

The Impact of Heterogeneity and Geometry on the Proof Complexity of Random Satisfiability

Thomas Bläsius¹, Tobias Friedrich¹, Andreas Göbel¹,
Jordi Levy², and Ralf Rothenberger¹

¹Hasso Plattner Institute, University of Potsdam, Potsdam, Germany
`firstname.lastname@hpi.de`

²IIIA, CSIC, Campus UAB, 08193 Bellaterra, Spain
`lastname@iia.csic.es`

Abstract

Satisfiability is considered the canonical NP-complete problem and is used as a starting point for hardness reductions in theory, while in practice heuristic SAT solving algorithms can solve large-scale industrial SAT instances very efficiently. This disparity between theory and practice is believed to be a result of inherent properties of industrial SAT instances that make them tractable. Two characteristic properties seem to be prevalent in the majority of real-world SAT instances, heterogeneous degree distribution and locality. To understand the impact of these two properties on SAT, we study the proof complexity of random k -SAT models that allow to control heterogeneity and locality. Our findings show that heterogeneity alone does not make SAT easy as heterogeneous random k -SAT instances have superpolynomial resolution size. This implies intractability of these instances for modern SAT-solvers. On the other hand, modeling locality with an underlying geometry leads to small unsatisfiable subformulas, which can be found within polynomial time.

A key ingredient for the result on geometric random k -SAT can be found in the complexity of higher-order Voronoi diagrams. As an additional technical contribution, we show a linear upper bound on the number of non-empty Voronoi regions, that holds for points with random positions in a very general setting. In particular, it covers arbitrary p -norms, higher dimensions, and weights affecting the area of influence of each point multiplicatively. This is in stark contrast to quadratic lower bounds for the worst case.

1 Introduction

Propositional satisfiability (SAT) is arguably among the most-studied problems for both theoretical and practical research. Nonetheless, the gap between theory and practice is huge. In theory, SAT is the prototypical hard problem and hardness of other problems is shown via reductions from SAT. Achieving even a running time of $O(2^{cn})$ for any $c < 1$ and n variables would be a major breakthrough and a somewhat surprising one at that. On the contrary, reductions to SAT are used to solve various problems appearing in practice, as state-of-the-art SAT solvers can easily handle industrial instances with millions of variables.

This theory-practice gap does not come from the lack of a sufficiently precise theoretical analysis of modern SAT solvers. They are actually provably slow on most instances, i.e., drawing an instance uniformly at random yields a hard instance with probability tending to 1 for $n \rightarrow \infty$, if the clause-variable ratio is not too low or way too high [7, 19]. Instead, the discrepancy comes from the fact that industrial instances have properties that make them easier than worst-case instances. In 2014, Vardi [52] wrote that “we have no understanding of why the specific sets of heuristics employed by modern SAT solvers are so effective in practice” and that we need this understanding to successfully advance SAT solving further.

In recent years, scientists have been studying properties of industrial SAT instances to gain this understanding. By modeling SAT instances as graphs, e.g., with edges indicating inclusion of variables in clauses, one can benefit from the extensive research conducted in the field of network science. Two properties commonly observed in real-world networks are heterogeneity and locality. *Heterogeneity* refers to the degree distribution, meaning that vertices have strongly varying degrees. In fact, one usually observes a heavy-tailed distribution with many vertices of low degree and few vertices of high degree. A common assumption is a power-law distribution [53], where the number of vertices of degree k is roughly proportional to $k^{-\beta}$. The constant β is called the *power-law exponent*. *Locality* refers to the fact that edges tend to connect vertices that are close in the sense that they remain well connected even when ignoring their direct connection. This can also be seen as having strong community structures, with high connectivity within communities and loose ties between communities.

With respect to these two properties, industrial SAT instances are similar to real-world networks. In many cases, the variable frequencies are heterogeneous [1] and there is a high level of locality [2]. The latter is often measured in terms of modularity. Inspired by network science, researchers have studied models that resemble industrial instances with respect to these properties. Particularly, Ansótegui et al. [3] introduced a power-law SAT model for heterogeneous instances, which has been theoretically studied in terms of satisfiability thresholds [28–30]. Moreover, Giráldez-Cru and Levy [35] introduced a model in which variable weights lead to heterogeneity while an underlying geometry facilitates locality. Comparing this to network models, the former model [3] is the SAT-variant of Chung-Lu graphs [17, 18]. The latter [35] is based on the popularity-similarity model [47], which is closely related to hyperbolic random graphs [39] and geometric inhomogeneous random graphs [16].

Besides serving as somewhat realistic benchmarks for SAT competitions [34], these SAT models can be used to study solver behavior depending on heterogeneity and locality. One can experimentally observe that a high level of heterogeneity improves the performance of SAT solvers that also perform well on industrial instances [3, 11]. Moreover, locality seems very beneficial as solvers appear to implicitly use the locality of a given instance [35]. This coincides with the findings of experiments on actual industrial instances that show that the locality (a.k.a. modularity) of an instance is a good predictor for solver performance [46, 54, 55].

Up to date, there are no theoretical results supporting these experimental observations. On the contrary, it has been shown that instances generated by the community attachment model [33], which enforces a community structure, are hard for modern SAT solvers [43].

With this paper, we provide a theoretical foundation that matches the observations in practice by studying the proof complexity of instances drawn from the power-law SAT model or from a very general model with underlying geometry. Our findings are that heterogeneous instances are hard asymptotically almost surely¹ in that modern SAT solvers require superpolynomial or even exponential running time to refute unsatisfiable instances. On the contrary, instances with a high level of locality facilitated by an underlying geometry are a. a. s. easy to solve. Besides these results on SAT, we provide insights on the complexity of weighted higher-order Voronoi diagrams in higher dimensions, which is of independent interest.

Outline. We state and discuss our main results and technical contributions in Section 2. Formal definitions are in Section 3. A short outline of our core arguments is in Section 4, followed by the formal proofs: lower bounds for the power-law model in Section 5, upper bounds on the complexity of Voronoi diagrams in Section 6, and upper bounds for the geometric SAT model in Section 7. To not distract from the core arguments, results we use that were either known before or are straight-forward to prove are outsourced to Appendix A.

2 Results, Technical Contribution, and Discussion

2.1 Power-Law SAT

The power-law SAT model has four parameters: the number of variables n , the number of clauses m , the number k of variables appearing in each clause, and a power-law exponent β . To draw a formula, power-law weights with exponent β are assigned to the variables and then each clause is generated independently by drawing k variables without repetition using probabilities proportional to the weights. Each literal is negated with probability $1/2$.

To discuss our first main contribution, let Φ be a formula drawn from the power-law model with density above the satisfiability threshold, i.e., Φ is a. a. s. unsatisfiable. We show that, although it is highly likely that Φ is unsatisfiable, it is highly unlikely that modern SAT solvers can figure that out in polynomial time. We prove this via a lower bound for the *resolution width*, which is the size of the largest clause appearing in a resolution proof. A lower bound w for the width implies lower bounds on the size of resolution proofs [7]: every resolution proof has size $\exp(\Omega(w^2/n))$ and every tree-like resolution proof has size $\exp(\Omega(w))$. Since resolution size and the running time of CDCL solvers are polynomially equivalent [5, 48], this also yields a lower bound for the running time of CDCL on power-law random k -SAT. For DPLL solvers, which use tree-like resolution, the bounds are even stronger.

Theorem 5.8. *Let Φ be a random power-law k -SAT formula with n variables, $m \in \Omega(n)$ clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-1}{k-1}$. Let $\Delta = m/n$ be large enough so that Φ is unsatisfiable at least with constant probability. Let $\varepsilon, \varepsilon_1, \dots, \varepsilon_3$ be constants with $\varepsilon > 0$, $\varepsilon_1 = \frac{k-\varepsilon}{2} - 1 > 0$, $\varepsilon_2 = (k-\varepsilon)\frac{\beta-2}{\beta-1} - 1 > 0$, and $0 < \varepsilon_3 < (\frac{k}{2} - 1)\frac{\beta-2}{\beta-1} - 1$. For the resolution width w of Φ , it holds a. a. s. that:*

(i) *If $\beta \in \left(\frac{2k-1}{k-1}, 3\right)$ and $\Delta \in o(n^{\varepsilon_2})$, then $w \in \Omega\left(n^{\varepsilon_2/\varepsilon_1} \Delta^{-1/\varepsilon_1}\right)$.*

(ii) *If $\beta = 3$ and $\Delta \in o\left(n^{\varepsilon_1}/\log^{1+\varepsilon_1} n\right)$, then $w \in \Omega\left(n \cdot \Delta^{-1/\varepsilon_1}/\log^{1+1/\varepsilon_1} n\right)$.*

(iii) *If $\beta > 3$ and $\Delta \in o(n^{\varepsilon_1})$, then $w \in \Omega\left(n \cdot \Delta^{-1/\varepsilon_1}\right)$.*

(iv) *If $\beta > \frac{2k-2}{k-2}$ and $\Delta \in o\left(n^{\varepsilon_3}/\log^{\varepsilon_3} n\right)$, then $w \in \Omega\left(n \cdot \Delta^{-1/\varepsilon_3}\right)$.*

¹*Asymptotically almost surely (a. a. s.)* refers to a probability that tends to 1 for $n \rightarrow \infty$. *With high probability (w. h. p.)* refers to the stronger requirement that the probability is in $1 - O(1/n)$.

The above lower bounds allow the density Δ to be super-constant (even polynomially), which is asymptotically above the satisfiability threshold. For the sake of simplicity, assume Δ to be constant in the following. Starting at the bottom (iii, iv), we get a linear bound for w if β is sufficiently large, i.e., greater than 3 or $(2k-2)/(k-2)$. For $\beta = 3$ (ii), the bound is still almost linear. Note that these results in particular imply exponential lower bounds on the resolution size and thus on the running time of CDCL and DPLL. For smaller β (i), we get a polynomial bound for the width with exponent $\varepsilon_2/\varepsilon_1$; see Figure 1 for a plot with ε close to 0.

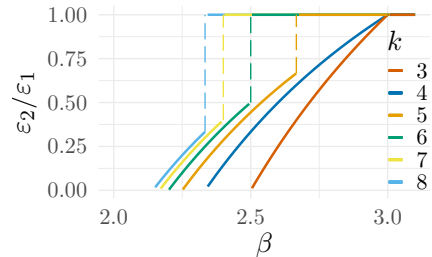


Figure 1: Exponent of the bound (i) in Theorem 5.8. Dashed vertical lines show where the bound (iv) takes over.

Interestingly enough, our bounds only hold for power law exponents $\beta > \frac{2k-1}{k-1}$. This is complemented by a previous result [29], which shows that asymptotically almost surely power-law random k -SAT instances with constant constraint densities are trivially unsatisfiable for power law exponents $\beta < \frac{2k-1}{k-1}$. Thus, their resolution width is constant.

Part iv of Theorem 5.8 is derived via lower bounds on the bipartite expansion of the clause-variable incidence graph of these instances. These results can be of independent interest for hypergraphs with edge size k and for random $(0, 1)$ -matrices. Additionally, these expansion properties yield lower bounds for the clause space complexity, which in turn gives lower bounds on the tree-like resolution of such formulas (Section 5.2). More precisely, this gives an exponential lower bound for the tree-like resolution for $\beta > \frac{2k-3}{k-2}$. This is an improvement of the bound obtained via the resolution width.

It is interesting to note that this result on the non-geometric model supports the claim that locality is a crucial factor for easy SAT instances. The lower bounds for the power-law model are solely based on the fact that every set of clauses covers a comparatively large set of variables. In other words, we only use that there are no clusters of clauses with similar variables, i.e., we explicitly use the lack of locality.

2.2 Geometric SAT

The geometric model has seven parameters: n , m , and k have the same meaning as for the power-law model. Moreover, w is a weight function assigning each variable v a weight w_v , d is the dimension of the underlying geometric space (we use the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$; see Section 3) equipped with a \mathfrak{p} -norm, and T is the so-called temperature that controls the strength of locality by varying the impact of the geometry. To draw a formula, the variables and clauses are assigned random positions in \mathbb{T}^d . Then, for each clause, k variables are drawn without repetition with probabilities depending on the variable weight and on the geometric distance between clause and variable. In the extreme case of $T = 0$, each clause deterministically includes the k closest variables (where closeness is a combination of geometric distance and weight), while increasing the temperature T increases the probability for the inclusion of more distant variables. For $T \rightarrow \infty$, the model converges to uniform random SAT. We have the following theorem, where W denotes the sum of all variable weights. The condition on the weights is in particular satisfied by power-law distributed weights.

Theorem 7.9. *Let Φ be a formula with n variables and $m \in \Theta(n)$ clauses drawn from the weighted geometric model with ground space \mathbb{T}^d equipped with a \mathfrak{p} -norm, temperature $T < 1$, $W \in O(n)$, and $w_v \in O(n^{1-\varepsilon})$ for every $v \in V$ and any constant $\varepsilon > 0$. Then, Φ contains a. a. s. an unsatisfiable subformula of constant size. It can be found in $O(n \log n)$ time.*

To briefly explain how we prove this, consider a simplified version where variables and clauses are points in the Euclidean plane and each clause contains the k variables geometrically

closest to it (temperature $T = 0$). Now consider the equivalence relation obtained by defining two points of the plane equivalent if and only if they have the same set of k closest variables. The equivalence classes of this relation are the regions of the order- k Voronoi diagram of the variable positions. With this connection, we can use upper bounds on the complexity of order- k Voronoi diagrams [41] to prove the existence of small and easy to find unsatisfiable subformulas. We note that this result is of asymptotic nature. In particular for small densities, the number of variables n has to be very large before the instances actually get as easy as stated in Theorem 7.9. Nevertheless, this results strongly suggests that an underlying geometry makes SAT instances more tractable.

To extend the above argument to the general statement in Theorem 7.9, we extend the complexity bounds for order- k Voronoi diagrams in various ways; see next section for more details. Moreover, for non-zero temperatures, clauses no longer include exactly the k closest variables but can, in principle, consist of any set of k variables. However, we can show that, with high probability, a linear fraction of clauses behaves as in the $T = 0$ case. We note that analyses of similar structures, such as hyperbolic random graphs, are often restricted to the simpler but less realistic $T = 0$ case, e.g., [9, 10, 12, 45]. We believe that our analysis provides insights on the non-zero temperature case that can be helpful for such related questions.

We note that our results seem to contradict the results of Mull et al. [43], stating that (i) a strong community structure is not sufficient to have tractable SAT instances and that (ii) the community attachment model [33], which enforces a community structure, generates hard instances. However, at a closer look, this is not a contradiction at all. Though measuring the community structure, e.g., via modularity, is a good indicator for locality, the concept of locality goes deeper. If the instance can be partitioned such that there are strong ties within each partition and loose ties between partitions, then the instance has a strong community structure. However, to have a high level of locality, this concept has to hierarchically repeat on different levels of magnitude, i.e., there needs to be community structure within each partition and between the partitions. To state this slightly differently, consider locality based on a notion of similarity between objects (here: variables or clauses). In this paper, we use distances between random points in a geometric space as a measure for similarity, which gives us a continuous range of more or less similar objects. In contrast to that, in the above mentioned papers focusing on a flat community structure [33, 43], similarity is a binary equivalence relation: two objects are either similar or they are not.

2.3 Voronoi Diagrams

Consider a finite set of points, called sites, in a geometric space. The most commonly studied type of Voronoi diagram assumes the 2-dimensional Euclidean plane as ground space and has one Voronoi region for each site, containing all points closer to this site than to any other site. We deviate from this default setting in four ways: (i) We allow an arbitrary constant dimension d , where the ground space is the torus or a hypercube in \mathbb{R}^d . (ii) We consider the order- k Voronoi diagram, which has for every subset A of sites with $|A| = k$ a (possibly empty) Voronoi region containing all points for which A are the k nearest sites. The number of non-empty order- k Voronoi regions is called the *complexity* of the diagram. (iii) The sites have multiplicative weights that scale the influence of the different sites. Without loss of generality, we assume the weights to be scaled such that the minimum is 1. (iv) We allow the \mathfrak{p} -norm for arbitrary $\mathfrak{p} \in \mathbb{N}^+ \cup \infty$.

Theorem 6.9. *Let S be a set of n sites with minimum weight 1, total weight W , and random positions on the d -dimensional torus equipped with a \mathfrak{p} -norm. For every fixed k , the expected number of regions of the weighted order- k Voronoi diagram of S is in $O(W)$. The same holds for random sites in a hypercube.*

To set this result into context, we briefly discuss previous work on the complexity of Voronoi diagrams in different settings. To this end, we use the following theorem that relates the complexity in terms of Voronoi regions (which is what we are concerned with in this paper) with the complexity in terms of vertices.²

Theorem 6.2. *Let S be a set of n weighted sites in general position in \mathbb{R}^d equipped with a \mathbf{p} -norm. If the order- k Voronoi diagram has ℓ vertices, then the order- $(k + d)$ Voronoi diagram has $\Omega(\ell)$ non-empty regions.*

We note that, using insights from previous work, this theorem is not hard to prove. One basically has to generalize the result by L e [40] bounding the number of d -spheres going through $d + 1$ points in d -dimensional space to weighted sites, and then observe how the Voronoi diagram changes in the construction by Lee [41] for $d = 2$, when going from order- k to order- $(k + 1)$. However, we are not aware of previous work stating this connection between vertices and non-empty regions in higher orders explicitly.

The four above-mentioned generalizations of the basic Voronoi diagram (higher dimension, higher order, multiplicative weights, and different \mathbf{p} -norms) have all been considered before. However, to the best of our knowledge, not all of them together.

Higher-order Voronoi diagrams have been introduced by Shamos and Hoey [51]. Lee [41] showed that the order- k Voronoi diagram in the plane (unweighted with Euclidean metric) has complexity $O(k(n - k))$ (in terms of number of regions), which is linear for constant k . For the 1- and ∞ -norm, Liu et al. [42] improved this bound to $O(\min\{k(n - k), (n - k)^2\})$. Closely related to the 1-norm, Gemsa et al. [32] showed similar complexity bounds for higher-order Voronoi diagrams on transportation networks of axis-parallel line segments. Bohler et al. [13] show an upper bound of $2k(n - k)$ for the much more general setting of abstract Voronoi diagrams. There, the metric is replaced by curves separating pairs of sites such that certain natural (but rather technical) conditions are satisfied. One obtains normal Voronoi diagrams when using perpendicular bisectors for these curves. This in particular shows that the $2k(n - k)$ bound on the number of regions in the order- k Voronoi diagram holds for arbitrary \mathbf{p} -norms in 2-dimensional space and for the hyperbolic plane. As the hyperbolic plane is closely related to 1-dimensional space with sites having multiplicative power-law weights [16], we suspect that the bound by Bohler et al. [13] also covers this case.

In general one can say that higher-order Voronoi diagrams of unweighted sites in 2-dimensional space are well-behaved in that they have linear complexity. This still holds true for arbitrary \mathbf{p} -norms. However, this picture changes for weighted sites or higher dimensions.

Voronoi diagrams with multiplicative weights were first considered by Boots [15]³ due to applications in economics. Beyond that, multiplicatively weighted Voronoi diagrams have applications in sensor networks [20], logistics [31] and the growth of crystals [21]. However, even in the most basic setting of 2-dimensional Euclidean space and order 1, weighted Voronoi diagrams can have quadratic complexity [4] (in terms of number of vertices). This comes from the fact that Voronoi cells are not necessarily connected; see Figure 2a for the construction of Aurenhammer and Edelsbrunner [4] that proves the lower bound. With Theorem 6.2, and as illustrated in Figure 2, this implies that even the order-3 Voronoi diagram of weighted sites in 2-dimensional Euclidean space has a quadratic number of non-empty regions. As a special case, Theorem 6.9 shows that this complexity is only linear in the unit square with randomly positioned sites. Moreover, this also implies that the number of vertices of the corresponding

²Although the Voronoi regions are not necessarily polytopes in the weighted setting, we adopt the notion for polytopes and call the corners of Voronoi regions *vertices*. I.e., vertices are the 0-dimensional elements (a.k.a. points) of the boundary, where higher-dimensional elements (a.k.a. edges, faces, etc.) intersect. They are represented as small black dots in Figure 2.

³In this paper, Voronoi regions are called *Thiessen polygons*.

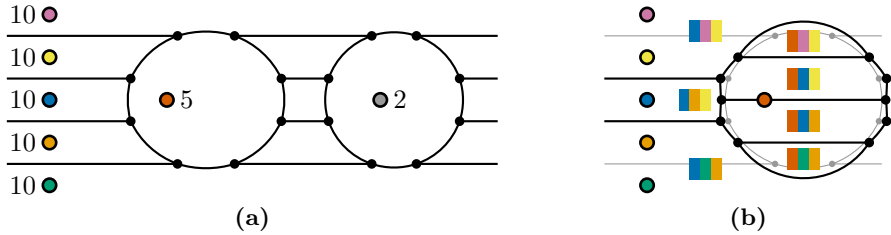


Figure 2: (a) Weighted Voronoi diagram (order-1) of the colored sites. Continuing the construction with $n/2$ high-weight sites on the left and $n/2$ low-weight sites towards the right yields $\Theta(n^2)$ vertices (small black dots). Note that each vertex lies on the boundary of three regions and has thus equal weighted distance to its three closest sites. (b) The order-3 Voronoi diagram for the same sites (excluding one). The colored boxes indicate the three closest sites. The order-1 diagram is shown in the background. Each order-1 vertex lies in the interior of an order-3 region as it has equal weighted distance to its three closest sites. As at most two order-1 vertices share an order-3 region, we get $\Omega(n^2)$ order-3 regions. Theorem 6.2 generalizes this observation.

order-1 Voronoi diagram is linear. This nicely complements the result by Har-Peled and Raichel [37], who show that the expected complexity of order-1 Voronoi diagrams of sites in 2-dimensional Euclidean space with random weights is $O(n \text{ polylog } n)$. We are not aware of any results concerning the complexity of Voronoi diagrams when combining multiplicative weights with higher dimension, higher order or other norms.

For higher dimensions, even normal (first order, unweighted) Voronoi diagrams in 3-dimensional Euclidean space can have $\Theta(n^2)$ [38, 50] vertices. Theorem 6.2 thus implies that the order-4 Voronoi diagram has a quadratic number of non-empty regions. Moreover, the complexity of higher-order Voronoi diagrams in higher dimensions has been considered before by Mulmuley [44], who obtains polynomial bounds with the degree of the polynomial depending on the dimension. Our Theorem 6.9 in particular shows that this complexity is much lower, namely linear, for the hypercube with randomly positioned sites. Moreover, via Theorem 6.2 this gives a linear bound on number of vertices in the normal order-1 Voronoi diagram in higher dimensions. We note that this special case of our result coincides with a previous result by Bienkowski et al. [8]. Similarly, Dwyer [24] showed that sites drawn uniformly from a higher dimensional unit sphere (instead of a hypercube) yield Voronoi diagrams of linear complexity in expectation. Moreover, due to Golin and Na [36] and Driemel et al. [22], the same is true for random sites on 3-dimensional polytopes and random sites on polyhedral terrains, respectively. Thus, though higher dimensional Voronoi diagrams can be rather complex in the worst case, these results indicate that one can expect most instances to be rather well behaved. An alternative explanation of why the complexity of practical instances is lower than the worst-case indicates is given by Erickson [25, 26], who studies the complexity of 3-dimensional Voronoi diagrams depending on the so-called spread of the sites.

The above results for higher dimensional Voronoi diagrams consider the Euclidean norm. For general \mathbf{p} -norms, L e [40] showed that the complexity of the Voronoi diagram is bounded by $O(n^c)$, where c is a constant independent of \mathbf{p} but dependent on the dimension d . With the same argument as above, Theorem 6.9 together with Theorem 6.2 implies a linear bound for this complexity that holds in expectation. Moreover, Boissonnat et al. [14] show more precise bounds of $\Theta(n^{\lceil d/2 \rceil})$ and $\Theta(n^2)$ for the ∞ - and the 1-norm, respectively. Again, our result implies linear bounds for random sites in this setting.

3 Formal Definitions

Power-Law Random k -SAT. The power-law model can be defined via the more general *non-uniform model*. To draw a k -SAT formula from the non-uniform model, let n and m be the number of vertices and clauses, respectively, and let w_1, \dots, w_n be variable weights. We sample m clauses independently at random. Each clause is sampled by drawing k variables without repetition with probabilities proportional to their weights. Then each of the k variables is negated independently at random with probability $1/2$.

The *power-law model* for a power-law exponent $\beta > 2$ is an instantiation of the non-uniform model with discrete power-law weights $w_i = i^{-1/(\beta-1)}$.

Graph Representation and Expansion. Let Φ be a SAT-formula with variable set V and clause set C . The *clause-variable incidence graph* $G(\Phi)$ of Φ has vertex set $C \cup V$, with an edge between a clause and a variable if and only if the clause contains the variable. Clearly, $G(\Phi)$ is bipartite. It is an (r, c) -*bipartite expander* if for all $C' \subset C$ with $|C'| \leq r$ it holds that $|N(C')| \geq (1 + c)|C'|$, where $N(C')$ is the neighborhood of C' .

Geometric Ground Space. We regularly deal with points with random positions in some geometric space. With *random point*, we refer to the uniform distribution in the sense that the probability for a point to lie in a region A is proportional to its volume $\text{vol}(A)$. For this to work, the volume of the ground space has to be bounded. Canonical options are, e.g., a unit-hypercube or a unit-ball. These, however, lead to the necessity of special treatment for points close to the boundary, which makes the analysis more tedious without giving additional insights. To circumvent this, we use a torus as ground space, which is completely symmetric.

The d -dimensional torus \mathbb{T}^d is defined as the d -dimensional hypercube $[0, 1]^d$ in which opposite borders are identified, i.e., a coordinate of 0 is identical to a coordinate of 1.⁴ It is equipped with the \mathbf{p} -norm as metric, for arbitrary but fixed $\mathbf{p} \in \mathbb{N}^+ \cup \{\infty\}$. To define it formally for the torus, let $\mathbf{p} = (p_1, \dots, p_d)$ and $\mathbf{q} = (q_1, \dots, q_d)$ be two points in \mathbb{T}^d . The circular difference between the i th coordinates is $|p_i - q_i|_{\circ} = \min\{|p_i - q_i|, 1 - |p_i - q_i|\}$. With this, the *distance* between \mathbf{p} and \mathbf{q} is

$$\|\mathbf{p} - \mathbf{q}\| = \begin{cases} \sqrt[p]{\sum_{i \in [d]} |p_i - q_i|_{\circ}^p} & \text{for } \mathbf{p} \neq \infty, \\ \max_{i \in [d]} \{|p_i - q_i|_{\circ}\} & \text{for } \mathbf{p} = \infty. \end{cases}$$

Random Points. We obtain the uniform distribution for a point $\mathbf{p} = (p_1, \dots, p_d)$ by drawing each coordinate p_i uniformly at random from $[0, 1]$. For two random points \mathbf{p} and \mathbf{q} , their distance $\|\mathbf{p} - \mathbf{q}\|$ is a random variable. Let $F_{\text{dist}}(x)$ be its *cumulative distribution function (CDF)*, i.e., $F_{\text{dist}}(x) = \Pr[\|\mathbf{p} - \mathbf{q}\| \leq x]$. To determine $F_{\text{dist}}(x)$, fix the position of \mathbf{p} . Then, for $x \leq 0.5$, the set of points of distance at most x to \mathbf{p} is simply the ball $B_{\mathbf{p}}(x)$ of radius x around \mathbf{p} , yielding

$$F_{\text{dist}}(x) = \text{vol}(B_{\mathbf{p}}(x)) = \Pi_{d, \mathbf{p}} \cdot x^d \quad \text{for } 0 \leq x \leq 0.5, \quad (1)$$

$$\text{with } \Pi_{d, \mathbf{p}} = \frac{\left(2\Gamma(1/\mathbf{p} + 1)\right)^d}{\Gamma(d/\mathbf{p} + 1)}.$$

where Γ is the gamma function. Note that $\Pi_{d, \mathbf{p}}$ only depends on d and \mathbf{p} but is constant in x . Moreover $\Pi_{2, 2} = \pi$ (thus the name Π), and $\Pi_{d, \infty} = \lim_{\mathbf{p} \rightarrow \infty} \Pi_{d, \mathbf{p}} = 2^d$. For distances $x > 0.5$, the formula for $F_{\text{dist}}(x)$ is more complicated (we basically have to subtract the parts reaching out of the hypercube). However, for our purposes, it suffices to know $F_{\text{dist}}(x)$ for $x \leq 0.5$ and use the obvious bound $F_{\text{dist}}(x) \leq 1$ for $x > 0.5$.

⁴For convenience reasons, we sometimes work with $[-0.5, 0.5]^d$ instead of $[0, 1]^d$.

Weighted Points and Distances. We regularly consider a fixed set of n points equipped with weights, which we call *sites*. For a site \mathbf{s}_i with weight w_i , the *weighted distance* of a point \mathbf{p} to \mathbf{s}_i is $\|\mathbf{s}_i - \mathbf{p}\|/w_i^{1/d}$. For a fixed value x , the set of points with weighted distance at most x are the points with $\|\mathbf{s}_i - \mathbf{p}\| \leq xw_i^{1/d}$. Note that the volume of this set is proportional to w_i . Intuitively, the region of influence of a site is thus proportional to its weight. To simplify notation in some places, we define *normalized weights* $\omega_i = w_i^{1/d}$.

Geometric Random k -SAT. In the *geometric model*, we sample positions for the variables and clauses uniformly at random in the d -dimensional torus \mathbb{T}^d . For $v \in V$ and $c \in C$, we use \mathbf{v} and \mathbf{c} to denote their positions, respectively. Let w_1, \dots, w_n be *variable weights* that are normalized such that the smallest weight is 1. Moreover, let $W = \sum_{v=1}^n w_v$. For a clause c and a variable v , define the *connection weight*

$$X(c, v) = \left(\frac{w_v}{\|\mathbf{c} - \mathbf{v}\|^d} \right)^{1/T}.$$

This is the reciprocal of the weighted distance between \mathbf{v} and \mathbf{c} raised to the power d/T . The k variables for the clause c are drawn without repetition with probabilities proportional to $X(c, v)$. Among all possible combinations, we choose which of the k variables to negate uniformly at random, without repetition if possible, i.e., we only get the same clause twice if we have more than 2^k clauses with the same variable set. For $T \rightarrow 0$ the model converges to the threshold case where c contains the k variables with smallest weighted distance.

The connection weight $X(c, v)$ is a random variable. We denote the CDF of $X(c, v)$ with $F_X(x)$. With the CDF for the distance between two random points in Equation (1), we obtain the following; see Lemma A.2 for a proof:

$$F_X(x) = 1 - \prod_{d,\mathbf{p}} w_v x^{-T} \quad \text{for } x \geq \left(2^d w_v\right)^{1/T}. \quad (2)$$

Voronoi Diagrams. Let $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$ be a set of sites with weights w_1, \dots, w_n . A point \mathbf{p} belongs to the (*open*) *Voronoi region* of a site \mathbf{s}_i if its weighted distance to \mathbf{s}_i is smaller than its weighted distance to any other site. The collection of all Voronoi regions is the *Voronoi diagram* of S . *Order- k Voronoi regions* are defined analogously for subsets $A \subseteq S$ with $|A| = k$, i.e., the region of A contains a point \mathbf{p} if and only if the weighted distances of \mathbf{p} to all sites in A is smaller than the weighted distance to any site not in A . More formally, \mathbf{p} belongs to the order- k Voronoi region of A if there exists a radius r such that $\|\mathbf{s}_i - \mathbf{p}\| \leq \omega_i r$ for $\mathbf{s}_i \in A$ and $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i r$ for $\mathbf{s}_i \notin A$. Note that the order- k Voronoi region of A is potentially empty. The *order- k Voronoi diagram* is the collection of all non-empty order- k Voronoi regions. Its *complexity* is the number of such non-empty regions.

4 Core Arguments

4.1 Power-Law SAT

We use a framework that Ben-Sasson and Wigderson [7] introduced for the uniform SAT model. We prove lower bounds for the resolution width, which implies lower bounds for the resolution size and the tree-like resolution size, which then implies lower bounds for the running times of CDCL and DPLL solvers, respectively.

To bound the resolution width, we essentially have to show that different clauses do not overlap too heavily. Specifically, a formula has resolution width $\Omega(w)$ if (1) every set S of at most w clauses contains at least $|S|$ different variables and (2) every set S of $\frac{1}{3}w \leq |S| \leq \frac{2}{3}w$ clauses contains at least a constant fraction of unique variables.

We achieve the bounds in Theorem 5.8 (i–iii) by showing the above two properties directly. For the bound in Theorem 5.8 (iv), we first observe that both properties are fulfilled if the clause-variable incidence graph of a k -CNF formula Φ has high enough bipartite expansion. Recall the definition of bipartite expansion from Section 3 and note how the requirement that the neighborhood of clause vertices is large resembles the requirement that clauses do not overlap too heavily. We show that $G(\Phi)$ is a bipartite expander asymptotically almost surely if Φ is drawn from the power-law model, which yields the lower bound of Theorem 5.8 (iv).

Compared to the uniform case, the weights make the properties required for the lower bounds less likely. Variables with high weight appear in many clauses, making the clauses less diverse. Thus, it is less likely that every clause set covers a large variety of variables.

4.2 Geometric SAT

To explain the core idea of our proof, consider the following simplified geometric model. Map n variables and m clauses to distinct points in the 2-dimensional Euclidean plane (randomly or deterministically). Build the SAT instance by including in each clause c the k variables with the smallest geometric distance to c . Now consider the order- k Voronoi diagram defined by the positions of the n variables. As a clause c contains the k closest variables, the k variables contained in c are exactly the k variables defining the Voronoi region of c 's position. Independent of the positions of the n variables, there are only at most $2k(n-k)$ regions in the order- k Voronoi diagram [13]. Thus, if we have at least $2^k 2k(n-k)$ clauses, then, by the pigeonhole principle, at least one Voronoi region contains 2^k clauses. As k is considered to be a constant, this number of clauses is linear in n , i.e., we still have constant density. Moreover, as repeating the same clause (with the same variable negations) is avoided whenever possible, there is a set of k variables that has a clause for every combination of literals. Thus, we have an unsatisfiable subformula of constant size 2^k , which implies low proof complexity.

This result can be varied and strengthened in multiple ways, e.g., by allowing weighted variables, a higher dimensional ground space, or by softening the requirement that every clause contains the k closest variables (model with higher temperature).

Abstract Geometric Spaces. The result by Bohler et al. [13] on the complexity of order- k Voronoi diagrams is very general in the sense that it holds for abstract Voronoi diagrams. Roughly speaking, abstract Voronoi diagrams are based on separating curves between pairs of points that take the role of perpendicular bisectors. In this way, one can abstract from the specific geometric ground space. Whether a point \mathbf{p} is closer to site \mathbf{s}_1 or to site \mathbf{s}_2 is no longer determined by comparing distances $\|\mathbf{s}_1 - \mathbf{p}\|$ and $\|\mathbf{s}_2 - \mathbf{p}\|$ but by the curve separating \mathbf{s}_1 from \mathbf{s}_2 . For this to work, the separating curves have to satisfy a handful of basic axioms. These are for example satisfied by perpendicular bisectors in the Euclidean or the hyperbolic plane. Thus, the above argumentation for low proof complexity directly carries over to the hyperbolic plane, or more generally, to any abstract geometric space satisfying the axioms.

Lower Density Via Random Clause Positions. Assume the variable positions are fixed. Now choose random positions for the clauses and observe in which regions of the order- k Voronoi diagram they end up. We want to know whether there is a region that contains at least 2^k clauses. This comes down to a balls into bins experiment. Each Voronoi region is a bin and each clause is a ball. Thus, there are $O(n)$ bins and m balls. Moreover, we are interested in the maximum load, i.e., the maximum number of balls that land in a single bin. Due to a result by Raab and Steger [49], the maximum load is a. a. s. in $\Omega(\frac{\log n}{\log \log n})$ if we throw $\Omega(\frac{n}{\text{polylog } n})$ balls. Thus, even for a slightly sublinear number of balls, the maximum load is superconstant. We note that this result holds for uniform bins. In our case, we have non-uniform bins, as the probability for a clause to end up in a particular Voronoi region is

proportional to the area of the region. However, it is not hard to see that the result by Raab and Steger [49] remains true for non-uniform bins; see Section A.5. Thus, even if the number of clauses m is slightly sublinear in the number of variables n , we get a small unsatisfiable subformula asymptotically almost surely if the Voronoi diagram has low complexity.

Positive or Negative Literals with Repetition. Above we assumed that we get the exact same clause with coinciding negations twice only if we already have more than 2^k clauses with the same set of k variables. Although this is arguably a reasonable assumption for the model, we can make a similar argument without it. Assume instead that for each variable, we choose the positive and negative literal uniformly at random, independently of all other choices. Moreover, assume for an increasing function f , that there are $f(n)$ clauses that have the same set of k variables. With the above balls into bins argument, we, e.g., have $f(n) \in \Omega(\frac{\log n}{\log \log n})$. Then the probability that there is a combination of positive and negative literals that we did not see at least once is at most $2^k(1 - 2^{-k})^{f(n)}$. This probability goes to 0 for $n \rightarrow \infty$, i.e., a. a. s., there is an unsatisfiable subformula of constant size 2^k .

Higher Dimension and Weighted Variables. At the core of our argument lies the fact that order- k Voronoi diagrams have linear complexity. As already mentioned in Section 2.3, this is no longer true for order- k Voronoi diagrams in higher dimensions or if the variables have multiplicative weights. A formal argument for why this property breaks is in Section 6.1. However, for sites distributed uniformly at random, we show in Section 6.2 that the complexity can be expected to be linear in the total weight, even in the more general setting. Thus, using that the variables have random positions (a requirement we did not need before), we can apply the above argument to obtain low proof complexity.

Non-Zero Temperature. Non-zero temperatures make it so that clauses do not necessarily contain the k closest variables. Instead, variables are included with probabilities depending on the distance. Thus, we cannot simply look at the order- k Voronoi diagram to determine which variables are contained in a given clause. We resolve this issue in Section 7. For this, we call a clause *nice*, if it behaves as it would in the $T = 0$ case, i.e., if it includes the k closest variables. In Section 7.1 we show that, in expectation, a constant fraction of clauses is actually nice. Moreover, in Section 7.2, we show that the number of nice clauses is concentrated around its expectation. With this, we can apply the same arguments as before to only the nice clauses, of which we have linearly many, to obtain a low proof complexity.

4.3 Voronoi Diagrams

The worst-case lower bounds for the complexity of order- k Voronoi diagrams follow from existing lower bounds on the number of vertices together with Theorem 6.2, which connects the complexity in terms of regions with the complexity in terms of vertices. This connection is obtained by observing how the order- k Voronoi diagram changes when increasing k .

For the average-case linear upper bound on the number of regions, the argument works roughly as follows, assuming the unweighted case for the sake of simplicity. For each size- k subset A of the sites, we devise an upper bound on the probability that A has non-empty order- k Voronoi region. This region is non-empty if and only if there are points that have A as the k closest sites, i.e., if there is a ball that contains the sites of A and no other sites. With this observation, we can use a win-win-style argument. Either the radius of this ball is small, which makes it unlikely that all sites of A lie in the ball, or the ball is large, which makes it unlikely that it contains no other sites.

5 Resolution Size of Power-Law Random k -SAT

5.1 The Direct Approach

As stated in Section 4.1, a formula has resolution width $\Omega(w)$ if (1) every set S of at most w clauses contains at least $|S|$ different variables and (2) every set S of $\frac{1}{3}w \leq |S| \leq \frac{2}{3}w$ clauses contains at least a constant fraction of unique variables. In this section we are going to show that both conditions are satisfied for power-law exponents $\beta > \frac{2k-1}{k-1}$ and clause-variable ratios $\Delta \in \Omega(n)$. The first condition can also be interpreted in terms of bipartite expansion. It states that the clause-variable incidence graph $G(\Phi)$ is a $(w, 0)$ -bipartite expander. The following lemma states bounds on w for which $G(\Phi)$ is a $(w, 0)$ -bipartite expander asymptotically almost surely. These bounds depend on the power-law exponent β as well as on the clause-variable ratio Δ . Our choices of k and β in the lemma ensure that $\varepsilon_1, \varepsilon_2 > 0$.

Lemma 5.1. *Let Φ be a random power-law k -SAT formula with n variables, $\Delta \cdot n = m \in \Omega(n)$ clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-1}{k-1}$. Let $\varepsilon_1 = k \cdot \frac{\beta-2}{\beta-1} - 1 > 0$ and $\varepsilon_2 = (k-2) \frac{\beta-2}{\beta-1} > 0$. Then $G(\Phi)$ has $(w, 0)$ -bipartite expansion a. a. s. if*

$$(i) \quad \beta \in \left(\frac{2k-1}{k-1}, 3 \right), \Delta \in o\left(n^{\varepsilon_1} / \log^{\varepsilon_2}(n)\right), \text{ and } w \in O\left(n^{\varepsilon_1/\varepsilon_2} \cdot \Delta^{-1/\varepsilon_2}\right)$$

$$(ii) \quad \beta = 3, \Delta \in o\left(n^{(k-2)/2} / \log^{(k-2)/2+1}(n)\right), \text{ and } w \in O\left(n \cdot (\Delta \cdot \ln n)^{-2/(k-2)}\right).$$

$$(iii) \quad \beta > 3, \Delta \in o\left(n^{\varepsilon_2} / \log^{\varepsilon_2} n\right), \text{ and } w \in O\left(n \cdot \Delta^{-1/\varepsilon_2}\right).$$

Proof. We are interested in showing $|N(C')| \geq |C'|$ for all $C' \subseteq C$ with $|C'| \leq w$. We consider a smallest C' such that $|N(C')| \leq |C'| - 1$ and denote it by \hat{C} . Let \mathcal{E}_i be the event that $|\hat{C}| = i$. Thus, \mathcal{E}_i implies that for all $C' \subseteq C$ with $|C'| < i$ it holds that $|N(C')| \geq |C'|$. This implies that every variable in $N(\hat{C})$ has to appear at least twice. Otherwise, one could delete a clause with a unique variable from \hat{C} to get a set \hat{C}' with $|\hat{C}'| = i - 1$ and $|N(\hat{C}')| \leq i - 2$. This would violate the minimality of \hat{C} . Also, \hat{C} must contain exactly $i - 1$ different variables. Otherwise, we could remove any clause from \hat{C} and violate minimality.

Now we bound

$$\sum_{i=1}^r \Pr(\mathcal{E}_i) \leq \sum_{i=1}^r \binom{m}{i} P_i,$$

where P_i is the probability to draw i clauses which contain at most $i - 1$ different variables and all of them at least twice. We can now imagine the $k \cdot i$ variables of the i clauses to be drawn independently with replacement. Now we consider the $i - 1$ different variables drawn. Then, we choose the $i - 1$ pairs of positions where each variable appears for the first and second time. As a rough upper bound we have at most

$$\binom{\binom{k \cdot i}{2}}{i-1} \leq \left(\frac{(k \cdot i)^2 \cdot e}{2 \cdot (i-1)} \right)^{i-1}$$

many possibilities by simply choosing $i - 1$ from all $\binom{k \cdot i}{2}$ possible pairs. Now we bound the probability that at these pairs of positions the same variables do appear. This is at most $\left(\sum_{j=1}^n p_j^2\right)$ per pair of positions. At the remaining $k \cdot i - 2 \cdot (i - 1)$ positions we can only choose from at most those $i - 1$ variables. Thus, the probabilities at all other positions are the sum of the $i - 1$ variable probabilities, which is at most the sum of the $i - 1$ highest

variable probabilities. Let $F(i)$ be the sum of the i highest variable probabilities. Then it holds that

$$\begin{aligned} P_i &\leq \left(\frac{(k \cdot i)^2 \cdot e}{2 \cdot (i-1)} \right)^{i-1} \cdot \left(\sum_{j=1}^n p_j^2 \right)^{i-1} \cdot F(i-1)^{k \cdot i - 2 \cdot (i-1)} \\ &\leq \kappa^{i-1} \cdot \left(\frac{i^2}{i-1} \right)^{i-1} \cdot \left(\sum_{j=1}^n p_j^2 \right)^{i-1} \cdot \left(\frac{i-1}{n} \right)^{(k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1}} \end{aligned}$$

for some constant $\kappa > 0$. According to Lemma A.1, $\sum_{j=1}^n p_j^2$ now depends on the power law exponent β . For $\beta < 3$ we get

$$\begin{aligned} P_i &\leq \kappa^i \cdot \left(\frac{i^2}{i-1} \right)^{i-1} \cdot n^{-(i-1) \cdot 2 \frac{\beta-2}{\beta-1}} \cdot \left(\frac{i-1}{n} \right)^{(k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1}} \\ &\leq \kappa^i \cdot n^{-k \cdot i \cdot \frac{\beta-2}{\beta-1}} \cdot i^{i-1} \cdot (i-1)^{(k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1}} \\ &\leq \kappa^i \cdot n^{-k \cdot i \cdot \frac{\beta-2}{\beta-1}} \cdot i^{(k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1} + (i-1)} \\ &= \kappa^i \cdot n^{-k \cdot i \cdot \frac{\beta-2}{\beta-1}} \cdot i^{(k \cdot i - 2 \cdot i) \frac{\beta-2}{\beta-1} + i + 2 \frac{\beta-2}{\beta-1} - 1} \\ &\leq \kappa^i \cdot n^{-k \cdot i \cdot \frac{\beta-2}{\beta-1}} \cdot i^{(k \cdot i - 2 \cdot i) \frac{\beta-2}{\beta-1} + i} = \kappa^i \cdot n^{-i \cdot (\varepsilon_1 + 1)} \cdot i^{i \cdot (\varepsilon_2 + 1)}, \end{aligned}$$

where we used $\left(\frac{i^2}{i-1} \right)^{i-1} \leq e \cdot i^{i-1}$ in the second line and upper-bounded $i-1$ in the base by i in the last line, which we can do since $(k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1} > 0$ due to $k \geq 3$ and $\beta > 2$. In the third line, we used $2 \frac{\beta-2}{\beta-1} - 1 < 0$, which holds since $\beta < 3$.

We can now see that

$$\begin{aligned} \sum_{i=1}^w \binom{m}{i} P_i &\leq \sum_{i=1}^w \binom{m}{i} \cdot \kappa^i \cdot n^{-i \cdot (\varepsilon_1 + 1)} \cdot i^{i \cdot (\varepsilon_2 + 1)} \\ &\leq \sum_{i=1}^w \kappa^i \cdot \Delta^i \cdot n^{-i \cdot \varepsilon_1} \cdot i^{i \cdot \varepsilon_2}, \end{aligned} \quad (3)$$

which holds since we assume $m = \Delta \cdot n$ and $\binom{m}{i} \leq \left(\frac{e \cdot m}{i} \right)^i$. In order to have a sum which is $o(1)$ we want to ensure that

$$\Delta \cdot n^{-\varepsilon_1} \cdot i^{\varepsilon_2}$$

is a small enough constant. It is easy to check that this holds for

$$i \in O\left(n^{\varepsilon_1/\varepsilon_2} \cdot \Delta^{-1/\varepsilon_2} \right).$$

Thus, we can set w to this value. If we split the sum in Equation (3) at $i_0 = \lfloor \varepsilon_1 \log n \rfloor$, the part with $i \leq i_0$ is upper-bounded by $O(\Delta \cdot n^{-\varepsilon_1} \cdot i_0^{\varepsilon_2}) \in O\left(\Delta (\log^{\varepsilon_2}(n)/n^{\varepsilon_1})\right)$ via a geometric series. The part with $i > i_0$ is upper-bounded by $O(2^{-i_0}) \in O(n^{-\varepsilon_1})$ if we choose $w \in \Theta(n^{\varepsilon_1/\varepsilon_2} \cdot \Delta^{-1/\varepsilon_2})$ sufficiently small. Thus, we get $\left(\Theta\left(n^{\varepsilon_1/\varepsilon_2} \cdot \Delta^{-1/\varepsilon_2}\right), 0\right)$ -expansion with probability at least $1 - \Theta\left(\Delta (\log^{\varepsilon_2}(n)/n^{\varepsilon_1})\right)$ or a. a. s. if $\Delta \in o(n^{\varepsilon_1}/(\log(n))^{\varepsilon_2})$.

For $\beta > 3$ we get

$$\begin{aligned} P_i &\leq \kappa^i \cdot \left(\frac{i^2}{i-1}\right)^{i-1} \cdot n^{-(i-1)} \cdot \left(\frac{i-1}{n}\right)^{(k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1}} \\ &\leq \kappa^i \cdot \left(\frac{i}{n}\right)^{i-1 + (k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1}}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=1}^r \binom{m}{i} P_i &\leq \sum_{i=1}^w \binom{n}{i}^i \cdot \kappa^i \cdot \Delta^i \cdot \left(\frac{i}{n}\right)^{i-1 + (k \cdot i - 2 \cdot (i-1)) \frac{\beta-2}{\beta-1}} \\ &= \sum_{i=1}^w \kappa^i \cdot \Delta^i \cdot \left(\frac{i}{n}\right)^{(k \cdot i - 2 \cdot i) \frac{\beta-2}{\beta-1} + 2 \frac{\beta-2}{\beta-1} - 1} \\ &\leq \sum_{i=1}^w \kappa^i \cdot \Delta^i \cdot \left(\frac{i}{n}\right)^{i \cdot (k-2) \frac{\beta-2}{\beta-1}} \\ &\leq \sum_{i=1}^w \kappa^i \cdot \Delta^i \cdot \left(\frac{i}{n}\right)^{i \cdot \varepsilon_2}, \end{aligned}$$

which holds since $\frac{i}{n} \leq 1$ and $2 \cdot \frac{\beta-2}{\beta-1} - 1 \geq 0$ for $\beta \geq 3$. It is now easy to show that $\Delta \cdot (i/n)^{\varepsilon_2}$ is at most a constant for $w \in \Theta(n \cdot \Delta^{-1/\varepsilon_2})$ sufficiently small. By splitting the sum as before, we can show $((n \cdot \Delta^{-1/\varepsilon_2}), 0)$ -expansion with probability at least $1 - \Theta(\Delta \cdot \log^{\varepsilon_2} n / n^{\varepsilon_2})$ or a. a. s. for $\Delta \in o(n^{\varepsilon_2} / \log^{\varepsilon_2} n)$.

For $\beta = 3$ we get the same result as for $\beta > 3$, except for an additional factor of $(\ln n)^{i-1}$. Thus,

$$\begin{aligned} \sum_{i=1}^w \binom{m}{i} P_i &\leq \sum_{i=1}^w \kappa^i \cdot \Delta^i \cdot \left(\frac{i}{n}\right)^{i \cdot (k-2) \frac{\beta-2}{\beta-1}} \ln^{i-1} n \\ &\leq \sum_{i=1}^w \kappa^i \cdot \Delta^i \cdot \left(i \cdot \frac{\ln^{2/(k-2)} n}{n}\right)^{i \cdot \frac{k-2}{2}} \end{aligned}$$

By assuming

$$w \in \Theta\left(n \cdot (\Delta \cdot \log n)^{-2/(k-2)}\right)$$

we can ensure that this sum is at most $O(\Delta \cdot \log n \cdot (\log(n)/n)^{(k-2)/2})$. Hence, we get $(\Theta(n \cdot (\Delta \cdot \log n)^{-2/(k-2)}), 0)$ -expansion with probability at least $1 - O(\Delta \cdot \ln n \cdot (\log(n)/n)^{(k-2)/2})$ or a. a. s. for $\Delta \in o(n^{(k-2)/2} / \log^{(k-2)/2+1}(n))$. \square

Now we want to show the second requirement of Theorem 5.3, that every set S of $\frac{1}{3}w \leq |S| \leq \frac{2}{3}w$ clauses contains at least a constant fraction of unique variables. Again, our choices of k and β in the lemma ensure that we can always choose an $\varepsilon > 0$ with $\varepsilon_1, \varepsilon_2 > 0$.

Lemma 5.2. *Let Φ be a random power-law k -SAT formula with n variables, $\Delta \cdot n = m \in \Omega(n)$ clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-1}{k-1}$. Let $\varepsilon, \varepsilon_1, \varepsilon_2$ be constant such that $\varepsilon > 0$, $\varepsilon_1 = \frac{k-\varepsilon}{2} - 1 > 0$, and $\varepsilon_2 = (k-\varepsilon) \frac{\beta-2}{\beta-1} - 1 > 0$. Let w be the largest value such that all sets C' of clauses from Φ with $\frac{1}{3}w \leq |C'| \leq \frac{2}{3}w$ contain at least $\varepsilon \cdot |C'|$ unique variables. For w it holds a. a. s. that:*

(i) If $\beta \in \left(\frac{2k-1}{k-1}, 3\right)$ and $\Delta \in o(n^{\varepsilon_2})$, then $w \in \Omega\left(n^{\varepsilon_2/\varepsilon_1} \cdot \Delta^{-1/\varepsilon_1}\right)$.

(ii) If $\beta = 3$ and $\Delta \in o\left(n^{\varepsilon_1}/\ln^{\varepsilon_1+1} n\right)$, then $w \in \Omega\left(n \cdot \Delta^{-1/\varepsilon_1}/\ln^{1+\frac{1}{\varepsilon_1}} n\right)$.

(iii) If $\beta > 3$ and $\Delta \in o(n^{\varepsilon_1})$, then $w \in \Omega\left(n \cdot \Delta^{-1/\varepsilon_1}\right)$.

Proof. Let $\varepsilon \in (0, \min\{k-1 - \frac{\beta-1}{\beta-2}, k-2\})$ be a constant. The upper bounds on ε ensure $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$. We want to bound the probability that there is a set of clauses C' with $\frac{1}{3}w \leq |C'| \leq \frac{2}{3}w$ and at most $\varepsilon \cdot |C'|$ many unique variables. Let P_i be the probability that there is a set C' of size i with that property. We assume the $k \cdot i$ Boolean variables to be drawn independently at random, i. e., we allow duplicate variables inside clauses. This only decreases the probability of having unique variables. Additionally, we split the probability into parts depending on the number j of different variables that appear in C' in addition to the $\varepsilon \cdot i$ unique ones. It holds that

$$\begin{aligned}
P_i &\leq \underbrace{\binom{m}{i}}_{\text{choices of clauses}} \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \underbrace{\binom{k \cdot i}{\varepsilon \cdot i}}_{\text{possible positions for the } \varepsilon \cdot i \text{ unique variables}} \underbrace{\binom{\binom{(k-\varepsilon) \cdot i}{2}}{j}}_{\text{possible positions for the first two appearances of the } j \text{ other variables}} \cdot \underbrace{1^{\varepsilon \cdot i}}_{\text{probability to draw a new variable}} \underbrace{\left(\sum_{x=1}^n p_x^2\right)^j}_{\text{probability that variables are same at positions for first two appearances}} \cdot \underbrace{F(j)^{k \cdot i - \varepsilon \cdot i - 2j}}_{\text{upper bound on probability to draw } j \text{ chosen variables again}} \\
&\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{k \cdot i}{\varepsilon \cdot i}\right)^{\varepsilon \cdot i} \left(\frac{(k-\varepsilon)^2 \cdot i^2}{j}\right)^j \cdot \left(\sum_{x=1}^n p_x^2\right)^j \cdot F(j)^{k \cdot i - \varepsilon \cdot i - 2j} \\
&\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot \left(\sum_{x=1}^n p_x^2\right)^j \cdot \left(\frac{j}{n}\right)^{(k \cdot i - \varepsilon \cdot i - 2j) \frac{\beta-2}{\beta-1}},
\end{aligned}$$

where $\kappa > 0$ is a constant that might depend on β . As in the proof of Lemma 5.1 we have to distinguish three cases depending on the power law exponent β . Using Lemma A.1 we see that for $\beta < 3$

$$\begin{aligned}
P_i &\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot \left(\sum_{x=1}^n p_x^2\right)^j \cdot \left(\frac{j}{n}\right)^{(k \cdot i - \varepsilon \cdot i - 2j) \frac{\beta-2}{\beta-1}} \\
&\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot n^{-2j \frac{\beta-2}{\beta-1}} \cdot \left(\frac{j}{n}\right)^{(k \cdot i - \varepsilon \cdot i - 2j) \frac{\beta-2}{\beta-1}} \\
&= \kappa^i \cdot \Delta^i \cdot n^{i \left(1 - (k-\varepsilon) \frac{\beta-2}{\beta-1}\right)} \cdot i^{-i} \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot j^{(k \cdot i - \varepsilon \cdot i - 2j) \frac{\beta-2}{\beta-1}}. \tag{4}
\end{aligned}$$

Now it remains to bound the inner sum. In order to do so, we will split it at $j_0 = \frac{3-\beta}{4}(k-\varepsilon) \cdot i$. It is easy to see that $0 < \frac{3-\beta}{4} < \frac{1}{4}$ for $2 < \beta < 3$, thus this choice of j is valid. For the first

part of the sum it holds that

$$\begin{aligned}
\sum_{j=1}^{\frac{3-\beta}{4}(k-\varepsilon)\cdot i} \left(\frac{i^2}{j}\right)^j \cdot j^{(k\cdot i-\varepsilon\cdot i-2j)\frac{\beta-2}{\beta-1}} &\leq \kappa^i \sum_{j=1}^{\frac{3-\beta}{4}(k-\varepsilon)\cdot i} i^{2\cdot j} \cdot j^{-j} \cdot i^{((k-\varepsilon)\cdot i-2j)\frac{\beta-2}{\beta-1}} \\
&\leq \kappa^i \cdot i^{(k-\varepsilon)\cdot i\frac{\beta-2}{\beta-1}} \sum_{j=1}^{\frac{3-\beta}{4}(k-\varepsilon)\cdot i} i^{\frac{2\cdot j}{\beta-1}} \\
&\leq \kappa^i \cdot i^{(k-\varepsilon)\cdot i\frac{\beta-2}{\beta-1}} \cdot i^{2\cdot\frac{3-\beta}{4}\cdot\frac{k-\varepsilon}{\beta-1}\cdot i} \\
&= \kappa^i \cdot i^{\frac{k-\varepsilon}{2}\cdot i},
\end{aligned}$$

where we used $j \leq \frac{3-\beta}{4}(k-\varepsilon)\cdot i$ and $((k-\varepsilon)\cdot i-2j) \geq 0$ in the first line and a geometric series in the third line. For the second part of the sum it holds that

$$\begin{aligned}
\sum_{j=\frac{3-\beta}{4}(k-\varepsilon)\cdot i}^{\frac{k-\varepsilon}{2}\cdot i} \left(\frac{i^2}{j}\right)^j \cdot j^{(k\cdot i-\varepsilon\cdot i-2j)\frac{\beta-2}{\beta-1}} &\leq \kappa^i \sum_{j=\frac{3-\beta}{4}(k-\varepsilon)\cdot i}^{\frac{k-\varepsilon}{2}\cdot i} i^{2\cdot j} \cdot j^{-j} \cdot i^{((k-\varepsilon)\cdot i-2j)\frac{\beta-2}{\beta-1}} \\
&\leq \kappa^i \cdot i^{(k-\varepsilon)\cdot i\frac{\beta-2}{\beta-1}} \sum_{j=\frac{3-\beta}{4}(k-\varepsilon)\cdot i}^{\frac{k-\varepsilon}{2}\cdot i} i^{j-2\cdot j\frac{\beta-2}{\beta-1}} \\
&\leq \kappa^i \cdot i^{(k-\varepsilon)\cdot i\frac{\beta-2}{\beta-1}} \cdot i^{\frac{3-\beta}{\beta-1}\cdot\frac{k-\varepsilon}{2}\cdot i} \\
&= \kappa^i \cdot i^{\frac{k-\varepsilon}{2}\cdot i},
\end{aligned}$$

where we used $j \geq \frac{3-\beta}{4}(k-\varepsilon)\cdot i$ in the second and a geometric series in the third line.

Thus,

$$\sum_{j=1}^{\frac{k-\varepsilon}{2}\cdot i} \left(\frac{i^2}{j}\right)^j \cdot j^{(k\cdot i-\varepsilon\cdot i-2j)\frac{\beta-2}{\beta-1}} \leq \kappa^i \cdot i^{\frac{k-\varepsilon}{2}\cdot i}$$

and plugging this into Equation (4) yields

$$P_i \leq \kappa^i \cdot \Delta^i \cdot n^{i\left(1-(k-\varepsilon)\frac{\beta-2}{\beta-1}\right)} \cdot i^{\left(\frac{k-\varepsilon}{2}\cdot i-1\right)} = \kappa^i \cdot \Delta^i \cdot n^{-\varepsilon_2\cdot i} \cdot i^{\varepsilon_1\cdot i}.$$

Since we want to sum over all P_i with $\frac{1}{3}w \leq i \leq \frac{2}{3}w$ for some w , it holds that

$$\begin{aligned}
\sum_{i=\frac{1}{3}w}^{\frac{2}{3}w} P_i &\leq \sum_{i=\frac{1}{3}w}^{\frac{2}{3}w} \kappa^i \cdot \Delta^i \cdot n^{-\varepsilon_2\cdot i} \cdot i^{\varepsilon_1\cdot i} \\
&\leq \sum_{i=\frac{1}{3}w}^{\frac{2}{3}w} \left(\kappa \cdot \Delta \cdot n^{-\varepsilon_2} \cdot w^{\varepsilon_1}\right)^i
\end{aligned}$$

This sums up to $o(1)$ as soon as $\Delta \cdot n^{-\varepsilon_2} \cdot w^{\varepsilon_1}$ is a suitably small constant and w is super-constant. In our case, we see that this holds for some

$$w \in O\left(n^{\varepsilon_2/\varepsilon_1} \Delta^{-1/\varepsilon_1}\right).$$

For $\beta = 3$ we get

$$\begin{aligned}
P_i &\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot \left(\sum_{x=1}^n p_x^2\right)^j \cdot \left(\frac{j}{n}\right)^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1}} \\
&\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot \left(\frac{\ln n}{n}\right)^j \cdot \left(\frac{j}{n}\right)^{((k-\varepsilon) \cdot i - 2j) \frac{1}{2}} \\
&= \kappa^i \cdot \Delta^i \cdot \left(\frac{n}{i}\right)^i \cdot n^{-\frac{k-\varepsilon}{2} i} \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} i^{2j} \cdot j^{\frac{k-\varepsilon}{2} i - 2j} \cdot (\ln n)^j. \tag{5}
\end{aligned}$$

We want to show that this inner sum is at most $\kappa^i \cdot (i \cdot \ln n)^{\frac{k-\varepsilon}{2} i}$. As before, we can split the sum. This time we split it at $j_0 = \frac{k-\varepsilon}{4} i$. For the first part we get

$$\begin{aligned}
\sum_{j=1}^{\frac{k-\varepsilon}{4} i} i^{2j} \cdot j^{\frac{k-\varepsilon}{2} i - 2j} \cdot (\ln n)^j &\leq \kappa^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{4} i} i^{2j} \cdot i^{\frac{k-\varepsilon}{2} i - 2j} \cdot (\ln n)^j \\
&\leq \kappa^i \cdot i^{\frac{k-\varepsilon}{2} i} \cdot \sum_{j=1}^{\frac{k-\varepsilon}{4} i} (\ln n)^j \\
&\leq \kappa^i \cdot i^{\frac{k-\varepsilon}{2} i} \cdot (\ln n)^{\frac{k-\varepsilon}{4} i},
\end{aligned}$$

where we used that $\frac{k-\varepsilon}{2} i - 2j \geq 0$ in the first line and we used $\sum_{j=1}^{\frac{k-\varepsilon}{4} i} (\ln n)^j \in O((\ln n)^{\frac{k-\varepsilon}{4} i})$ in the last line. The second part of the sum yields

$$\begin{aligned}
\sum_{j=\frac{k-\varepsilon}{4} i}^{\frac{k-\varepsilon}{2} i} i^{2j} \cdot j^{\frac{k-\varepsilon}{2} i - 2j} \cdot (\ln n)^j &\leq \kappa^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{4} i} i^{2j} \cdot i^{\frac{k-\varepsilon}{2} i - 2j} \cdot (\ln n)^j \\
&\leq \kappa^i \cdot i^{\frac{k-\varepsilon}{2} i} \sum_{j=1}^{\frac{k-\varepsilon}{4} i} (\ln n)^j \leq \kappa^i \cdot i^{\frac{k-\varepsilon}{2} i} (\ln n)^{\frac{k-\varepsilon}{4} i},
\end{aligned}$$

since $j \in \Theta(i)$. Plugging this into Equation (5) gives us

$$P_i \leq \kappa^i \cdot \Delta^i \cdot \left(\frac{n}{i}\right)^{i(1-\frac{k-\varepsilon}{2})} \cdot (\ln n)^{\frac{k-\varepsilon}{2} i} = \kappa^i \cdot \Delta^i \cdot \left(\frac{n}{i}\right)^{-\varepsilon_1 i} \cdot (\ln n)^{(\varepsilon_1+1) \cdot i}.$$

As before, we can see that this is at most κ^i for some constant $\kappa \in (0, 1)$ if

$$w \in O\left(n / \ln^{1+\frac{1}{\varepsilon_1}} n \cdot \Delta^{-1/\varepsilon_1}\right)$$

is small enough.

For $\beta > 3$ we get

$$\begin{aligned}
P_i &\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot \left(\sum_{x=1}^n p_x^2\right)^j \cdot \left(\frac{j}{n}\right)^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1}} \\
&\leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^i \cdot \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} \left(\frac{i^2}{j}\right)^j \cdot n^{-j} \cdot \left(\frac{j}{n}\right)^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1}} \\
&= \kappa^i \cdot \Delta^i \cdot \left(\frac{n}{i}\right)^i \cdot n^{-(k-\varepsilon) \frac{\beta-2}{\beta-1} i} \sum_{j=1}^{\frac{k-\varepsilon}{2} \cdot i} i^{2j} \cdot n^{j(2 \frac{\beta-2}{\beta-1} - 1)} \cdot j^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1} - j}. \tag{6}
\end{aligned}$$

This time we are going to show that the inner sum is bounded by $i^{(k-\varepsilon) \frac{\beta-2}{\beta-1} i} \cdot \left(\frac{n}{i}\right)^{\frac{k-\varepsilon}{2} i (2 \frac{\beta-2}{\beta-1} - 1)}$. Again, we split the sum. This time at

$$j_0 = \frac{(k-\varepsilon) \frac{\beta-2}{\beta-1} i}{1 + 2 \frac{\beta-2}{\beta-1}}.$$

Our choice ensures $((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1} - j \geq 0$ for $j \leq j_0$. Thus, the first part of the sum yields

$$\begin{aligned}
&\sum_{j=1}^{j_0} i^{2j} \cdot n^{j(2 \frac{\beta-2}{\beta-1} - 1)} \cdot j^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1} - j} \\
&\leq \kappa^i \cdot \sum_{j=1}^{j_0} i^{2j} \cdot n^{j(2 \frac{\beta-2}{\beta-1} - 1)} \cdot i^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1} - j} \\
&\leq \kappa^i \cdot i^{(k-\varepsilon) \frac{\beta-2}{\beta-1} i} \cdot \sum_{j=1}^{j_0} \left(\frac{n}{i}\right)^{j(2 \frac{\beta-2}{\beta-1} - 1)} \\
&\leq \kappa^i \cdot i^{(k-\varepsilon) \frac{\beta-2}{\beta-1} i} \cdot \left(\frac{n}{i}\right)^{\frac{k-\varepsilon}{2} i (2 \frac{\beta-2}{\beta-1} - 1)},
\end{aligned}$$

where the last line holds since

$$j_0 = \frac{(k-\varepsilon) \frac{\beta-2}{\beta-1} i}{1 + 2 \frac{\beta-2}{\beta-1}} i \leq \frac{k-\varepsilon}{2} i$$

for $\beta > 3$. For the second part of the sum we get

$$\begin{aligned}
&\sum_{j=j_0}^{\frac{k-\varepsilon}{2} \cdot i} i^{2j} \cdot n^{j(2 \frac{\beta-2}{\beta-1} - 1)} \cdot j^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1} - j} \\
&\leq \kappa^i \cdot \sum_{j=j_0}^{\frac{k-\varepsilon}{2} \cdot i} i^{2j} \cdot n^{j(2 \frac{\beta-2}{\beta-1} - 1)} \cdot i^{((k-\varepsilon) \cdot i - 2j) \frac{\beta-2}{\beta-1} - j} \\
&\leq \kappa^i \cdot i^{(k-\varepsilon) \frac{\beta-2}{\beta-1} i} \cdot \left(\frac{n}{i}\right)^{\frac{k-\varepsilon}{2} i (2 \frac{\beta-2}{\beta-1} - 1)},
\end{aligned}$$

since $\frac{k-\varepsilon}{6}i \leq \frac{(k-\varepsilon)^{\frac{\beta-2}{\beta-1}}}{1+2^{\frac{\beta-2}{\beta-1}}}i \leq j \leq \frac{k-\varepsilon}{2}i$. If we plug our estimate into Equation (6) this gives us

$$P_i \leq \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^{\left(1-\frac{k-\varepsilon}{2}\right)i} = \kappa^i \cdot \Delta^i \left(\frac{n}{i}\right)^{-\varepsilon_1 \cdot i}.$$

We can now find a $w \in \Theta(n \cdot \Delta^{-1/\varepsilon_1})$ small enough such that the property holds as desired.

In all three cases we can choose w in such a way that the probability for the property not to hold is at most $\kappa^{\frac{w}{3}}$ for some constant $\kappa \in (0, 1)$. This means, the property holds a. a. s. for $w \in \Omega(1)$. \square

The two properties we showed in Lemma 5.1 and Lemma 5.2 can be used to derive lower bounds on resolution width via the following theorem by Ben-Sasson and Wigderson [7].

Theorem 5.3 ([7]). *Let Φ be an unsatisfiable k -CNF formula with $k \geq 3$. If there is a $w \in \mathbb{N}$ such that*

- (i) *for all sets of clauses C' with $|C'| \leq w$ it holds that C' contains at least $|C'|$ different Boolean variables and*
- (ii) *for all sets of clauses C' with $\frac{1}{3}w \leq |C'| \leq \frac{2}{3}w$ it holds that C' contains at least $\varepsilon \cdot |C'|$ unique variables for some constant $\varepsilon > 0$.*

then the resolution width of Φ is $\Omega(w)$.

The theorem implies the following bounds on resolution width for power-law random k -SAT.

Lemma 5.1 and Lemma 5.2 together with Theorem 5.3 imply the following corollary. However, Theorem 5.3 only works for unsatisfiable instances. Since these lemmas do not condition on instances being unsatisfiable, we need to make sure that the probability for instances being unsatisfiable is high enough. Thus, we require the clause-variable ratio Δ to be high enough for instances to be unsatisfiable with at least constant probability.

Corollary 5.4. *Let Φ be a random power-law k -SAT formula with n variables, $m \in \Omega(n)$ clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-1}{k-1}$ constant. Let $\Delta = m/n$ be large enough so that Φ is unsatisfiable at least with constant probability. Let $\varepsilon, \varepsilon_1, \varepsilon_2$ be constants with $\varepsilon > 0$, $\varepsilon_1 = \frac{k-\varepsilon}{2} - 1 > 0$, and $\varepsilon_2 = (k-\varepsilon)^{\frac{\beta-2}{\beta-1}} - 1 > 0$. For the resolution width w of Φ , it holds a. a. s. that:*

- (i) *If $\beta \in \left(\frac{2k-1}{k-1}, 3\right)$ and $\Delta \in o(n^{\varepsilon_2})$, then $w \in \Omega\left(n^{\varepsilon_2/\varepsilon_1} \cdot \Delta^{-1/\varepsilon_1}\right)$.*
- (ii) *If $\beta = 3$ and $\Delta \in o\left(n^{\varepsilon_1}/\log^{\varepsilon_1+1} n\right)$, then $w \in \Omega\left(n \cdot \Delta^{-1/\varepsilon_1}/\log^{1+\frac{1}{\varepsilon_1}} n\right)$.*
- (iii) *If $\beta > 3$ and $\Delta \in o\left(n^{\varepsilon_1+1}\right)$, then $w \in \Omega\left(n \cdot \Delta^{-1/\varepsilon_1}\right)$.*

Proof. If both Lemma 5.1 and Lemma 5.2 hold, we can use Theorem 5.3 to get the desired bound on resolution width. We are going to show that the values of w from Lemma 5.2 are smaller than those from Lemma 5.1. The expansion bound from Lemma 5.1 also holds for those smaller values of w due to the definition of bipartite expansion. Thus, the bound from Lemma 5.2 gives us the maximum w we can achieve.

First, consider the case $\beta \in \left(\frac{2k-1}{k-1}, 3\right)$. Let $\varepsilon_3 = k^{\frac{\beta-2}{\beta-1}} - 1$ and $\varepsilon_4 = (k-2)^{\frac{\beta-2}{\beta-1}}$. We want to show that

$$n^{\varepsilon_2/\varepsilon_1} \cdot \Delta^{-1/\varepsilon_1} \leq n^{\varepsilon_3/\varepsilon_4} \cdot \Delta^{-1/\varepsilon_4}. \quad (7)$$

Both bounds only hold for

$$\Delta \in o(n^{\varepsilon_2}) \subseteq o(n^{\varepsilon_3}/\log^{\varepsilon_4}(n)),$$

since $\varepsilon_2 = (k - \varepsilon)\frac{\beta-2}{\beta-1} - 1 < k\frac{\beta-2}{\beta-1} - 1 = \varepsilon_3$. It holds that

$$\varepsilon_2/\varepsilon_1 = \frac{(k - \varepsilon) \cdot \frac{\beta-2}{\beta-1} - 1}{\frac{k-\varepsilon}{2} - 1} < \frac{k \cdot \frac{\beta-2}{\beta-1} - 1}{\frac{k}{2} - 1} < \frac{k\frac{\beta-2}{\beta-1} - 1}{(k-2) \cdot \frac{\beta-2}{\beta-1}} = \varepsilon_3/\varepsilon_4.$$

We can now distinguish four cases. First, assume $\Delta \geq 1$. If $\varepsilon_1 \leq \varepsilon_4$, then $\Delta^{-1/\varepsilon_1} \leq \Delta^{-1/\varepsilon_4}$, which implies Equation (7). If $\varepsilon_1 > \varepsilon_4$, we need to ensure

$$\Delta \leq n^{(\frac{\varepsilon_3}{\varepsilon_4} - \frac{\varepsilon_2}{\varepsilon_1})/(\frac{1}{\varepsilon_4} - \frac{1}{\varepsilon_1})}.$$

This is already the case, since we assume $\Delta \in o(n^{\varepsilon_2})$ and $\varepsilon_2 \leq (\frac{\varepsilon_3}{\varepsilon_4} - \frac{\varepsilon_2}{\varepsilon_1})/(\frac{1}{\varepsilon_4} - \frac{1}{\varepsilon_1})$ due to $\varepsilon_1 > \varepsilon_4$ and $\varepsilon_3 \geq \varepsilon_2$. Thus, Equation (7) holds.

Now assume $\Delta < 1$. If $\varepsilon_1 \leq \varepsilon_4$, we need to ensure that

$$\Delta \geq n^{(\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_3}{\varepsilon_4})/(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_4})}.$$

This already holds, since we assume $\Delta \in \Omega(1)$ and $(\frac{\varepsilon_2}{\varepsilon_1} - \frac{\varepsilon_3}{\varepsilon_4})/(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_4}) \leq 0$ due to $\varepsilon_1 \leq \varepsilon_4$ and $\varepsilon_2/\varepsilon_1 \leq \varepsilon_3/\varepsilon_4$. Thus, Equation (7) holds. If $\varepsilon_4 \leq \varepsilon_1$, then $\Delta^{-1/\varepsilon_1} \leq \Delta^{-1/\varepsilon_4}$ and Equation (7) holds as well.

Now consider $\beta = 3$. We need to show that

$$n/(\ln n)^{\frac{\varepsilon_1+1}{\varepsilon_1}} \Delta^{-1/\varepsilon_1} = n/(\ln n)^{\frac{k-\varepsilon}{k-\varepsilon-2}} \Delta^{-2/(k-\varepsilon-2)} \in O(n \cdot (\Delta \cdot \ln n)^{-2/(k-2)}).$$

Again, the left-hand side is from Lemma 5.2 and the right-hand side is from Lemma 5.1. This holds, due to our assumption $\Delta \in \Omega(1)$ and since $\varepsilon_1 = \frac{k-\varepsilon}{2} - 1 > 0$ implies $0 < \varepsilon < k - 2$ and thus $\frac{k-\varepsilon}{k-\varepsilon-2} > \frac{2}{k-\varepsilon-2} > \frac{k-\varepsilon}{k-2}$. Additionally, the bound only holds up to $\Delta \in o(n^{(k-\varepsilon-2)/2}/\ln^{(k-\varepsilon)/2}(n)) \subseteq o(n^{(k-2)/2}/\log^{(k-2)/2+1}(n))$.

For $\beta > 3$ we have to show

$$n \cdot \Delta^{-1/\varepsilon_1} \in O(n \cdot \Delta^{-1/\varepsilon_4})$$

as well as $\Delta \in o(n^{\varepsilon_1}) \subseteq o((n/\log n)^{\varepsilon_4})$. This holds since $\varepsilon_1 = \frac{k-\varepsilon}{2} - 1 \leq (k-2)\frac{\beta-2}{\beta-1} = \varepsilon_4$ due to $\beta > 3$. This shows that in all three cases the bounds from Lemma 5.2 are smaller, thus giving us the resulting lower bounds on resolution width. \square

This is nearly the statement of Theorem 5.8. However, we can already show linear resolution width at constant clause-variable ratios for $\beta > \frac{2k-2}{k-2}$ instead of $\beta > 3$ via bipartite expansion. This gives a better bound for $k \geq 5$. The bounds on bipartite expansion and the resulting bounds on resolution width will be derived in the next section.

5.2 A Lower Bound on Bipartite Expansion

In this section we show an improved bound on the bipartite expansion. We will use it to obtain a lower bound for the resolution width that is already linear for $\beta \leq \frac{2k-2}{k-2}$, which is potentially smaller than 3, and therefore improve the previous bound. Recall that linear resolution width implies exponential resolution size, and thus also exponential tree-like resolution size. Moreover, our bound on the bipartite expansion can also be used to bound the so-called resolution clause space, which additionally yields an exponential lower bound for the resolution size even if $\beta \leq \frac{2k-3}{k-2}$. The following lemma shows the expansion property.

Lemma 5.5. *Let Φ be a random power-law k -SAT formula with n variables, m clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-3}{k-2}$. If $\Delta = m/n \in o(n^\varepsilon / \log^\varepsilon n)$, then there exists an $r \in \Theta(n \cdot \Delta^{-1/\varepsilon})$ such that the clause-variable incidence graph $G(\Phi)$ is an (r, c) -bipartite expander a. a. s. for $c = (k-1) - (1+\varepsilon)\frac{\beta-1}{\beta-2}$.*

Proof. As in the proof of [6, Lemma 5.1], we define a bad event \mathcal{E} , that $G(\Phi)$ is not a $(\Theta(r), c)$ -bipartite expander. If \mathcal{E} happens, then there is a set $C' \subseteq C$ with $1 \leq |C'| \leq r$ such that $|N(C')| < (1+c)|C'|$. Given a set $C' \subseteq C = [m]$ of clause indices with $|C'| = i$ we want to bound the probability P_i that the $k \cdot i$ indices of variables appearing in those clauses contain at most $(1+c) \cdot i$ different variables. Since clauses contain variables without repetition, it holds that P_i is dominated by the probability to draw at most $(1+c) \cdot i$ different variables when drawing $k \cdot i$ Boolean variables independently at random. Now imagine sampling these $k \cdot i$ variables in some arbitrary, but fixed order. It holds that the probability to draw a new variable is at most 1, while the probability to draw an old variable is at most the probability to draw one of the $(1+c) \cdot i$ variables of maximum probability. As before, the sum of these probabilities is denoted by $F((1+c) \cdot i)$. This gives us

$$P_i \leq \binom{m}{i} \cdot \binom{k \cdot i}{(1+c) \cdot i} \cdot 1^{(1+c) \cdot i} \cdot F((1+c) \cdot i)^{k \cdot i - (1+c) \cdot i}.$$

Note that this expression also captures the case that we draw fewer than $(1+c) \cdot i$ different variables, since the probability to draw a new variable is bounded by one and thus also captures the probability that this new variable is in fact an old one. In the case of a power-law distribution, we have

$$F((1+c) \cdot i) \sim \left(\frac{(1+c) \cdot i}{n} \right)^{\frac{\beta-2}{\beta-1}}$$

due to Lemma A.1 and thus

$$\begin{aligned} P_i &\leq \binom{m}{i} \cdot \binom{k \cdot i}{(1+c) \cdot i} \cdot \left(\frac{(1+c) \cdot i}{n} \right)^{(k-(1+c)) \cdot \frac{\beta-2}{\beta-1} \cdot i} \\ &\leq \left(\frac{e \cdot m}{i} \right)^i \cdot \left(\frac{e \cdot k}{1+c} \right)^{(1+c) \cdot i} \cdot \left(\frac{(1+c) \cdot i}{n} \right)^{(k-(1+c)) \cdot \frac{\beta-2}{\beta-1} \cdot i} \\ &= \kappa(c, \beta, k)^i \cdot \Delta^i \left(\frac{i}{n} \right)^{i((k-(1+c)) \cdot \frac{\beta-2}{\beta-1} - 1)} \\ &= \kappa(c, \beta, k)^i \cdot \Delta^i \left(\frac{i}{n} \right)^{i \cdot \varepsilon} \end{aligned}$$

for some constant $\kappa(c, \beta, k) > 0$, $m = \Delta \cdot n$, and $c = (k-1) - (1+\varepsilon)\frac{\beta-1}{\beta-2}$.

Summing over all $i \geq 1$ now yields

$$\Pr[\mathcal{E}] \leq \sum_{i=1}^r \kappa(c, \beta, k)^i \cdot \Delta^i \cdot \left(\frac{i}{n} \right)^{i \cdot \varepsilon}$$

We split this sum into two parts, the first part from $i = 1$ to $\lfloor \varepsilon \cdot \log n \rfloor$ and the second part

from $\lceil \varepsilon \cdot \log n \rceil$ to r . For the first part we get

$$\begin{aligned} \sum_{i=1}^{\lceil \varepsilon \cdot \log n \rceil} \kappa(c, \beta, k)^i \cdot \Delta^i \cdot \left(\frac{i}{n}\right)^{i \cdot \varepsilon} &\leq \sum_{i=1}^{\lceil \varepsilon \cdot \log n \rceil} \kappa(c, \beta, k)^i \cdot \Delta^i \cdot \left(\frac{\varepsilon \cdot \log n}{n}\right)^{i \cdot \varepsilon} \\ &\leq 2 \cdot \kappa(c, \beta, k) \cdot \Delta \cdot \left(\frac{\varepsilon \cdot \log n}{n}\right)^\varepsilon \\ &\in O\left(\Delta \left(\frac{\log n}{n}\right)^\varepsilon\right), \end{aligned}$$

which holds, since $\sum_{i=1}^m \alpha^i \leq 2 \cdot \alpha$ for all $m \geq 1$ and $\alpha < \frac{1}{2}$. This holds for big enough values of n and for $\Delta \in o(n^\varepsilon / \log^\varepsilon n)$. For the second part we get

$$\sum_{i=\lceil \varepsilon \cdot \log n \rceil}^r \kappa(c, \beta, k)^i \cdot \Delta^i \cdot \left(\frac{i}{n}\right)^{i \cdot \varepsilon} \leq \sum_{i=\lceil \varepsilon \cdot \log n \rceil}^r 2^{-i} \in O\left(\left(\frac{1}{n}\right)^\varepsilon\right),$$

which holds if we choose

$$r \in O\left(n \cdot \Delta^{-1/\varepsilon}\right)$$

small enough so that $\Delta \cdot \left(\frac{r}{n}\right)^\varepsilon < \frac{1}{2 \cdot \kappa(c, \beta, k)}$. \square

This notion of bipartite expansion is connected to the resolution width of a formula. The following corollary, implicitly stated by Ben-Sasson and Wigderson [7], formalizes this connection.

Corollary 5.6 ([7]). *Let $k \geq 3$ integer and constant, let $\varepsilon > 0$ constant, and let Φ be an unsatisfiable k -CNF. If $G(\Phi)$ is a $\left(r, \frac{k+\varepsilon}{2} - 1\right)$ -bipartite expander, then Φ has resolution width at least $\Omega(r)$.*

Proof. Due to the definition of bipartite expansion, $\frac{k+\varepsilon}{2} > 1$ ensures the first condition of Theorem 5.3. We will show that the second condition is fulfilled as well. Let $G(\Phi) = (C, V, E)$ and let $C' \subseteq C$ with $\frac{1}{3}r \leq |C'| \leq \frac{2}{3}r$. Let $\delta C'$ denote the set of unique variables from C' , i. e. $\delta C' = \{v \in N(C') \mid |N(v) \cap C'| = 1\}$. As Ben-Sasson and Wigderson state in [7, proof of Theorem 6.5] it holds that:

$$|N(C')| - |\delta C'| \leq (k \cdot |C'| - |\delta C'|)/2,$$

which implies

$$|\delta C'| \geq 2|N(C')| - k \cdot |C'| \geq \varepsilon \cdot |C'|$$

due to the $\left(r, \frac{k+\varepsilon}{2} - 1\right)$ -bipartite expansion. These two properties imply a resolution width of $\Omega(r)$. \square

This result on the bipartite expansion of power-law random k -SAT allows us to derive the following corollary on resolution width. Again, we require the clause-variable ratio Δ to be high enough for instances to be unsatisfiable with at least constant probability.

Corollary 5.7. *Let Φ be a random power-law k -SAT formula with n variables, $m \in \Omega(n)$ clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-2}{k-2}$. Let $\Delta = m/n$ be large enough so that Φ is unsatisfiable at least with constant probability. For $0 < \varepsilon < \left(\frac{k}{2} - 1\right) \frac{\beta-2}{\beta-1} - 1$ constant and $\Delta \in o(n^\varepsilon / \log^\varepsilon n)$ it holds a. a. s. that Φ has resolution width $w \in \Omega(n \cdot \Delta^{-1/\varepsilon})$.*

Proof. Due to Lemma 5.5 $G(\Phi)$ is a $(\Omega(n \cdot \Delta^{-1/\varepsilon}), c)$ -bipartite expander for $c = (k-1) - (1+\varepsilon)\frac{\beta-1}{\beta-2}$. With $\beta > \frac{2k-2}{k-2}$, it holds that we can choose an $\varepsilon > 0$ so that $c > \frac{k}{2} - 1$. This means, the requirement of Corollary 5.6 is fulfilled and implies the statement. \square

Together with Corollary 5.4 the former corollary implies Theorem 5.8.

Theorem 5.8. *Let Φ be a random power-law k -SAT formula with n variables, $m \in \Omega(n)$ clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-1}{k-1}$. Let $\Delta = m/n$ be large enough so that Φ is unsatisfiable at least with constant probability. Let $\varepsilon, \varepsilon_1, \dots, \varepsilon_3$ be constants with $\varepsilon > 0$, $\varepsilon_1 = \frac{k-\varepsilon}{2} - 1 > 0$, $\varepsilon_2 = (k-\varepsilon)\frac{\beta-2}{\beta-1} - 1 > 0$, and $0 < \varepsilon_3 < (\frac{k}{2} - 1)\frac{\beta-2}{\beta-1} - 1$. For the resolution width w of Φ , it holds a. a. s. that:*

(i) *If $\beta \in (\frac{2k-1}{k-1}, 3)$ and $\Delta \in o(n^{\varepsilon_2})$, then $w \in \Omega(n^{\varepsilon_2/\varepsilon_1} \Delta^{-1/\varepsilon_1})$.*

(ii) *If $\beta = 3$ and $\Delta \in o(n^{\varepsilon_1/\log^{1+\varepsilon_1} n})$, then $w \in \Omega(n \cdot \Delta^{-1/\varepsilon_1} / \log^{1+1/\varepsilon_1} n)$.*

(iii) *If $\beta > 3$ and $\Delta \in o(n^{\varepsilon_1})$, then $w \in \Omega(n \cdot \Delta^{-1/\varepsilon_1})$.*

(iv) *If $\beta > \frac{2k-2}{k-2}$ and $\Delta \in o(n^{\varepsilon_3} / \log^{\varepsilon_3} n)$, then $w \in \Omega(n \cdot \Delta^{-1/\varepsilon_3})$.*

Additionally, Ben-Sasson and Galesi [6] state a theorem that directly connects bipartite expansion and tree-like resolution size. An application of this theorem yields a slightly better bound on tree-like resolution size than the ones derived from resolution width.

Theorem 5.9 ([6]). *Let Φ be an unsatisfiable CNF and let $G(\Phi) = (U \cup V, E)$ be the clause-variable incidence graph of Φ . If $G(\Phi)$ is a (r, c) -bipartite expander then Φ has resolution clause space of at least $\frac{c \cdot r}{2+c}$ and tree-like resolution size of at least $\exp(\frac{c \cdot r}{2+c})$.*

Proof. [6, Theorems 4.2 and 3.3] state together that any bipartite graph that is an (r, c) -bipartite expander has a resolution clause space of at least $\frac{c \cdot r}{2+c}$. Thus, with [27, Theorem 1.6], it holds that the resolution size for formulas whose clause-variable incidence graph is an (r, c) -bipartite expander, is at least $\exp(\frac{c \cdot r}{2+c})$. \square

This leads to the following corollary, which already asserts exponential tree-like resolution size for constant clause-variable ratios at $\beta > \frac{2k-3}{k-2}$.

Corollary 5.10. *Let Φ be a random power-law k -SAT formula with n variables, $m = \Omega(n)$ clauses, $k \geq 3$, and power-law exponent $\beta > \frac{2k-3}{k-2}$. Let $\Delta = m/n$ be large enough so that Φ is unsatisfiable at least with constant probability. For $0 < \varepsilon < (k-1)\frac{\beta-2}{\beta-1} - 1$ constant and $\Delta \in o((n/\log n)^\varepsilon)$, it holds that Φ has tree-like resolution size $\exp(\Omega(n \cdot \Delta^{-1/\varepsilon}))$.*

Proof. Using Lemma 5.5 we see that for $\beta > \frac{2k-3}{k-2}$ the clause-variable incidence graph a. a. s. is a $(\Theta(n \cdot \Delta^{-1/\varepsilon}), c)$ -bipartite expander for some constant $c > 0$. Thus, Theorem 5.9 implies the statement. \square

6 The Complexity of Voronoi Diagrams

We first show quadratic lower bounds on the complexity (number of non-empty regions) of order- k Voronoi diagrams that already hold in rather basic settings. Afterwards, we consider random point sets and prove a linear upper bound.

6.1 Worst-Case Lower Bounds

In this section, we show worst-case lower bounds on the number of non-empty regions of higher-order Voronoi diagrams. As already mentioned in Section 2.3, our lower bounds are based on previously known lower bounds on the number of vertices of Voronoi, in conjunction with a new theorem connecting the number of vertices with the number of regions in higher orders. This theorem relies on the fact that there are not too many different points with equal distance to a set of $d + 1$ sites in d -dimensional space. For the unweighted case and for $\mathfrak{p} \neq \infty$, the result in the next lemma was shown by Lê [40]. We extend it to weighted sites and $\mathfrak{p} = \infty$, following along the lines of Lê's proof [40] (at least for $\mathfrak{p} \neq \infty$): (i) Observe that the points with equal distance to the $d + 1$ sites is the set of solutions to a system of polynomial equations. (ii) Show that the so-called *additive complexity* of these polynomial equations is bounded by a constant only depending on d . (iii) Apply [40, Proposition 3], giving an upper bound on the number of solutions to a system of equations that only depends on d and on the additive complexities of the equations.

Lemma 6.1. *Let A be a set of $d + 1$ weighted sites in general position⁵ in \mathbb{R}^d equipped with a \mathfrak{p} -norm. Then, the number of points with equal weighted distance to all sites in A only depends on d .*

Proof. Assume $\mathfrak{p} \neq \infty$, and let $\mathbf{s}_0, \dots, \mathbf{s}_d$ be $d + 1$ sites with normalized weights $\omega_0, \dots, \omega_d$. Recall that the weighted distance between \mathbf{s}_i and a point \mathbf{p} is $\|\mathbf{s}_i - \mathbf{p}\|/\omega_i$. Thus, \mathbf{p} has the same distance to all $d + 1$ sites if, for all $i \in [d]$, it satisfies

$$\frac{\|\mathbf{s}_0 - \mathbf{p}\|}{\omega_0} - \frac{\|\mathbf{s}_i - \mathbf{p}\|}{\omega_i} = 0. \quad (8)$$

We note that this polynomial has the same form in the unweighted case [40, Equation 10], except we have the additional factors $1/\omega_0$ and $1/\omega_i$.

Concerning (ii), it thus suffices to note that these additional factors do not significantly increase the so-called additive complexity. We do not fully define the additive complexity here, but rather cite the properties crucial for this proof. The additive complexity $L_+(P)$ of a polynomial P is defined to be 0 if P is a monomial. Moreover, by [40, Lemma 4], it holds that

$$\begin{aligned} L_+(P_1 + \dots + P_n) &\leq n - 1 + L_+(P_1) + \dots + L_+(P_n), \\ L_+(P^m) &\leq L_+(P), \text{ for any } m \in \mathbb{N}, \text{ and} \\ L_+(PQ) &\leq L_+(P) + L_+(Q), \end{aligned}$$

where all P_i , P , and Q are polynomials. With this, it is easy to see that the additive complexity of the polynomial in Equation (8) is bounded by a constant only depending on d . In fact, the last bound, $L_+(PQ) \leq L_+(P) + L_+(Q)$ in conjunction with the property that constants are monomials with additive complexity 0, makes it so that the additional constant factors ω_0 and ω_i do not increase the additive complexity at all. Thus, the additive complexity is bounded by $4d - 1$ [40, Lemma 5].

Finally, applying [40, Proposition 3] directly yields the claim, which concludes the proof for $\mathfrak{p} \neq \infty$.

For $\mathfrak{p} = \infty$, we cannot use the same argument, as Equation (8) is not polynomial: $\|\mathbf{s}_i - \mathbf{p}\|$ involves the maximum over all coordinates. However, for each s_i , there are only d possibilities to which coordinate the maximum is evaluated, leading to d^{d+1} combinations. For each of

⁵For a formal definition what *general position* means in this context, see [40]. As usual, the configurations excluded by the assumption of general position have measure 0.

these combinations, we consider its own system of equations. Denote the resulting set of systems of equations with \mathcal{E} . Clearly, every solution for the system of equations in (8) is a solution to at least one system in \mathcal{E} . Thus, the number of solutions to (8) is bounded by the total number of solutions to systems in \mathcal{E} . Clearly, with the same argument as above, the number of solutions to each system of equations in \mathcal{E} is bounded by a constant only depending on d . As \mathcal{E} contains only d^{d+1} systems, this bounds the number of solutions to (8) by a constant only depending on d . \square

With this, we can now prove the theorem establishing the connection between vertices and non-empty regions.

Theorem 6.2. *Let S be a set of n weighted sites in general position in \mathbb{R}^d equipped with a \mathbf{p} -norm. If the order- k Voronoi diagram has ℓ vertices, then the order- $(k + d)$ Voronoi diagram has $\Omega(\ell)$ non-empty regions.*

Proof. We first show that a vertex of the order- k Voronoi diagram is an interior point of a non-empty region of the order- $(k + d)$ Voronoi diagram. Afterwards, we show that only a constant number of different vertices can end up in the same region.

Let $\mathbf{p} \in \mathbb{R}^d$ be a vertex of the order- k Voronoi diagram. Then \mathbf{p} has equal weighted distance to exactly $d + 1$ sites (the sites are in general position). Let $\{\mathbf{s}_1, \dots, \mathbf{s}_{d+1}\} = A \subseteq S$ be these sites and let P be the ε -environment of \mathbf{p} , i.e., a ball with sufficiently small radius ε centered at \mathbf{p} . For a point $\mathbf{p}' \in P$, sort all sites in S by weighted distance from \mathbf{p}' . Then all sites in A appear consecutive in this order. Moreover, we obtain almost the same order of S for every $\mathbf{p}' \in P$. The only difference is that the sites of A might be reordered. Also, as \mathbf{p} is a vertex of the order- k Voronoi diagram, at least one site from A belongs to the k sites with smallest weighted distance to \mathbf{p} . It follows that the first $k + d$ sites in this order completely include all sites from A . Thus, the $k + d$ closest sites are the same for all points in the ε -environment P around \mathbf{p} ; let B be the set of these sites. It follows that B has non-empty Voronoi region in the order- $(k + d)$ Voronoi diagram as this region has \mathbf{p} in its interior.

It remains to show that only a constant number of vertices of the order- k Voronoi diagram can be contained in the same region of the order- $(k + d)$ Voronoi diagram, i.e., the order- $(k + d)$ region belonging to B includes only a constant number of order- k vertices. As stated above, every order- k vertex belongs to a subset $A \subseteq B$ with $|A| = d + 1$. There are only $\binom{|B|}{|A|} \leq \binom{k+d}{d+1}$ such subsets A , which is constant for constant k and d . Moreover, every fixed subset A of $d + 1$ sites is responsible for only a constant number of vertices due to Lemma 6.1. Thus, only a constant number of order- k vertices end up in the same order- $k + d$ region, which concludes the proof. \square

Theorem 6.2 transfers some known lower bounds on the number of vertices of Voronoi diagrams to lower bounds on the number of non-empty regions of order- k Voronoi diagrams. In particular, we get the following corollaries.

Corollary 6.3. *In the worst case, the order-4 Voronoi diagram of n (unweighted) sites in 3-dimensional Euclidean space has $\Omega(n^2)$ non-empty regions.*

Proof. In the worst case, the ordinary (order-1, unweighted) Voronoi diagram of n sites in 3-dimensional Euclidean space has $\Omega(n^2)$ vertices [38, 50]. Applying Theorem 6.2 yields the claim. \square

Corollary 6.4. *In the worst case, the order-3 Voronoi diagram of n weighted sites in 2-dimensional Euclidean space has $\Omega(n^2)$ non-empty regions.*

Proof. In the worst case, the order-1 Voronoi diagram of n weighted sites in 2-dimensional Euclidean space has $\Omega(n^2)$ vertices [4]; also see Figure 2. Applying Theorem 6.2 yields the claim. \square

6.2 Upper Bounds for Sites with Random Positions

Let $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\} \subseteq \mathbb{T}^d$ be n randomly positioned sites with weights w_1, \dots, w_n . In the following, we bound the complexity of the weighted order- k Voronoi diagram in terms of non-empty regions. Recall from Section 3 that the torus \mathbb{T}^d is the hypercube $[0, 1]^d$ that wraps around in every dimension in the sense that opposite sides are identified. However, the following arguments do not require this property. Thus, the exact same results hold for Voronoi diagrams in hypercubes.

For the normalized weights $\omega_1, \dots, \omega_n$, recall from Section 3, that the point $\mathbf{p} \in \mathbb{T}^d$ belongs to the Voronoi region corresponding to $A \subseteq S$ with $|A| = k$ if there exists a radius r such that $\|\mathbf{p} - \mathbf{s}_i\| \leq \omega_i r$ if $\mathbf{s}_i \in A$ and $\|\mathbf{p} - \mathbf{s}_i\| > \omega_i r$ if $\mathbf{s}_i \notin A$. Thus, A has non-empty order- k Voronoi region if and only if there exists such a point \mathbf{p} . Our goal in the following is to bound the probability for its existence.

Our general approach to achieve such a bound is the following. The condition $\|\mathbf{p} - \mathbf{s}_i\| \leq \omega_i r$ for $\mathbf{s}_i \in A$ basically tells us the sites in A are either close together or that r has to be large. In contrast to that, the condition $\|\mathbf{p} - \mathbf{s}_i\| > \omega_i r$ for $\mathbf{s}_i \notin A$ tells us that many sites (namely all $n - k$ sites in $S \setminus A$) have to lie sufficiently far away from \mathbf{p} , which is unlikely if r is large. How unlikely this is of course depends on r and thus on how close the sites in A lie together. Therefore, to follow this approach, we first condition on how close the sites in A lie together.

To formalize this, consider a size- k subset $A \subseteq S$ and assume without loss of generality that $A = \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$. The site in A with the lowest weight, without loss of generality \mathbf{s}_1 , will play a special role. We define the random variable R_A to be

$$R_A = \max_{i \in [k]} \frac{\|\mathbf{s}_1 - \mathbf{s}_i\|}{\omega_1 + \omega_i}. \quad (9)$$

The intuition behind the definition of R_A is the following. The weighted center between \mathbf{s}_1 and \mathbf{s}_i is the point \mathbf{p} on the line between them such that $\|\mathbf{s}_1 - \mathbf{p}\| = \omega_1 r$ and $\|\mathbf{s}_i - \mathbf{p}\| = \omega_i r$ for a radius $r \in \mathbb{R}$. Then R_A is the maximum value for r over $i \in [k]$. In the unweighted setting, R_A is just half the maximum distance between \mathbf{s}_1 and any other site \mathbf{s}_i . In a sense, R_A describes how close the sites in A lie together. Thus, it provides a lower bound on r .

Based on R_A , we slightly relax the condition on A having non-empty Voronoi region. We call A *relevant* if there exists a point $\mathbf{p} \in \mathbb{T}^d$ and a radius $r \geq R_A$ such that $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 r$ and $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i r$ for $i > k$. The following lemma states that being relevant is in fact a weaker condition than having non-empty order- k Voronoi region. Thus, bounding the probability that a set is relevant from above also bounds the probability for a non-empty Voronoi region from above.

Lemma 6.5. *A subset of k sites that has a non-empty order- k Voronoi region is relevant.*

Proof. Assume $A = \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$ has a non-empty order- k Voronoi region. Then there exists a point \mathbf{p} and a radius r such that $\|\mathbf{s}_i - \mathbf{p}\| \leq \omega_i r$ if and only if $i \leq k$. Thus, $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 r$ and $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i r$ for $i > k$ clearly holds, and it remains to show $r \geq R_A$. From $\|\mathbf{s}_i - \mathbf{p}\| \leq \omega_i r$ for $i \in [k]$ it follows that $\|\mathbf{s}_1 - \mathbf{p}\| + \|\mathbf{s}_i - \mathbf{p}\| \leq \omega_1 r + \omega_i r$ holds for any $i \in [k]$. Thus, by rearranging and applying the triangle inequality, we obtain $r \geq (\|\mathbf{s}_1 - \mathbf{p}\| + \|\mathbf{s}_i - \mathbf{p}\|) / (\omega_1 + \omega_i) \geq \|\mathbf{s}_1 - \mathbf{s}_i\| / (\omega_1 + \omega_i)$. This immediately yields $r \geq R_A$. \square

Now we proceed to bound the probability that a set A is relevant. The following lemma bounds this probability conditioned on the random variable R_A . At its core, we have to bound the probability of the event $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i r$ for $\mathbf{s}_i \notin A$. For a fixed point \mathbf{p} and a fixed radius r , this is rather easy. Thus, most of the proof is concerned with eliminating the existential quantifiers for \mathbf{p} and r .

Lemma 6.6. For constants c_1 and c_2 depending only on d and \mathbf{p} , it holds that

$$\Pr [A \text{ is relevant} \mid R_A] \leq c_1 \min_{\mathbf{s}_i \in A} \{w_i\} \exp \left(-c_2 R_A^d \sum_{\mathbf{s}_i \notin A} w_i \right).$$

Proof. As before, we assume that $A = \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$ and that \mathbf{s}_1 has minimum weight among sites in A , i.e., $\min_{\mathbf{s}_i \in A} \{w_i\} = w_1$. By definition, A is relevant conditioned on R_A , if and only if there exists a radius $r \geq R_A$ and point $\mathbf{p} \in \mathbb{T}^d$ such that $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 r$ and $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i r$ for $i > k$, i.e., formally we have

$$\exists r \geq R_A \exists \mathbf{p} \in \mathbb{T}^d \forall i > k: \|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 r \wedge \|\mathbf{s}_i - \mathbf{p}\| > \omega_i r. \quad (10)$$

The core difficulties of bounding the probability for this event are the existential quantifiers that quantify over the continuous variables r and \mathbf{p} . In both cases, we resolve this by using an appropriate discretization, for which we then apply the union bound.

We get rid of the existential quantifier for r by dividing the interval $[R_A, \infty)$, which covers the domain of r , into pieces of length at most R_A . More formally, we split the event $\exists r \geq R_A$ with the desired property into the disjoint events $\exists r \in [jR_A, (j+1)R_A)$ for $j \in \mathbb{N}^+$. For a fixed j , $r \geq jR_A$ and $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i r$ implies $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i jR_A$. Moreover, $r \leq (j+1)R_A$ and $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 r$ implies $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 (j+1)R_A$. Note that this completely eliminates the variable r from the event, which lets us drop the existential quantifier for r . Thus, the event in Equation (10) implies

$$\exists j \in \mathbb{N}^+ \exists \mathbf{p} \in \mathbb{T}^d \forall i > k: \|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 (j+1)R_A \wedge \|\mathbf{s}_i - \mathbf{p}\| > \omega_i jR_A. \quad (11)$$

Note that the new existential quantifier for j is not an issue: as j is discrete, we can simply use the union bound and sum over the probabilities we obtain for the different values of j . We will later see that this sum is dominated by the first term corresponding to $j = 1$.

To deal with the existential quantifier for \mathbf{p} , assume $j \in \mathbb{N}^+$ to be a fixed number. First note that $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 (j+1)R_A$ implies that \mathbf{p} lies somewhat close to \mathbf{s}_1 . We discretize the space around \mathbf{s}_1 using a grid such that the point \mathbf{p} is guaranteed to lie inside a grid cell. By choosing the distance between neighboring grid vertices sufficiently small, we guarantee that \mathbf{p} lies close to a grid vertex. Then, instead of considering \mathbf{p} itself, we deal with its closest grid vertex. To define the grid formally, let $\omega_{\min} = \min_{i=k+1}^n \{w_i\}$ be the minimum weight of sites not in A and let $x = \omega_{\min} j R_A / \sqrt[d]{d}$ (x will be the width of our grid cells). To simplify notation, assume that \mathbf{s}_1 is the origin. Otherwise, we can simply translate the grid defined in the following to be centered at \mathbf{s}_1 to obtain the same result. Let $\Gamma = \{\ell x \mid \ell \in \mathbb{Z} \wedge |(\ell-1)x| \leq \omega_1 (j+1)R_A\}$ be the set of all multiples of x that are not too large. We use the grid defined by the Cartesian product Γ^d . Then the following three properties of Γ^d are easy to verify.

- (i) A point \mathbf{p} with $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 (j+1)R_A$ lies in a grid cell.
- (ii) The maximum distance between a point in a grid cell and its closest grid vertex is $\sqrt[d]{d}x/2 = \omega_{\min} j R_A / 2$.
- (iii) Γ^d has at most $c'_1 (\omega_1 j R_A / x)^d = c_1 (\omega_1 / \omega_{\min})^d \leq c_1 \omega_1^d$ vertices for constants c_1 and c'_1 only depending on d and \mathbf{p} .

Going back to the event in Equation (11), let \mathbf{p} be a point with $\|\mathbf{s}_1 - \mathbf{p}\| \leq \omega_1 (j+1)R_A$ and $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i j R_A$ (for all $i > k$). By the first inequality and Property i, \mathbf{p} lies in a grid

cell of Γ^d . Let $\mathbf{p}' \in \Gamma^d$ be the grid vertex with minimum distance to \mathbf{p} . Then, by Property ii, $\|\mathbf{p} - \mathbf{p}'\| \leq \omega_{\min} j R_A / 2$. Thus, using the triangle inequality and $\|\mathbf{s}_i - \mathbf{p}\| > \omega_i j R_A$, we obtain

$$\|\mathbf{s}_i - \mathbf{p}'\| \geq \|\mathbf{s}_i - \mathbf{p}\| - \|\mathbf{p} - \mathbf{p}'\| > \omega_i j R_A - \frac{\omega_{\min} j R_A}{2} \geq \frac{\omega_i j R_A}{2}.$$

It follows that the event in Equation (11) implies

$$\exists j \in \mathbb{N}^+ \exists \mathbf{p}' \in \Gamma^d \forall i > k: \|\mathbf{s}_i - \mathbf{p}'\| > \frac{\omega_i j R_A}{2}.$$

For this event, we can now bound the probability. First note that $\|\mathbf{s}_i - \mathbf{p}'\| > \omega_i j R_A / 2$ implies that the ball of radius $\omega_i j R_A / 2$ around \mathbf{p}' does not contain \mathbf{s}_i . By Lemma A.3, the volume of this ball intersected with $[-0.5, 0.5]^d$ is $\min\{1, c_2 (\omega_i j R_A)^d\}$ for a constant c_2 depending only on d and \mathbf{p} . As the \mathbf{s}_i are chosen independently and using that $1 - x \leq \exp(-x)$ for $0 \leq x \leq 1$, we obtain

$$\begin{aligned} \Pr \left[\forall i > k: \|\mathbf{s}_i - \mathbf{p}'\| > \frac{\omega_i j R_A}{2} \right] &= \prod_{i=k+1}^n \max \left\{ 0, 1 - c_2 (\omega_i j R_A)^d \right\} \\ &\leq \prod_{i=k+1}^n \exp \left(-c_2 (\omega_i j R_A)^d \right) \\ &= \exp \left(-c_2 j^d R_A^d \sum_{i=k+1}^n \omega_i^d \right). \end{aligned}$$

We resolve the two existential quantifiers for j and \mathbf{p}' using the union bound. Recall from Property iii that the grid Γ^d contains only $c_1 \omega_1^d$ vertices. Using that $\omega_i = w_i^{1/d}$, we obtain

$$\begin{aligned} \Pr [A \text{ is relevant} \mid R_A] &\leq \Pr \left[\exists j \in \mathbb{N}^+ \exists \mathbf{p}' \in \Gamma^d \forall i > k: \|\mathbf{s}_i - \mathbf{p}'\| > \frac{\omega_i j R_A}{2} \right] \\ &\leq \sum_{j=1}^{\infty} c_1 w_1 \exp \left(-c_2 j^d R_A^d \sum_{i=k+1}^n w_i \right). \end{aligned}$$

To conclude the proof, it remains to show that the sum over j is dominated by the first term corresponding to $j = 1$. For this, note that

$$\begin{aligned} \sum_{j=1}^{\infty} \exp(-x j^d) &= \exp(-x) \cdot \sum_{j=1}^{\infty} \frac{\exp(-x j^d)}{\exp(-x)} \\ &= \exp(-x) \cdot \sum_{j=1}^{\infty} \exp \left(-x (j^d - 1) \right) \\ &\leq \exp(-x) \cdot \sum_{j=1}^{\infty} (\exp(-x))^{j-1} \end{aligned}$$

As x is positive in our case, the sum is bounded by a constant due to the convergence of the geometric series. This concludes the proof. \square

Now that we know the probability that $A \subseteq S$ is relevant conditioned on R_A , we want to understand how R_A is distributed. The following lemma gives an upper bound on its density function.

Lemma 6.7. *There exists a constant c depending only on k , d , and \mathfrak{p} , such that the density function $f_{R_A}(x)$ of the random variable R_A satisfies*

$$f_{R_A}(x) \leq cx^{dk-d-1} \frac{1}{\min_{\mathbf{s}_i \in A} \{w_i\}} \prod_{\mathbf{s}_i \in A} w_i.$$

Proof. The density function $f_{R_A}(x)$ is the derivative of the distribution function $F_{R_A}(x) = \Pr[R_A \leq x]$. Thus, we have to upper bound the slope of $\Pr[R_A \leq x]$. As before, we assume that $A = \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$ and that \mathbf{s}_1 has minimum weight among sites in A , i.e., $\min_{\mathbf{s}_i \in A} \{w_i\} = w_1$. Recall the definition of R_A in Equation (9). It follows directly that $R_A \leq x$ if and only if $\|\mathbf{s}_1 - \mathbf{s}_i\|/(\omega_1 + \omega_i) \leq x$ for all $i \in [k]$. Note that this clearly holds for $i = 1$. For greater i , this is the case if and only if \mathbf{s}_i lies in the ball $B_{\mathbf{s}_1}((\omega_1 + \omega_i)x)$ of radius $(\omega_1 + \omega_i)x$ around \mathbf{s}_1 . To simplify notation, we denote this ball with $B(x_i)$ in the following. Note that the volume $\text{vol}(B(x_i))$ is exactly the probability for \mathbf{s}_i to lie sufficiently close to \mathbf{s}_1 . As the positions of the different sites \mathbf{s}_i are independent, we obtain

$$F_{R_A}(x) = \Pr[R_A \leq x] = \prod_{i=2}^k \text{vol}(B(x_i)).$$

To upper bound the derivative of this, we have to upper bound the growth of $\text{vol}(B(x_i))$ depending on x_i . For sufficiently small x_i , this volume is given by the volume of a ball in \mathbb{R}^d . For larger x_i , due to the fact that our ground space⁶ is bounded, the growth of this volume slows down. Thus, to get an upper bound on the derivative, we can simply use the volume of a ball in \mathbb{R}^d . Thus, for appropriate constants c_1 and c_2 only depending on d and \mathfrak{p} , we obtain

$$\frac{d}{dx} \text{vol}(B(x_i)) \leq \frac{d}{dx} c_1 ((\omega_1 + \omega_i)x)^d \leq \frac{d}{dx} c_2 (\omega_i x)^d = \frac{d}{dx} c_2 w_i x^d.$$

With this, it follows that

$$f_{R_A}(x) = \frac{d}{dx} F_{R_A}(x) \leq \frac{d}{dx} \prod_{i=2}^k (c_2 w_i x^d),$$

which immediately yields the claimed bound. \square

By Lemma 6.6, we know the probability for a set A to be relevant conditioned on R_A and by Lemma 6.7 we know how R_A is distributed. Based on this, we can bound the unconditional probability that A is relevant.

Lemma 6.8. *Let $A \subseteq S$. For a constant c only depending on k , d , and \mathfrak{p} , the probability that A is relevant satisfies*

$$\Pr[A \text{ is relevant}] \leq c \frac{\prod_{\mathbf{s}_i \in A} w_i}{\left(\sum_{\mathbf{s}_i \notin A} w_i\right)^{k-1}}.$$

Proof. Let $A \subseteq S$ and let R_A be the random variable as defined before; see Equation (9). Note that $0 \leq R_A \leq \sqrt[d]{d}$. By the law of total probability, we have

$$\Pr[A \text{ is relevant}] = \int_0^{\sqrt[d]{d}} \Pr[A \text{ is relevant} \mid R_A = x] \cdot f_{R_A}(x) dx.$$

⁶Again, this is true for the torus as well as for the Hypercube.

Using Lemma 6.6 and Lemma 6.7, we obtain

$$\Pr [A \text{ is relevant} \mid R_A = x] \cdot f_{R_A}(x) \leq c_1 \prod_{\mathbf{s}_i \in A} w_i \exp \left(-c_2 x^d \sum_{\mathbf{s}_i \notin A} w_i \right) x^{dk-d-1},$$

for constants c_1 and c_2 only depending on k , d , and \mathbf{p} . Ignoring the factors independent of x for now, this expression has the form

$$x^{\alpha d-1} \exp \left(-\beta x^d \right) \text{ with } \alpha = k-1 \text{ and } \beta = c_2 \sum_{\mathbf{s}_i \notin A} w_i,$$

which lets us apply Lemma A.4 to bound the integral. We obtain

$$\begin{aligned} \Pr [A \text{ is relevant}] &= c_1 \prod_{\mathbf{s}_i \in A} w_i \cdot \int_0^{\sqrt[d]{d}} x^{\alpha d-1} \exp \left(-\beta x^d \right) dx \\ &\leq c_1 \prod_{\mathbf{s}_i \in A} w_i \cdot \frac{\Gamma(\alpha)}{\beta^\alpha d}. \end{aligned}$$

As k is an integer, $\Gamma(\alpha) = \Gamma(k-1) = (k-2)!$, which is constant. Thus, substituting α and β by its corresponding values and aggregating all constant factors into c yields

$$\Pr [A \text{ is relevant}] \leq c \frac{\prod_{\mathbf{s}_i \in A} w_i}{\left(\sum_{\mathbf{s}_i \notin A} w_i \right)^{k-1}},$$

which is exactly the bound we wanted to prove. \square

Having bound the probability that a specific subset of sites $A \subseteq S$ of size k is relevant, we can now bound the expected total number of relevant subsets. By Lemma 6.5, this also bounds the number of non-empty Voronoi regions.

Theorem 6.9. *Let S be a set of n sites with minimum weight 1, total weight W , and random positions on the d -dimensional torus equipped with a \mathbf{p} -norm. For every fixed k , the expected number of regions of the weighted order- k Voronoi diagram of S is in $O(W)$. The same holds for random sites in a hypercube.*

Proof. For every subset $A \subseteq S$ with $|A| = k$, let X_A be the indicator random variable that has value 1 if and only if A has non-empty order- k Voronoi region. Moreover, let X be the sum of these random variables. Note that $\mathbb{E}[X]$ is exactly the quantity, we are interested in. Using linearity of expectation, we obtain

$$\mathbb{E} [\text{number of regions}] = \mathbb{E}[X] = \sum_{\substack{A \subseteq S \\ |A|=k}} \mathbb{E}[X_A]$$

Due to Lemma 6.5, a subset A with non-empty Voronoi region is also relevant. Thus, $\mathbb{E}[X_A] \leq \Pr [A \text{ is relevant}]$ and Lemma 6.8 yields

$$\sum_{\substack{A \subseteq S \\ |A|=k}} \mathbb{E}[X_A] \leq \sum_{\substack{A \subseteq S \\ |A|=k}} c \frac{\prod_{\mathbf{s}_i \in A} w_i}{\left(\sum_{\mathbf{s}_i \notin A} w_i \right)^{k-1}}. \quad (12)$$

For technical reasons, we assume c to be the maximum of 1 and the constant from Lemma 6.8. We continue by proving the following claim:

$$\sum_{\substack{A \subseteq S \\ |A|=k}} \mathbb{E}[X_A] \leq 2^{k^2} cW \quad (13)$$

In addition of implying the theorem, this claim specifies a constant that comes on top of c , which is crucial for the rest of the proof.

We first prove the claim for the situation, in which W is not dominated by the highest k weights. Afterwards, we deal with the other somewhat special case. More formally, let the weights w_1, \dots, w_n be sorted increasingly and consider the case that $\sum_{i=1}^{n-k} w_i \geq 2^{-k}W$, i.e., if we leave out the k largest weights, we still have a significant portion of the total weight. We can use this to estimate the denominator in Equation (12):

$$\begin{aligned} \sum_{\substack{A \subseteq S \\ |A|=k}} c \frac{\prod_{s_i \in A} w_i}{\left(\sum_{s_i \notin A} w_i\right)^{k-1}} &\leq \sum_{\substack{A \subseteq S \\ |A|=k}} c \frac{\prod_{s_i \in A} w_i}{(2^{-k}W)^{k-1}} \\ &= 2^{k(k-1)} c \cdot \frac{\sum_{A \subseteq S, |A|=k} \prod_{s_i \in A} w_i}{W^{k-1}}. \end{aligned}$$

To bound the fraction by W , observe that the binomial theorem yields

$$W^k = \left(\sum_{i=1}^n w_i \right)^k \geq \sum_{\substack{A \subseteq S \\ |A|=k}} \prod_{s_i \in A} w_i,$$

as each summand on the on the right-hand side also appears on the left-hand side. This proves the claim in Equation (13) for the case $\sum_{i=1}^{n-k} w_i \geq 2^{-k}W$.

For $\sum_{i=1}^{n-k} w_i < 2^{-k}W$, assume for contradiction that the claim in Equation (13) does not hold for every set of n weights. Then there exists a minimum counterexample, i.e., a smallest number of n weights such that the expected number of non-empty regions exceeds $2^{k^2} cW$. We show that, based on this assumption, we can construct an even smaller counterexample; a contradiction. First note that $n > 2k$ for every counterexample, as there are less than $2^{k^2} cW$ subsets otherwise (recall that $c \geq 1$).

Now let w_1, \dots, w_n be the minimum counterexample and again assume that the weights are ordered increasingly. Moreover, fix the coordinates of the sites s_1, \dots, s_n and consider two order- k Voronoi diagrams: one on the set of all sites $S = \{s_1, \dots, s_n\}$, and the one on all but the k heaviest sites $S' = \{s_1, \dots, s_{n-k}\}$ (note that this is well defined as $n > 2k$). In the following, we call the former Voronoi diagram \mathcal{V} and the latter \mathcal{V}' . We define a mapping from the non-empty regions of \mathcal{V} to non-empty regions of \mathcal{V}' . Let $A \subseteq \{s_1, \dots, s_n\}$ be a subset of size k with non-empty region in \mathcal{V} and let \mathbf{p} be an arbitrary point in this region. Moreover, let A' be the set of sites corresponding to the region of \mathcal{V}' containing \mathbf{p} . Then we map the region of A to the region of A' . Note that A and A' share all sites that have not been deleted: $A \cap A' = A \cap S'$. Thus, if we have a second non-empty region in \mathcal{V} , corresponding to a subset of sites $B \subseteq S$, that is also mapped to the region corresponding to A' in \mathcal{V}' , then $B \cup S' = A \cup S'$. This limits the number of different regions in \mathcal{V} that are mapped to the same region of \mathcal{V}' to at most 2^k . Thus, the number of regions in \mathcal{V}' is at least 2^{-k} times the number of regions in \mathcal{V} . As this holds for arbitrary coordinates, this also holds for the expected number of non-empty regions when choosing random coordinates.

As we assumed w_1, \dots, w_n to be a counterexample for Equation (13), the expected number of regions with these weights is more than $2^{k^2} cW$. Thus, by the above argument, the expected

number of regions for the weights w_1, \dots, w_{n-k} is at least $2^{-k} \cdot 2^{k^2} cW$. As we consider the case $\sum_{i=1}^{n-k} w_i < 2^{-k}W$, we can substitute W to obtain that the weights w_1, \dots, w_{n-k} lead to at least $2^{k^2} c \sum_{i=1}^{n-k} w_i$ non-empty regions in expectation. Thus, the weights w_1, \dots, w_{n-k} also form a counterexample for the claim in Equation (13), which is a contradiction to the assumption that w_1, \dots, w_n is the minimum counterexample and thus to the assumption that there is a counterexample at all. \square

7 Geometric SAT with Non-Zero Temperature

In the case with temperature $T = 0$, we used the fact that every clause contains the k variables with smallest weighted distance; recall Section 4.2. This is no longer true for higher temperatures: for $T > 0$, a clause can, in principle, contain any variable. However, the probability to contain a variables that is far away is rather small. In the remainder of this section, we show that a constant fraction of clauses actually behave just like in the $T = 0$ case, i.e., they contain the k closest variables. With this, we can then apply the argument outlined in Section 4.2.

7.1 Expected Number of Nice Clauses

Recall that a clause c is generated by drawing k variables without repetition with probabilities proportional to the connection weights. We call c *nice* if the i th variable drawn for c has the i th highest connection weight with c , i.e., c does not only contain the k variables with highest connection weight but they are drawn in descending order. This is a slightly stronger property than just requiring c to contain the k variables with lowest weighted distance.

Let \bar{x} be the connection weight of a variable v that has rather high connection weight with c . To show that the probability for $v \in c$ is reasonably high, we prove that \bar{x} is large compared to the sum of connection weights over all variables with smaller weight. The following lemma bounds this sum for a given \bar{x} . We use the Iverson bracket to exclude the variables with weight larger than \bar{x} from the sum, i.e., $[X(c, v) \leq \bar{x}]$ evaluates to 1 if $X(c, v) \leq \bar{x}$ and to 0 otherwise.

Lemma 7.1. *Let c be a clause at any position and let V be a set of n weighted variables with random positions in \mathbb{T}^d . For $T < 1$ and $\bar{x} \in \Omega(W^{1/T})$, the expected sum of connection weights smaller than \bar{x} is in $O(\bar{x})$, i.e.,*

$$\mathbb{E} \left[\sum_{v \in V} X(c, v) \cdot [X(c, v) < \bar{x}] \right] \in O(\bar{x}).$$

Proof. Using linearity of expectation, the term in the lemma's statement equals to the sum over the expectations $\mathbb{E} [X(c, v) \cdot [X(c, v) \leq \bar{x}]]$. To bound this expectation, we consider the three events $X(c, v) \leq (2^d w_v)^{1/T}$, $(2^d w_v)^{1/T} < X(c, v) < \bar{x}$, and $\bar{x} \leq X(c, v)$. Note that $[X(c, v) < \bar{x}]$ is 0 in the last event and 1 in the former two. Thus, we obtain

$$\begin{aligned} & \mathbb{E} [X(c, v) \cdot [X(c, v) < \bar{x}]] \\ &= \Pr [X(c, v) \leq (2^d w_v)^{1/T}] \cdot \mathbb{E} [X(c, v) \mid X(c, v) \leq (2^d w_v)^{1/T}] \end{aligned} \quad (14)$$

$$+ \Pr [(2^d w_v)^{1/T} < X(c, v) < \bar{x}] \cdot \mathbb{E} [X(c, v) \mid (2^d w_v)^{1/T} < X(c, v) < \bar{x}] \quad (15)$$

We bound the first term from above by assuming $X(c, v) = (2^d w_v)^{1/T}$ whenever $X(c, v) \leq (2^d w_v)^{1/T}$. Moreover, using the CDF for $X(c, v)$ (2) yields

$$\begin{aligned}
(14) &\leq \Pr \left[X(c, v) \leq (2^d w_v)^{1/T} \right] \cdot (2^d w_v)^{1/T} \\
&= \left(1 - \Pi_{d, \mathfrak{p}} w_v (2^d w_v)^{-1} \right) \cdot (2^d w_v)^{1/T} \\
&= \left(1 - \Pi_{d, \mathfrak{p}} 2^{-d} \right) \cdot (2^d w_v)^{1/T} \in \Theta(w_v^{1/T}).
\end{aligned}$$

For the second term, we have to integrate over the probability density function (PDF) $f_X(x)$ of the connection weights $X(c, v)$, which is the derivative of $F_X(x)$ (2). Thus, $f_X(x) = T \Pi_{d, \mathfrak{p}} w_v x^{-T-1}$ for $x \geq (2^d w_v)^{1/T}$, and we obtain

$$\begin{aligned}
(15) &= \Pr \left[(2^d w_v)^{1/T} < X(c, v) < \bar{x} \right] \cdot \int_{(2^d w_v)^{1/T}}^{\bar{x}} \frac{x \cdot f_X(x)}{\Pr \left[(2^d w_v)^{1/T} < X(c, v) < \bar{x} \right]} dx \\
&= T \Pi_{d, \mathfrak{p}} w_v \cdot \int_{(2^d w_v)^{1/T}}^{\bar{x}} x^{-T} dx.
\end{aligned}$$

For $T < 1$, this evaluates to

$$\begin{aligned}
&= T \Pi_{d, \mathfrak{p}} w_v \cdot \left[\frac{x^{1-T}}{1-T} \right]_{(2^d w_v)^{1/T}}^{\bar{x}} \\
&= \frac{T \Pi_{d, \mathfrak{p}} w_v}{1-T} \cdot \left[\bar{x}^{1-T} - (2^d w_v)^{1/T-1} \right] \\
&\leq \frac{T \Pi_{d, \mathfrak{p}} w_v}{1-T} \cdot \bar{x}^{1-T} \\
&\in \Theta \left(w_v \bar{x}^{1-T} \right).
\end{aligned}$$

Putting these bounds together yields

$$\begin{aligned}
\mathbb{E} \left[\sum_v X(c, v) \cdot [X(c, v) \leq \bar{x}] \right] &= \sum_v ((14) + (15)) \\
&\in O \left(\sum_v w_v^{1/T} + \sum_v w_v \bar{x}^{1-T} \right) \\
&\subseteq O \left(\left(\sum_v w_v \right)^{1/T} + \bar{x}^{1-T} \cdot \sum_v w_v \right) \\
&\in O \left(W^{1/T} + \bar{x}^{1-T} W \right).
\end{aligned}$$

As, $\bar{x} \in \Omega(W^{1/T})$, we have $W^{1/T} \in O(\bar{x})$, which handles the first term. The second term is also in $O(\bar{x})$, as $\bar{x} \in \Omega(W^{1/T})$ implies $W \in O(\bar{x}^T)$. Thus, this yields the claimed bound of $O(\bar{x})$. \square

This lets us show that each clause is nice with constant probability. The only assumption we need for this is the fact that no single weight is too large, i.e., every weight w_i has to be asymptotically smaller than the total weight W .

Theorem 7.2. *Let Φ be a random formula drawn from the weighted geometric model with ground space \mathbb{T}^d equipped with a \mathfrak{p} -norm, with temperature $T < 1$, and with $w_v/W \in o(1)$ for $v \in V$. Let c be a clause of Φ . Then c is nice with probability $\Omega(1)$.*

Proof. We prove two things. First, we show that, with probability $\Omega(1)$, there are at least k variables sufficiently close to c that they have connection weight $\Omega(W^{1/T})$. Second, we use Lemma 7.1 to show that the k variables with highest connection weight are chosen for c with constant probability (in descending order).

For the first part, we show that there is a constant a such that, with constant probability, at least k variables have connection weight at least $aW^{1/T}$. For a fixed variable v , we can use the CDF of $X(c, v)$ (Equation (2)) to obtain

$$\begin{aligned} \Pr \left[X(c, v) \geq aW^{1/T} \right] &= \Pi_{d, \mathfrak{p}} w_v \left(aW^{1/T} \right)^{-T} \\ &= \frac{\Pi_{d, \mathfrak{p}} w_v}{a^T W} \\ &= 2k \frac{w_v}{W}, \text{ for } a = \left(\frac{\Pi_{d, \mathfrak{p}}}{2k} \right)^{1/T}. \end{aligned}$$

Note that this is a valid probability, as $w_v/W \in o(1)$ implies that it is below 1. For the above choice of a , we obtain that the expected number of variables with connection weight at least $aW^{1/T}$ is $2k$. As the connection weights for the different variables are independent, we can apply the Chernoff-Hoeffding bound in Theorem A.7 to obtain that at least k variables have connection weight $aW^{1/T}$ with constant probability.

For the second part of the proof, let \bar{x} be the connection weight of the k th closest variable. With the argument above, we can assume $\bar{x} \in \Omega(W^{1/T})$ with constant probability, which lets us apply Lemma 7.1. To do so, consider the experiment of drawing the first variable for our clause c . Let v be the variable that maximizes the connection weight $X(c, v)$. The probability of drawing v equals $X(c, v)$ divided by the sum of all connection weights. By Lemma 7.1, the sum of all connection weights smaller than \bar{x} is in $O(\bar{x})$. Thus, the sum of all connection weights is in $O(X(c, v))$, which implies that v is chosen with constant probability. As we draw variables without repetition, the exact same argument applies for the second closest variable and so on. Thus, the probability that c contains the k closest variables drawn in order of descending connection weights is at least a constant, if there are k sufficiently close variables. As the latter holds with constant probability, c is nice with constant probability. \square

By the linearity of expectation, this immediately yields the following bound on the expected number of nice clauses.

Corollary 7.3. *Let Φ be a random formula with m clauses drawn from the weighted geometric model with ground space \mathbb{T}^d equipped with a \mathfrak{p} -norm, with temperature $T < 1$, and with $w_v/W \in o(1)$ for $v \in V$. The expected number of nice clauses in Φ is $\Theta(m)$.*

7.2 Concentration of Nice Clauses

We show that the number of nice clauses is concentrated around its expectation, i.e., with high probability, a constant fraction of clauses is nice. Our main tool for this will be the method of bounded differences; see Section A.6.2. To this end, we consider several random variables, e.g., the coordinates of clauses and variables, that together determine the whole process of generating a random formula. The number of nice clauses is then a function f of these random variables and its expectation is $\Theta(m)$, due to Corollary 7.3. Roughly speaking, the method of bounded differences then states that the probability that f deviates too much from its expectation is low if changing a single random variable only slightly changes f .

7.2.1 The Random Variables

So far, we viewed the generation of a random formula as a two-step process: first, sample coordinates for the variables and clauses; second, sample the variables contained in each clause based on their distances. The first step can be easily expressed via random variables. Let V_1, \dots, V_n and C_1, \dots, C_m be the coordinates⁷ of the n variables and m clauses, respectively. Though the second step heavily depends on the distances determined by the first, we can determine all random choices in advanced. For all $i \in [m]$ and $j \in [k]$, let X_i^j be a random variable uniformly distributed in $[0, 1)$. The variable X_i^j determines the j th variable of the i th clause c_i in the following way. We partition the interval $[0, 1)$ such that each variable v not already chosen for c_i corresponds to a subinterval of length proportional to the connection weight $X(c_i, v)$. We order these subinterval by length such that the largest interval comes first. The j th variable of c_i is then the variable whose interval contains X_i^j . Note that this samples k different variables for each clause, with probabilities proportional to the connection weights $X(c_i, v)$. Note further that the whole generation process of a random formula is determined by evaluating the independent random variables $V_1, \dots, V_n, C_1, \dots, C_m, X_1^1, \dots, X_m^k$.

To formalize the concept of nice clauses in this context, we require some more notation. For $i \in [m]$, let \mathcal{V}_i be the sequence of all variables ordered decreasingly by connection weight with the clause c_i . Moreover, let $\mathcal{V}_i[a, b]$ denote the subsequence from the a th to the b th variable in this sequence, including the boundaries. To simplify notation, we abbreviate the unique element in $\mathcal{V}_i[a, a]$ with $\mathcal{V}_i[a]$. Recall that clause c_i is nice if, for each of k steps, we choose the variable with highest connection weight that has not been chosen before. With respect to the random variables, this happens if, for each $j \in [k]$, X_i^j is smaller than the weight of $\mathcal{V}_i[j]$ divided by the sum of all weights of the remaining variables $\mathcal{V}_i[j, n]$. We thus define the indicator variable

$$N_i = \begin{cases} 1, & \text{if } \forall j \in [k]: X_i^j < \frac{X(c_i, \mathcal{V}_i[j])}{\sum_{v \in \mathcal{V}_i[j, n]} X(c_i, v)}, \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

which is 1 if and only if the i th clause is nice. With this, we can define the number of nice clauses as $f(V_1, \dots, V_n, C_1, \dots, C_m, X_1^1, \dots, X_m^k) = \sum_{i \in [m]} N_i$.

7.2.2 Bounding the Effect on the Number of Nice Clauses

To apply Theorem A.9, we have to bound the effect of changing the value of only one of these random variables on f . For the variables C_1, \dots, C_m , this is easy: Changing C_i moves the position of the clause c_i , which only makes a difference for c_i . Thus, the number of nice clauses changes by at most 1. Similarly, changing X_i^j only impacts the clause c_i , which implies that it changes the number of nice clauses by at most 1.

For the variables V_1, \dots, V_n , one can actually construct situations in which changing only a single position drops f from m to 0. There are basically two situations in which this can happen. First, if a single variable is close to many clauses, changing its position potentially impacts many clauses. Second, if many inequalities in Equation (16) are rather tight, then moving a single variable slightly closer to many clauses can increase the denominator on the right hand side by enough to change N_i for many clauses. We exclude both situations by defining unlikely bad events. By assuming these bad events do not happen, we can bound the effect of moving a single variable v by

$$\delta_v = w_v^{\frac{1}{1+T}} n^{\frac{T}{1+T}} \log^{\frac{2}{1+T}} n. \quad (17)$$

⁷Technically, these are multivariate random variables, as they represent d -dimensional points in \mathbb{T}^d .

The following lemma states that, with high probability, no variable is too close to too many clauses. This eliminates the first problematic situation (and will also help with the second).

Lemma 7.4. *Let $m \in O(n)$, $0 < T < 1$, and $w_v \in O(n^{1-\varepsilon})$ for every $v \in V$ and arbitrary constant $\varepsilon > 0$. With high probability, for every variable v , the number of clauses with connection weight at least $w_v^{\frac{1}{1+T}} n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n$ is in $O(\delta_v)$.*

Proof. Let $x_0 = w_v^{\frac{1}{1+T}} n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n$ and assume v to have a fixed position. To bound the probability $\Pr [X(c, v) \geq x_0]$ for a fixed clause c , we can use the CDF of $X(c, v)$ in Equation (2), assuming that $x_0 \geq (2^d w_v)^{1/T}$. This can be verified as follows:

$$\begin{aligned} x_0 &\geq \left(2^d w_v\right)^{1/T}, \\ \Leftrightarrow w_v^{\frac{1}{1+T}} n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n &\geq 2^{\frac{d}{T}} w_v^{\frac{1}{T}}, \\ \Leftrightarrow n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n &\geq 2^{\frac{d}{T}} w_v^{\frac{1}{T} - \frac{1}{1+T}}, \\ \Leftrightarrow n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n &\geq 2^{\frac{d}{T}} w_v^{\frac{1}{T(1+T)}}. \end{aligned}$$

Note that, due to $w_v \in O(n^{1-\varepsilon})$ for $\varepsilon > 0$, the left hand side is asymptotically bigger than the right hand side. Thus, we can use Equation (2) to obtain

$$\begin{aligned} \Pr [X(c, v) \geq x_0] &= \Pi_{d,p} w_v x_0^{-T} \\ &= \Pi_{d,p} w_v^{1 - \frac{T}{1+T}} n^{-\frac{1}{1+T}} \log^{\frac{2}{1+T}} n \\ &= \Pi_{d,p} \left(\frac{w_v}{n}\right)^{\frac{1}{1+T}} \log^{\frac{2}{1+T}} n. \end{aligned}$$

As there are $m \in O(n)$ clauses, the expected number of clauses with weight at least x_0 is

$$m \Pi_{d,p} \left(\frac{w_v}{n}\right)^{\frac{1}{1+T}} \log^{\frac{2}{1+T}} n \in O\left(w_v^{\frac{1}{1+T}} n^{1 - \frac{1}{1+T}} \log^{\frac{2}{1+T}} n\right) = O\left(w_v^{\frac{1}{1+T}} n^{\frac{T}{1+T}} \log^{\frac{2}{1+T}} n\right),$$

which is already the claimed bound of $O(\delta_v)$. As $0 < T < 1$, this upper bound grows polynomially in n . Thus, by the Chernoff-Hoeffding bound in Corollary A.8, it holds asymptotically with probability at least $1 - n^{-2}$. Applying the union bound over all variables yields the claim. \square

For the second issue, consider a clause c_i that is nice, i.e., $N_i = 1$, and let v be a variable. We call c_i *v-critical* if for one of the k inequalities in Equation (16) the difference between the left and right hand side is at most δ_v/n . The second problematic situation mentioned above arises when there are too many critical clauses. In the following lemma, we first bound the number of critical clauses. Afterwards, we show that the concept of critical clauses works as intended in the sense that only few non-critical clauses change from being nice to not being nice by moving a distant variable closer.

Lemma 7.5. *Let $m \in O(n)$, $0 < T < 1$ and let v be a variable. With high probability, there are only $O(\delta_v)$ *v-critical* clauses.*

Proof. A clause c_i can only be *v-critical* if one of the random variables X_i^j for $j \in [k]$ differs by at most δ_v/n to the right hand side of the inequality in Equation (16). The probability for this to happen for a single X_i^j is δ_v/n . As k is constant, c_i is *v-critical* with probability

$O(\delta_v/n)$. Thus, as $m \in O(n)$, the expected number of v -critical clauses is in $O(\delta_v)$. As the event of being v -critical is independent for the different clauses, and as this bound is polynomial in n for $T > 0$ (see Equation (17)), the Chernoff-Hoeffding bound in Corollary A.8 yields the claim. \square

To show that non-critical clauses are not an issue, we argue along the following lines. Let v be the variable we move and consider a nice but non- v -critical clause c . As c is not v -critical, the difference between the right and left hand side of the inequality in Equation (16) is at least $\delta_v/n \geq (w_v/n)^{\frac{1}{1+T}}$. Thus, if moving v does not decrease the right hand side too much, then c remains nice. Assuming that v is not one of the k variables closest to c , the right hand side decreases when moving v closer to c , as this increases the denominator. To bound this increase, we can use Lemma 7.4 by excluding the rare cases where v is very close to c . As the increase of the denominator has to be considered relative to the numerator, we also have to give an upper bound on the numerator, which is done by the following lemma.

Lemma 7.6. *Let $w_v \in O(n^{1-\varepsilon})$ for every $v \in V$ and arbitrary constant $\varepsilon > 0$. With high probability every clause has k variables with weight at least $W^{\frac{1}{T}} \log^{-\frac{2}{T}} n$.*

Proof. Let, $x_0 = W^{\frac{1}{T}} \log^{-\frac{2}{T}} n$ be the above weight and let c be a clause with fixed position. For every variable v , the probability for $X(c, v) \geq x_0$ is $\Pi_{d,p} w_v x_0^{-T} = \Pi_{d,p} w_v W^{-1} \log^2 n$ by Equation (2). Note that we can apply Equation (2) as $x_0 \geq (2^d w_v)^{1/T}$ due to the condition $w_v \in O(n^{1-\varepsilon})$ and the fact that $W \geq n$. Summing this over all variables yields that the expected number of variables with weight at least x_0 is $\Pi_{d,p} \log^2 n$. By Corollary A.8, c has $\Omega(\log^2 n)$ variables with weight at least x_0 with probability $1 - n^{-2}$. Applying the union bound over all clauses and the fact that k is constant while $\log^2 n$ grows with n yields the claim. \square

With these prerequisites, we can now bound the effect of moving a single variable on the number of nice clauses. The focus on non- v -critical clauses is justified by Lemma 7.5, stating that there are only few v -critical clauses.

Lemma 7.7. *Let $m \in O(n)$, $0 < T < 1$, and $w_v \in O(n^{1-\varepsilon})$ for every $v \in V$ and arbitrary constant $\varepsilon > 0$. Assume we move a single variable v . With high probability, among the nice and non- v -critical clauses, all but $O(\delta_v)$ clauses remain nice.*

Proof. Let v be the variable we move and consider a nice and non- v -critical clause c such that $X(c, v) < w_v^{\frac{1}{1+T}} n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n$ holds before and after the movement. By Lemma 7.4, this bound on $X(c, v)$ holds for all but $O(\delta_v)$ clauses. Thus, we can simply ignore all clauses not satisfying this condition.

As $w_v \in O(n^{1-\varepsilon})$, $X(c, v)$ is asymptotically smaller than $W^{\frac{1}{T}} \log^{-\frac{2}{T}} n$. Thus, by Lemma 7.6, with high probability, v does not belong to the k variables closest to c and is thus not contained in c (as c is nice).

To see that c remains nice with high probability, consider Equation (16). If we move v away from c , then $X(c, v)$ decreases, which decreases the denominator of the right hand side of the inequality. In this case, c clearly remains nice. Assume in the following that v moves closer to c , increasing $X(c, v)$ and thus the denominator. As c is not v -critical, the difference between the right and left hand side in the inequality is at least δ_v/n . Thus, if the right hand side decreases by less than this value for all $j \in [k]$, then c remains nice. To bound this decrease, let x be the numerator, let y be the denominator, and let z be amount by which $X(c, v)$ increases. Then the right hand side decreases by

$$\frac{x}{y} - \frac{x}{y+z} = \frac{x(y+z) - xy}{y(y+z)} = \frac{xy + xz - xy}{y^2 + yz} = \frac{xz}{y^2 + yz} \leq \frac{xz}{y^2}.$$

Note that $\frac{x}{y} \leq 1$, as the numerator appears as a term in the sum of the denominator. Thus, the decrease is upper bounded by $\frac{z}{y}$. Recall that we assumed an upper bound for $X(c, v)$, which yields an upper bound for the weight increase: $z < w_v^{\frac{1}{1+T}} n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n$. Moreover, as the numerator includes the weight of one of the k closest variables and since $W \geq n$, we have $y \geq n^{\frac{1}{T}} \log^{-\frac{2}{T}} n$ with high probability by Lemma 7.6. Putting this together yields

$$\frac{xz}{y^2} \leq \frac{z}{y} < \frac{w_v^{\frac{1}{1+T}} n^{\frac{1}{T(1+T)}} \log^{-\frac{2}{T(1+T)}} n}{n^{\frac{1}{T}} \log^{-\frac{2}{T}} n} = \left(\frac{w_v}{n}\right)^{\frac{1}{1+T}} \log^{\frac{2}{1+T}} n = \frac{\delta_v}{n}$$

which is exactly the bound we needed so that c remains nice. \square

Theorem 7.8. *Let Φ be a random formula with n variables and $m \in \Theta(n)$ clauses drawn from the weighted geometric model with ground space \mathbb{T}^d equipped with a \mathfrak{p} -norm, with temperature $0 < T < 1$, with $W \in O(n)$, and with $w_v \in O(n^{1-\varepsilon})$ for every $v \in V$ and arbitrary constant $\varepsilon > 0$. With high probability, $\Theta(m)$ clauses are nice.*

Proof. To apply Theorem A.9, we first define a bad event \mathcal{B} . Lemmas 7.4–7.7 show bounds that hold with high probability. We define \mathcal{B} to be the event the statement of one of these lemmas does not hold. Clearly, as the lemmas hold with high probability, $\Pr[\mathcal{B}] \in O(n^{-1})$.

We now bound the maximum effect of f when changing a single random variable assuming the complement \mathcal{B}^c of the bad event. As argued before, changing the position C_i of the clause c_i or changing X_i^j for $j \in k$ only effects the N_i of c_i , and thus changes f by at most 1. Now consider how f changes if we move a single variable v . We only consider the case that f is decreased by this movement. By symmetry reasons (one can always consider the reverse movement), this also covers the case that f is increased. Thus, the only thing we have to count are clauses that were nice before the movement and are no longer nice afterwards. Let c be a clause that was nice before the movement. There are three possibilities: $X(c, v)$ increases, decreases, or remains unchanged. In the last case, clause c clearly remains nice.

If $X(c, v)$ decreases, then the right hand side in the inequality of Equation (16) only decreases for a $j \in [k]$ if v is among the k closest variables. However, by Lemma 7.4 and Lemma 7.6 v is among the k closest variables for only $O(\delta_v)$ clauses (when assuming \mathcal{B}^c). Thus, f can decrease by at most $O(\delta_v)$ due to clauses for which $X(c, v)$ decreases.

If $X(c, v)$ increases, there are the two options that c is critical or non-critical. Also recall, that c is nice. By Lemma 7.5 there are only $O(\delta_v)$ critical clauses (assuming \mathcal{B}^c). Moreover, by Lemma 7.7, only $O(\delta_v)$ of the non-critical clauses become non-nice (again assuming \mathcal{B}^c).

In total, moving a single variable decreases f by only $O(\delta_v)$, assuming that the bad event \mathcal{B} does not happen. As $w_v \in O(n^{1-\varepsilon})$ for a constant $\varepsilon > 0$, we can bound δ_v (see Equation (16)) by

$$\begin{aligned} \delta_v &= w_v^{\frac{1}{1+T}} n^{\frac{T}{1+T}} \log^{\frac{2}{1+T}} n \\ &= w_v^{\frac{1}{2}} w_v^{\frac{1}{1+T} - \frac{1}{2}} n^{\frac{T}{1+T}} \log^{\frac{2}{1+T}} n \\ &\in O\left(w_v^{\frac{1}{2}} n^{(1-\varepsilon)\left(\frac{1}{1+T} - \frac{1}{2}\right) + \frac{T}{1+T}} \log^{\frac{2}{1+T}} n\right). \end{aligned}$$

Thus, for another constant $\varepsilon' > 0$ the exponent of n simplifies to

$$(1 - \varepsilon) \left(\frac{1}{1+T} - \frac{1}{2}\right) + \frac{T}{1+T} = \frac{1}{1+T} - \frac{1}{2} + \frac{T}{1+T} - \varepsilon' = \frac{1}{2} - \varepsilon',$$

yielding the bound

$$\delta_v \in O\left(\sqrt{w_v n} n^{-\varepsilon'} \log^{\frac{2}{1+T}} n\right) \subseteq O\left(\frac{\sqrt{w_v n}}{\log n}\right).$$

To summarize, for the upper bounds Δ_i on the effect of changing a single random variable in Theorem A.9, we obtain $\Delta_i = 1$ for the random variables $C_1, \dots, C_m, X_1^1, \dots, X_m^k$, and $\Delta_i \in O(\sqrt{w_v n} / \log n)$ for the random variable corresponding to the position of the variable v . Thus, the sum of their squares is

$$\Delta \in O\left(m + km + \sum_{v \in V} \left(\frac{\sqrt{w_v n}}{\log n}\right)^2\right) = O\left(\frac{n}{\log^2 n} \sum_{v \in V} w_v\right) = O\left(\frac{n^2}{\log^2 n}\right),$$

where the last equality follows from the fact that we assume the sum of all weights W to be linear. Now, we can conclude the proof by applying Corollary A.10: We just proved that $\Delta \in O(\mathbb{E}[f]^2 / \log^2 n)$. We argued before that $\Pr[\mathcal{B}] \in O(n^{-1})$. Moreover, the number of nice clauses is at most m and at least 0, i.e., $u - \ell \leq m$. Thus, $(u - \ell)\Pr[\mathcal{B}]$ bounded from above by a constant and thus at most $c\mathbb{E}[f]$ for $c < 1$. \square

7.2.3 Putting Things Together

Now we are ready to prove our main theorem for the geometric model.

Theorem 7.9. *Let Φ be a formula with n variables and $m \in \Theta(n)$ clauses drawn from the weighted geometric model with ground space \mathbb{T}^d equipped with a \mathfrak{p} -norm, temperature $T < 1$, $W \in O(n)$, and $w_v \in O(n^{1-\varepsilon})$ for every $v \in V$ and any constant $\varepsilon > 0$. Then, Φ contains a. a. s. an unsatisfiable subformula of constant size. It can be found in $O(n \log n)$ time.*

Proof. Let m' be the number of clauses in Φ that consist of the k variables with minimum weighted distance. By Theorem 7.8 we have $m' \in \Theta(m) = \Theta(n)$. In the following, we consider only these clauses.

Consider the weighted order- k Voronoi diagram of the n variables and let n' be the number of non-empty regions. By Theorem 6.9 and due to $W \in O(n)$, we have $\mathbb{E}[n'] \in O(n)$. Moreover, it follows from Markov's inequality that $n' \leq n \log n$ holds asymptotically almost surely:

$$\Pr[n' \geq n \log n] \leq \frac{\mathbb{E}[n']}{n \log n} \in O\left(\frac{1}{\log n}\right).$$

Now, determining the k variables of a clause c is equivalent to observing which region of the order- k Voronoi diagram contains c , or more precisely, which k variables define this region. Thus, choosing random positions for the clauses is like throwing m' balls into n' (non-uniform) bins. Thus, if $m' \in \Omega(n' / \text{polylog } n')$, we can apply Corollary A.5. With the above bounds, which hold asymptotically almost surely, it is not hard to see that this condition in fact holds: If $n' \leq n$, it clearly holds as $m' \in \Omega(n)$. Otherwise, we have $n' \leq n \log n \leq n \log n'$, which implies $n \geq n' / \log n'$, and thus $m \in \Omega(n' / \log n')$.

Applying Corollary A.5 tells us that, asymptotically almost surely, there is a bin with a superconstant number of balls. In other words, there is a superconstant number of clauses that share the same set of k variables. For sufficiently large n , this is bigger than 2^k , which implies an unsatisfiable subformula consisting of only 2^k clauses. Clearly, it can be found in $O(n \log n)$ time by sorting the clauses lexicographically with respect to the contained variables. \square

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A Basic Technical Tools

This section is a collection of tools we use throughout the paper that were either known before or are straight-forward to prove but distract from the core arguments we make in the paper.

A.1 Discrete Power-Law Weights

The following lemma summarizes some properties of the probability distribution given by the discrete power-law weights.

Lemma A.1. *Let*

$$p_i = \frac{i^{-1/(\beta-1)}}{\sum_{j=1}^n j^{-1/(\beta-1)}}$$

for $\beta > 2$. It holds that

$$\sum_{j=1}^n j^{-1/(\beta-1)} = (1 + o(1)) \frac{\beta - 1}{\beta - 2} \cdot n^{(\beta-2)/(\beta-1)},$$

$$F(i) := \sum_{j=1}^i p_j = O\left(\left(\frac{i}{n}\right)^{(\beta-2)/(\beta-1)}\right),$$

and

$$\sum_{j=1}^n p_j^2 = \begin{cases} \Theta\left(n^{-2\frac{\beta-2}{\beta-1}}\right), & \beta < 3; \\ \Theta(\ln n/n), & \beta = 3; \\ \Theta(n^{-1}), & \beta > 3. \end{cases} \quad (18)$$

Proof. Since $j^{-1/(\beta-1)}$ is monotonically decreasing, it holds that

$$\begin{aligned} \sum_{j=1}^n j^{-1/(\beta-1)} &\leq 1 + \int_{j=1}^n j^{-1/(\beta-1)} dj \\ &= 1 + \frac{\beta - 1}{\beta - 2} \left(n^{(\beta-2)/(\beta-1)} - 1 \right) = \frac{\beta - 1}{\beta - 2} n^{(\beta-2)/(\beta-1)} - \frac{1}{\beta - 2} \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^n j^{-1/(\beta-1)} &\geq n^{-1/(\beta-1)} + \int_{j=1}^n j^{-1/(\beta-1)} dj \\ &= \frac{\beta - 1}{\beta - 2} n^{(\beta-2)/(\beta-1)} - \frac{\beta - 1}{\beta - 2} + n^{-1/(\beta-1)}. \end{aligned}$$

Equivalently, we get

$$\begin{aligned}
F(i) &= \sum_{j=1}^i p_j = \frac{\sum_{j=1}^i j^{-1/(\beta-1)}}{\sum_{j=1}^n j^{-1/(\beta-1)}} \\
&\leq \frac{1}{\sum_{j=1}^n j^{-1/(\beta-1)}} \left(1 + \int_{j=1}^i j^{-1/(\beta-1)} dj \right) \\
&\leq \frac{1}{\sum_{j=1}^n j^{-1/(\beta-1)}} \left(\frac{\beta-1}{\beta-2} i^{(\beta-2)/(\beta-1)} - \frac{1}{\beta-2} \right) \\
&= O\left(\left(\frac{i}{n} \right)^{(\beta-2)/(\beta-1)} \right).
\end{aligned}$$

Finally, we want to bound

$$\sum_{j=1}^n p_j^2 = \frac{\sum_{j=1}^n j^{-2/(\beta-1)}}{\left(\sum_{j=1}^n j^{-1/(\beta-1)} \right)^2}.$$

First, note that for $\beta = 3$ this equation yields

$$\sum_{j=1}^n p_j^2 = \frac{H_n}{\left(\sum_{j=1}^n j^{-1/(\beta-1)} \right)^2} = \Theta(\ln n/n),$$

where H_n denotes the n -th harmonic number. For $\beta \neq 3$ we can achieve

$$\begin{aligned}
\sum_{j=1}^n p_j^2 &\leq \frac{1}{\left(\sum_{j=1}^n j^{-1/(\beta-1)} \right)^2} \left(1 + \int_{j=1}^n j^{-2/(\beta-1)} dj \right) \\
&= \frac{1}{\left(\sum_{j=1}^n j^{-1/(\beta-1)} \right)^2} \left(1 + \frac{\beta-1}{\beta-3} \left(n^{(\beta-3)/(\beta-1)} - 1 \right) \right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^n p_j^2 &\geq \frac{1}{\left(\sum_{j=1}^n j^{-1/(\beta-1)} \right)^2} \left(n^{-2/(\beta-1)} + \int_{j=1}^n j^{-2/(\beta-1)} dj \right) \\
&= \frac{1}{\left(\sum_{j=1}^n j^{-1/(\beta-1)} \right)^2} \left(n^{-2/(\beta-1)} + \frac{\beta-1}{\beta-3} \left(n^{(\beta-3)/(\beta-1)} - 1 \right) \right)
\end{aligned}$$

If $\beta < 3$, the expressions above yield

$$\sum_{j=1}^n p_j^2 \in \Theta \left(\frac{1}{\left(\sum_{j=1}^n j^{-1/(\beta-1)} \right)^2} \right) \subseteq \Theta \left(n^{-2(\beta-2)/(\beta-1)} \right).$$

For $\beta > 3$, they yield

$$\sum_{j=1}^n p_j^2 \in \Theta \left(\frac{n^{(\beta-3)/(\beta-1)}}{n^{2(\beta-2)/(\beta-1)}} \right) \subseteq \Theta \left(n^{-1} \right).$$

This proves all statements of the lemma. \square

A.2 CDF of Connection Weights in the Geometric Model

The CDF $F_X(x)$ of the connection weights $X(c, v)$ in the geometric SAT model satisfies the following lemma.

Lemma A.2. $F_X(x) = 1 - \Pi_{d,p} w_v x^{-T}$ for $x \geq (2^d w_v)^{1/T}$.

Proof. Inserting the definition of the connection weight and rearranging slightly yields

$$\begin{aligned} F_X(x) &= \Pr [X(c, v) \leq x] \\ &= \Pr \left[\left(\frac{w_v}{\|c - v\|^d} \right)^{1/T} \leq x \right] \\ &= \Pr \left[\|c - v\| \geq w_v^{1/d} x^{-T/d} \right] \\ &= 1 - \Pr \left[\|c - v\| < w_v^{1/d} x^{-T/d} \right]. \end{aligned}$$

As c and v are two random points, we can use the CDF for the distances between random points in Equation (1) to obtain

$$F_X(x) = 1 - \Pi_{d,p} w_v x^{-T} \quad \text{for } x \geq (2^d w_v)^{1/T},$$

which concludes the proof. \square

A.3 Volume of Balls in a Hypercube

We are regularly concerned with the asymptotic behavior of a ball's volume depending on its radius. The following lemma helps us to deal with the edge case, where the ball stretches beyond the boundary of our ground space.

Lemma A.3. *Let H be a d -dimensional unit-hypercube in \mathbb{R}^d equipped with a \mathbf{p} -norm. There exists a constant $c > 0$ such that, for every $\mathbf{p} \in H$ and $r > 0$, the intersection of H with the ball $B_{\mathbf{p}}(r)$ of radius r around \mathbf{p} has volume at least $\min\{1, cr^d\}$.*

Proof. In this following, we assume $H = [-0.5, 0.5]^d$ (rather than $[0, 1]^d$), as it makes the proof more convenient. If r is sufficiently small, then $B_{\mathbf{p}}(r)$ is completely contained in H . Thus, in this case, the claim follows from the fact that the volume of a ball with radius r in d -dimensional space is proportional to r^d . Thus, we have to prove that the parts of $B_{\mathbf{p}}(r)$ outside of H are asymptotically not relevant.

Let p_1, \dots, p_d be the coordinates of \mathbf{p} and assume without loss of generality that \mathbf{p} lies in the all-negative orthant, i.e., $p_i \leq 0$ for $i \in [d]$. We prove the claim by defining a box B with the following three properties. First, the box B has volume proportional to r^d . Second, B is a subset of the ball $B_{\mathbf{p}}(r)$. Third, B is a subset of the hypercube H or H is a subset of $B_{\mathbf{p}}(r)$. Note that the lemma's statement clearly holds if H is a subset of $B_{\mathbf{p}}(r)$ as the intersection has volume 1 in this case. If H is not a subset of $B_{\mathbf{p}}(r)$, the second and third property imply that B is a subset of the intersection of $B_{\mathbf{p}}(r)$ and H . Thus, the volume of B given by the first property is a lower bound for the volume of the intersection, which proves the claim.

It remains to define B and prove the three properties. The box B has \mathbf{p} as corner and extends from there in the direction of the all-positive orthant. The side lengths are chosen proportional to the distance from the edge of H in this direction. Formally, the corners of B are $\{p_1, p_1 + r(0.5 - p_1)/\sqrt[d]{d}\} \times \dots \times \{p_d, p_d + r(0.5 - p_d)/\sqrt[d]{d}\}$.

To prove the first property, note that the side length of B in dimension i is $r(0.5 - p_i)/\sqrt[d]{d}$. As $p_i \leq 0$, this is at least $0.5r/\sqrt[d]{d}$, which implies that the volume of B is at least $(0.5r/\sqrt[d]{d})^d$.

For the second property, note that the point of B with maximum distance from \mathbf{p} is the opposite corner, i.e., the point with coordinates $(p_i + r(0.5 - p_i)/\sqrt[p]{d})$. The distance from \mathbf{p} is given by

$$\sqrt[p]{\sum_{i=1}^d \left(\frac{r(0.5 - p_i)}{\sqrt[p]{d}} \right)^p} \leq \sqrt[p]{\sum_{i=1}^d \frac{r^p}{d}} = r.$$

Finally, for the third property, assume $r = \sqrt[p]{d}$. Then the coordinates $p_i + r(0.5 - p_i)/\sqrt[p]{d}$ of the corners of B simplify to 0.5. Thus, all corners of B are still in the hypercube H if $r \leq \sqrt[p]{d}$. On the other hand, if $r \geq \sqrt[p]{d}$, then H is completely contained in $B_{\mathbf{p}}(r)$, which concludes the proof of the last property. We note that it is easy to verify that all above arguments also hold for the limit $\mathbf{p} = \infty$. \square

A.4 Derivative of the Incomplete Gamma Function

We need the following somewhat technical bound that is easy to verify.

Lemma A.4. *Let Γ be the gamma function. For any $\alpha, \beta, \gamma, d \in \mathbb{R}$ with $\beta, \gamma, d > 0$,*

$$\int_0^\gamma x^{\alpha d - 1} \exp(-\beta x^d) dx \leq \frac{\Gamma(\alpha)}{\beta^\alpha d}.$$

Proof. Let $\Gamma(\alpha, x)$ be the incomplete gamma function. Its derivative is

$$\frac{\partial \Gamma(\alpha, x)}{\partial x} = -x^{\alpha - 1} \exp(-x).$$

Thus, it follows that

$$\frac{\partial}{\partial x} \frac{\Gamma(\alpha, \beta x^d)}{\beta^\alpha d} = \beta d x^{d-1} \frac{(\beta x^d)^{\alpha-1} \exp(-\beta x^d)}{\beta^\alpha d} = x^{\alpha d - 1} \exp(-\beta x^d).$$

Using this, the given integral evaluates to

$$\begin{aligned} \int_0^\gamma x^{\alpha d - 1} \exp(-\beta x^d) dx &= \left[-\frac{\Gamma(\alpha, \beta x^d)}{\beta^\alpha d} \right]_0^\gamma \\ &= \frac{1}{\beta^\alpha d} \left(\Gamma(\alpha, 0) - \Gamma(\alpha, \beta \gamma^d) \right) \\ &\leq \frac{\Gamma(\alpha, 0)}{\beta^\alpha d}. \end{aligned} \quad \square$$

A.5 Balls Into Heterogeneous Bins

Consider throwing m balls into n uniform bins, i.e., for each ball we draw one of the n bins uniformly at random and place the ball into the drawn bin. The *maximum load* L is the random variable that describes the maximum number of balls that are together in the same bin. From the analysis by Raab and Steger [49, Theorem 1], we immediately get the following corollary.

Corollary A.5 ([49], Theorem 1). *Throw m balls into n uniform bins and let L be the maximum load. If $m \in \Omega(\frac{n}{\text{polylog } n})$, then $L \in \Omega(\frac{\log n}{\log \log n})$ asymptotically almost surely.*

Now assume we have non-uniform bins, i.e., the probability for each ball to end up in the i th bin is p_i with $\sum_i p_i = 1$. Intuitively, Corollary A.5 should still hold in this setting, as increasing the probability of some bins only makes it more likely that a bin gets many balls. Making this argument formal yields the following theorem.

Theorem A.6. *Corollary A.5 also holds for the non-uniform bins.*

Proof. Let $B = [n]$ be the set of all bins and let B' be the subset of bins with probability at least $1/(2n)$. These are the bins whose probability either increased, or decreased by a factor of at most 2. Without loss of generality, let $B' = [n']$. Note that the probability for a ball to land in a bin of B' is at least a constant, as every bin not in B' has probability at most $1/(2n)$. Thus, by the Chernoff-Hoeffding bound in Corollary A.8, a constant fraction of the balls end up in a bin of B' with high probability. We make a case distinction on how large n' is.

First, assume $n' \leq m/\log n$. Thus, with high probability, we end up with $\Theta(m)$ balls in at most $m/\log n$ bins, which means that at least one bin contains $\Omega(\log n)$ balls. Thus, clearly $L \in \Omega(\log n/\log \log n)$.

Second, assume $n' > m/\log n$. Recall that each bin in B' has probability at least $1/(2n)$. We consider the alternative experiment where, for every ball, each bin in B' has probability exactly $1/(2n)$ to get the ball. Balls not landing in B' are discarded. Let L' denote the maximum number of balls that share a bin in B' . Clearly, we can couple the two experiments such that $L \geq L'$ holds in every outcome. It remains to show that $L' \in \Omega(\log n/\log \log n)$. For this, let m' be the number of balls ending up in B' . Note that m' is a random variable. However, if we condition on m' , then we are back to the normal homogeneous balls into bins, except that we throw m' balls into n' bins. If we show that $m' \in \Omega(n'/\text{polylog } n')$, then Corollary A.5 tells us that $L' \in \Omega(\log n'/\log \log n')$. First note that this is sufficient for our purpose: as $n' > m/\log n$ and $m \in \Omega(n/\text{polylog } n)$, we get

$$\begin{aligned} \frac{\log n'}{\log \log n'} &> \frac{\log m - \log \log n}{\log(\log m - \log \log n)} \\ &\in \Omega\left(\frac{\log n - \log \text{polylog } n - \log \log n}{\log(\log n - \log \text{polylog } n - \log \log n)}\right) \\ &\subseteq \Omega\left(\frac{\log n}{\log \log n}\right). \end{aligned}$$

It remains to show that $m' \in \Omega(n'/\text{polylog } n')$ so that we can actually apply Corollary A.5. To do so, recall that B' has n' bins, each with probability $1/(2n)$. Thus, the probability that a single ball lands in B' is $n'/(2n)$, which shows that m' is $mn'/(2n)$ in expectation. As n' is almost m (up to logarithmic factors) and m is almost n , this expectation is almost linear in n . Thus, by the Chernoff-Hoeffding bound in Corollary A.8, we can assume that

$$m' \in \Theta\left(\frac{mn'}{n}\right)$$

holds with high probability. Using that $n' > m/\log n$ and $m \in \Omega(n/\text{polylog } n)$, we obtain

$$\frac{mn'}{n} > \frac{m^2}{n \log n} \in \Omega\left(\frac{n}{\text{polylog } n}\right).$$

As $n' \leq n$, it follows that $m \in \Omega(n'/\text{polylog } n')$, which concludes the proof. \square

A.6 Concentration Bounds

For a random experiment, we say that an event happens *with high probability (w. h. p.)* if the probability is at least $1 - O(1/n)$. The type of event we are usually interested in is that a random variable assumes a value close to its expectation, i.e., that the random variable is concentrated. In the following, we state two well known techniques to prove concentration, namely a Chernoff-Hoeffding bound and the method of bounded differences. In both cases we derive asymptotic variants that suite our purpose better than the original exact bounds.

A.6.1 Chernoff-Hoeffding

Theorem A.7 (Theorem 1.1 in [23]). *Let X_1, \dots, X_n be independent random variables with values in $\{0, 1\}$ and let $X = \sum_{i \in [n]} X_i$ be their sum. Then, for all $0 < \varepsilon < 1$,*

$$\begin{aligned} \Pr [X > (1 + \varepsilon)\mathbb{E}[X]] &\leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right), \text{ and} \\ \Pr [X < (1 - \varepsilon)\mathbb{E}[X]] &\leq \exp\left(-\frac{\varepsilon^2}{2}\mathbb{E}[X]\right). \end{aligned}$$

We use this bound multiple times in a similar way, which is captured by the following direct corollary.

Corollary A.8. *Let X_1, \dots, X_n , and X be as in Theorem A.7. Let $f(n) \in \omega(\log n)$ be an upper or lower bound for $\mathbb{E}[X]$ and let c be a constant. With probability $1 - n^{-c}$, $X \in O(f(n))$ and $X \in \Omega(f(n))$, respectively.*

Proof. Assume $f(n)$ is a lower bound, i.e., $f(n) \leq \mathbb{E}[X]$. We show $X \in \Omega(f(n))$ with the desired probability. By the second inequality of Theorem A.7, we have

$$\begin{aligned} \Pr [X < (1 - \varepsilon)f(n)] &\leq \Pr [X < (1 - \varepsilon)\mathbb{E}[X]] \\ &\leq \exp\left(-\frac{\varepsilon^2}{2}\mathbb{E}[X]\right) \\ &\leq \exp\left(-\frac{\varepsilon^2}{2}f(n)\right) = n^{-\omega(1)}, \end{aligned}$$

where the last equality is due to the fact that $f(n) \in \omega(\log n)$. Thus, for any constant c , this probability is below n^{-c} for sufficiently large n . Hence, for any constant $\varepsilon \in (0, 1)$, $X \geq (1 - \varepsilon)f(n)$ with probability $1 - n^{-c}$.

Assume $f(n)$ is an upper bound, i.e., $\mathbb{E}[X] \leq f(n)$. Let X' be a random variable with $f(n) = \mathbb{E}[X']$ such that X' dominates X in the sense that $X \leq X'$ for every outcome. We show that $X' \in O(f(n))$ with probability $1 - n^{-c}$, which implies $X \in O(f(n))$ with at least the same probability. The first inequality of Theorem A.7 yields

$$\begin{aligned} \Pr [X' > (1 + \varepsilon)f(n)] &= \Pr [X' > (1 + \varepsilon)\mathbb{E}[X']] \\ &\leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X']\right) \\ &= \exp\left(-\frac{\varepsilon^2}{3}f(n)\right) = n^{-\omega(1)}. \end{aligned}$$

As before, the last inequality comes from the fact that $f(n) \in \omega(\log n)$. All remaining arguments are as in the case where $f(n)$ was a lower bound. \square

A.6.2 Method of Bounded Differences

We use the following concentration bound that allows us to exclude a bad event \mathcal{B} . Moreover, for a sequence of random variables X_1, \dots, X_n , let \mathbf{X}_i with $i \leq n$ be the short form for X_1, \dots, X_i .

Theorem A.9 (Theorem 7.1 in [23]). *Let f be a function of n random variables X_1, \dots, X_n , such that $\mathbb{E}[f]$ is bounded and $\ell \leq f \leq u$. Let \mathcal{B} be any event, and let Δ_i be the maximum effect of f assuming \mathcal{B}^c (the complement of \mathcal{B}):*

$$\left| \mathbb{E}[f \mid \mathbf{X}_{i-1}, X_i = a_i, \mathcal{B}^c] - \mathbb{E}[f \mid \mathbf{X}_{i-1}, X_i = a'_i, \mathcal{B}^c] \right| \leq \Delta_i,$$

where a_i and a'_i are valid evaluations for X_i . Then for $\Delta = \sum_i \Delta_i^2$ and $t \geq 0$

$$\Pr[f > \mathbb{E}[f] + t + (u - \ell)\Pr[\mathcal{B}]] \leq \exp\left(-\frac{2t^2}{\Delta}\right) + \Pr[\mathcal{B}],$$

and

$$\Pr[f < \mathbb{E}[f] - t - (u - \ell)\Pr[\mathcal{B}]] \leq \exp\left(-\frac{t^2}{\Delta}\right) + \Pr[\mathcal{B}].$$

Corollary A.10. *Assume for Theorem A.9 that $\Delta \in O(\mathbb{E}[f]^2 / \log^2 n)$, that $\Pr[\mathcal{B}] \in O(n^{-1})$, and that $(u - \ell)\Pr[\mathcal{B}] \leq c\mathbb{E}[f]$ for a constant $c < 1$. Then w. h. p. $f \in \Theta(\mathbb{E}[f])$.*

Proof. We only show the lower bound $f(\mathbf{X}_n) \in \Omega(\mathbb{E}[f])$. The upper bound follows analogously. Let c_1 be a constant such that $c + c_1 < 1$. We show that, with high probability, f is at least $(1 - c_1 - c)\mathbb{E}[f]$, which proves the claim. By the assumption $(u - \ell)\Pr[\mathcal{B}] \leq c\mathbb{E}[f]$, we have

$$(1 - c_1 - c)\mathbb{E}[f] \leq \mathbb{E}[f] - c_1\mathbb{E}[f] - (u - \ell)\Pr[\mathcal{B}].$$

Thus, by using the bound of Theorem A.9 with $t = c_1\mathbb{E}[f]$, we obtain

$$\begin{aligned} \Pr[f \leq (1 - c_1 - c)\mathbb{E}[f]] &\leq \Pr[f \leq \mathbb{E}[f] - c_1\mathbb{E}[f] - (u - \ell)\Pr[\mathcal{B}]] \\ &\leq \exp\left(-\frac{c_1^2\mathbb{E}[f]^2}{\Delta}\right) + \Pr[\mathcal{B}]. \end{aligned}$$

Now let c_2 be the constant hidden in the O -notation for Δ , i.e., $\Delta \leq c_2\mathbb{E}[f]^2 / \log^2 n$. Then, for $c_3 = c_1^2/c_2$, we have

$$\begin{aligned} \exp\left(-\frac{c_1^2\mathbb{E}[f]^2}{\Delta}\right) &\leq \exp\left(-\frac{c_1^2\mathbb{E}[f]^2}{c_2\mathbb{E}[f]^2 / \log^2 n}\right) \\ &= \exp\left(-c_3 \log^2 n\right) \\ &= n^{-c_3 \log n} \in O\left(n^{-1}\right), \end{aligned}$$

which concludes the proof. \square