a formula of* $\Pi(C)$*. Then the following conditions are equivalent:*

- $\vdash_{\Pi_\Delta(C)} \varphi$
- $\models_\mathcal{A} \varphi$ *for all* $\Pi_\Delta(C)$*-algebra* $\mathcal{A}$
- $\models_\mathcal{A} \varphi$ *for all linearly ordered* $\Pi_\Delta(C)$*-algebra* $\mathcal{A}$

The *standard* $\Pi_\Delta(C)$*-algebra* is the algebra $[0,1]_{\Pi_\Delta(C)}$ over the unit real interval [0, 1] where the truth constants are interpreted by their own values, i.e.

$$[0,1]_{\Pi(C)} = ([0,1], \min, \max, \cdot, \Rightarrow_\Pi, \Delta, \{r\}_{r \in C})$$

Contrary to $\Pi(C)$-chains, truth constants cannot collapse.

Lemma 11. *Let* $\mathcal{A} = (A, \wedge, \vee, \odot, \Rightarrow, \Delta, \{\overline{r}^A\}_{r \in C\}}$ *be a* $\Pi_\Delta(C)$*-chain. Then* $\overline{r}^A < \overline{s}^A$ *for any* $r, s \in C$ *such that* $r < s$*. Hence,* A *cannot be finite.*

Proof. Let $r < s$. If $\overline{r}^A = \overline{s}^A$, then $\overline{s}^A \Rightarrow \overline{r}^A = \overline{1}^A$, hence $\overline{t}^A = \overline{1}^A$, where $t = r/s$, hence $\Delta(\overline{t}^A) = \overline{1}^A$. But, since $t < 1$, this is in contradiction with the fact that $\Delta(\overline{t}^A) = \overline{0}^A$ for all $t < 1$. □

Lemma 12. *Let* $\mathcal{A}$ *be a* $\Pi_\Delta(C)$*-algebra over* $[0,1]$*, where* $C = N_a$ *or* $C = R_a$*. Then* $\mathcal{A}$ *is isomorphic to the standard algebra* $[0,1]_{\Pi_\Delta(C)}$*.*

Proof. Let $\alpha = \overline{a}^A$. There exists a real β such that $\alpha^\beta = a$. Then the mapping $f : [0,1] \to [0,1]$ defined by $f(x) = x^\beta$ is a $\Pi(C)$-algebra isomorphism such that $f(\overline{r}^A) = r$ for all $r \in C$. □

Theorem 5 (Standard Completeness). *For* $C = N_a$ *and* $C = R_a$ *(and for any* $0 < a < 1$*), the logic* $\Pi_\Delta(C)$ *is weak standard complete, that is, for any formula* φ*,* $\vdash_{\Pi_\Delta(C)} \varphi$ *if and only if* $\models_{[0,1]_{\Pi_\Delta(C)}} \varphi$*.*

Proof. Analogous to the proof of T heorem 2, taking into account the simplification due to the fact that there are no $\Pi_\Delta(C)$-algebra of Type II. □

The issue of (finite) strong completeness now is easier. Indeed, recalling the form of local deduction theorem for Π_Δ, $\psi \vdash_{\Pi_\Delta} \varphi$ iff $\vdash_{\Pi_\Delta} \Delta\psi \to \varphi$, one can easily prove the following.

Corollary 13 (Finite strong standard Completeness). *For any formulas* φ *and* ψ *of* $\Pi_\Delta(C)$*, it holds that* $\psi \models_{[0,1]_{\Pi_\Delta(C)}} \varphi$ *iff* $\psi \vdash_{\Pi_\Delta(C)} \varphi$*.*

6 FINAL REMARKS

After the study of Lukasiewicz logic and Gödel logic with truth constants in [13, 10] and in [8], in this paper we have started the study of the expansion of Product logic, the other main fuzzy logic based on a continuous t-norm, with truth constants. We have done some first steps for the case where the set of truth constants correspond to powers (either natural or rational) of a given real value in $[0,1]$. Nevertheless, it is a matter of current research the case where truth constants correspond to the set of all rationals in $[0,1]$, which was the case studied for the expansions of Lukasiewicz and Gödel logics. In addition, we have considered the expansions of those product logics with Baaz's Δ connective, obtaining finite strong standard completeness. In fact, this result could be proved in an analogous way for the corresponding expansions of Lukasiewicz and Gödel logics with Δ. Finally, let us remark that a recent result by Cintula [4, Lemma 3.4.4] implies that if a logic L (without addtional truth constants) is standard complete, then its expansion with some suitable set of truth constants C is complete with respect to the class of $L(C)$-algebras whose reduct is a standard L-algebra. This may shed new light on getting standard completeness for $L(C)$ logics, in particular, it seems interesting to be investigated for the case of $L = \Pi_\Delta$ and $C = \mathbb{Q} \cap [0,1]$.

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