

A Probabilistic Extension of the Logic of Evidence and Truth $\mathbf{LET}^{+}_{\mathbf{K}}$

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Abstract. Logics of Evidence and Truth (LET) are paradefinite logics (that is, logics that are both paraconsistent and paracomplete) which extend the logic of first-degree entailment (FDE), also known as Belnap-Dunn four-valued logic. These extensions incorporate a unary operator which recovers the property of excluded middle and explosion for sentences under its scope. In this paper we focus on the logic $\mathrm{LET}^{+}_{\mathrm{K}}$, which possesses rules for propagating classicality and has a six-valued semantics captured by the lattice L6, known as the crystal lattice. We formulate a two-layer fuzzy probabilistic logic based on the $[0, 1]$-valued Łukasiewicz logic to reason about the probability of formulas of $\mathrm{LET}^{+}_{\mathrm{K}}$.

1 Introduction

The logic of first-degree entailment (FDE), also known as the Belnap–Dunn four-valued logic, features, in addition to the truth-values *true* (T_0) and *false* (F_0), two further truth-values: *neither* (n) and *both* (b). These additional values allow for the representation of incomplete or contradictory evidence concerning the truth status of a given proposition. Logics of Evidence and Truth (LETs) are systems that are both paraconsistent and paracomplete, and which extend FDE by incorporating a unary operator $\circ$ that recovers the principle of excluded middle and the property of explosion for sentences falling under its scope. More precisely, $\circ$ validates the inferences

$$\circ\varphi \vdash \varphi \vee \neg\varphi \ (1) \quad \text{and} \quad \circ\varphi, \varphi, \neg\varphi \vdash \psi \ (2),$$

respectively. The logics of evidence and truth can be traced back to the logics of formal inconsistency (LFIs) [3]. In LFIs, $\circ\varphi$ is intended to mean that φ is *consistent*. However, in a LET, which is designed to handle positive and negative evidence, the intended meaning of $\circ\varphi$ is that there is *conclusive evidence* for φ, or that the information provided by φ, whether positive or negative, is *reliable*. In such a case, φ is assumed to behave classically.

The logic $\mathrm{LET}^{+}_{\mathrm{K}}$ arises from first extending FDE with a classical implication, obtaining the logic called $\mathrm{FDE}^{\rightarrow}$, then extending $\mathrm{FDE}^{\rightarrow}$ with the inference rules

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(1) and (2) given above, yielding the logic $\mathrm{LET}_{\mathrm{K}}$, and finally extending $\mathrm{LET}_{\mathrm{K}}$ with propagation rules of classicality for the operator $\circ$, which express how $\circ$ is transmitted from less complex to more complex sentences and vice-versa [4]. A natural deduction system for $\mathrm{LET}^{+}_{\mathrm{K}}$ given in [4] is shown to have a sound, complete, and decidable deterministic six-valued semantics. $\mathrm{LET}^{+}_{\mathrm{K}}$ admits as well a Hilbert-style axiomatisation sound and complete with respect to the previously mentioned 6-valued semantics, see [3]. In the literature, one finds works on probabilities over FDE (e.g., [8]), over LETs (e.g., [9]), and particularly over $\mathrm{LET}^{+}_{\mathrm{K}}$ [1]. Additionally, two-layered logics formalizing reasoning with probabilities over Belnap-Dunn four-valued logic have been proposed, see e.g. [2].

In this short paper, based on seminal ideas introduced in [6,7] and later gathered in [5], we define a probabilistic extension of $\mathrm{LET}^{+}_{\mathrm{K}}$ by introducing a $[0,1]$-valued modal logic (over the infinite-valued Łukasiewicz logic Ł), called $\mathrm{FP}(\mathrm{LET}^{+}_{\mathrm{K}})$, with a fuzzy modality P whose intended semantics is given by the $\mathrm{LET}^{+}_{\mathrm{K}}$-probability functions introduced in [1]. In this logic, the truth values (in $[0,1]$) of formulas of the form $P\varphi$, $P\neg\varphi$, or $P\circ\varphi$ account for the probability of having positive evidence for φ, negative evidence for φ, and the reliability of such evidence. We assume the reader has some familiarity with Łukasiewicz logic Ł, see e.g. [6].

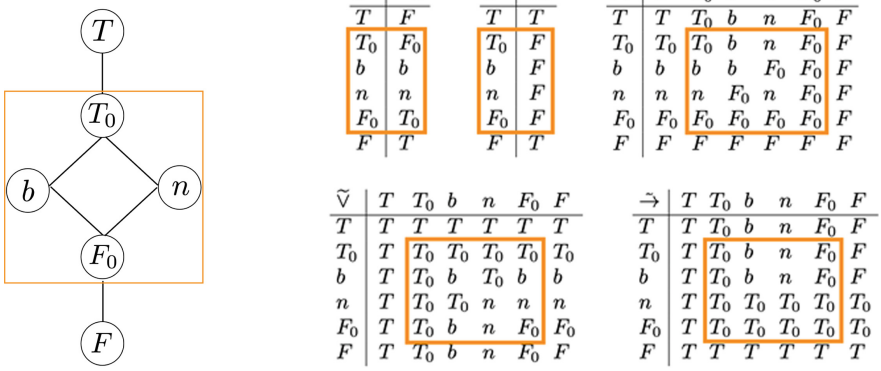
2 Preliminaries on the Logic of Evidence and Truth $\mathbf{LET}^{+}_{\mathbf{K}}$

For simplicity reasons we assume a language $\mathcal{L}$ for $\mathrm{LET}^{+}_{\mathrm{K}}$ built over a finite set of propositional variables Var, together with the set of connectives $\{\wedge, \vee, \rightarrow, \neg, \circ\}$.

The semantics for $\mathrm{LET}^{+}_{\mathrm{K}}$ we will focus on in this paper is the deterministic 6-valued semantics induced by the following logical matrix $\mathcal{M}_6 = (\mathbf{6}, D, O)$, where $\mathbf{6} = \{F, F_0, n, b, T_0, T\}$ is the set of six truth-values, $D = \{b, T_0, T\}$ is the set of designated values, and O is the set of truth functions over $\mathbf{6}$ for the above connectives, defined by the truth-tables on the right-hand side of Fig. 1. The lattice structure induced on $\mathbf{6}$ by the lattice operations $\widetilde{\wedge}$ and $\widetilde{\vee}$ is the lattice L6 depicted on the left-hand side in Fig. 1. Notice that this semantics extends the Belnap-Dunn semantics over the subset of truth-values $BD = \{F_0, n, b, T_0\}$, with $\{b, T_0\}$ as set of designated values. In Fig. 1 we have highlighted in orange the restriction of the lattice and operations on the subset of Belnap-Dunn truth-values BD.

A $\mathcal{M}_6$-valuation is a mapping $v : Var \rightarrow 6$ that extends to any formula from $\mathcal{L}$ by using the truth-tables from Fig. 1. We will denote by Ω the set of all $\mathcal{M}_6$-valuations, i.e. $\Omega = 6^{Var}$. For any φ, we will denote by $Mod(\varphi)$ the set of models of φ, that is, $Mod(\varphi) = \{v \in \Omega \mid v(\varphi) \in D\}$. We define the following logical equivalence relation on the language $\mathcal{L}$ of $\mathrm{LET}^{+}_{\mathrm{K}}$: $\varphi \approx \psi$ iff $\varphi \models_{\mathcal{M}_6} \psi$ and $\psi \models_{\mathcal{M}_6} \varphi$. In other words, $\varphi \approx \psi$ iff $Mod(\varphi) = Mod(\psi)$.

The corresponding notion of logical consequence is defined in the natural way: for any set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}$, one defines that φ follows from Γ, written $\Gamma \models_{\mathcal{M}_6} \varphi$, whenever for every $\mathcal{M}_6$-valuation v, if $v(\psi) \in D$ for every $\psi \in \Gamma$, then $v(\varphi) \in D$



$\tilde{\neg}$	
T	F
T_0	F_0
b	b
n	n
F_0	T_0
F	T

$\tilde{\circ}$	
T	T
T_0	F
b	F
n	F
F_0	F
F	T

$\tilde{\wedge}$	T	T_0	b	n	F_0	F
T	T	T_0	b	n	F_0	F
T_0	T_0	T_0	b	n	F_0	F
b	b	b	b	F_0	F_0	F
n	n	n	F_0	n	F_0	F
F_0	F_0	F_0	F_0	F_0	F_0	F
F	F	F	F	F	F	F

$\tilde{\vee}$	T	T_0	b	n	F_0	F
T	T	T	T	T	T	T
T_0	T	T_0	T_0	T_0	T_0	T_0
b	T	T_0	b	T_0	b	b
n	T	T_0	T_0	n	n	n
F_0	T	T_0	b	n	F_0	F_0
F	T	T_0	b	n	F_0	F

$\tilde{\rightarrow}$	T	T_0	b	n	F_0	F
T	T	T_0	b	n	F_0	F
T_0	T	T_0	b	n	F_0	F
b	T	T_0	b	n	F_0	F
n	T	T_0	T_0	T_0	T_0	T_0
F_0	T	T_0	T_0	T_0	T_0	T_0
F	T	T	T	T	T	T

Fig. 1. Left: the crystal lattice L6; Right: truth-tables for the 6-valued semantics $\mathcal{M}_6$.

as well. Equivalently, $\Gamma \models_{\mathcal{M}_6} \varphi$ iff $(\bigcap_{\psi\in\Gamma} Mod(\psi)) \subseteq Mod(\varphi)$. We note that $\models_{\mathcal{M}_6}$ enjoys the deduction theorem: $\Gamma, \varphi \models_{\mathcal{M}_6} \psi$ iff $\Gamma \models_{\mathcal{M}_6} \varphi \to \psi$.

The authors in [4] show that this semantics is sound and complete for a natural deduction system introduced there. In [3] it was introduced a Hilbert-style system for $\mathrm{LET}^{+}_{\mathrm{K}}$, obtained by extending an axiomatic system for positive classical logic (over the signature $\{\wedge, \vee, \to\}$)[1] by adding the following axioms, where $\alpha \leftrightarrow \beta := (\alpha \to \beta) \wedge (\beta \to \alpha)$, $\chi^T := \chi \wedge \circ\chi$ and $\chi^F := \neg\chi \wedge \circ\chi$:

$\neg\neg\varphi \leftrightarrow \varphi$
$\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
$\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
$\neg(\varphi \to \psi) \leftrightarrow (\varphi \wedge \neg\psi)$
$\circ\varphi \to (\varphi \to (\neg\varphi \to \psi))$
$\circ\varphi \to (\varphi \vee \neg\varphi)$
$\circ\circ\varphi$
$\circ\neg\varphi \leftrightarrow \circ\varphi$
$\circ(\varphi \wedge \psi) \leftrightarrow (\varphi^T \wedge \psi^T) \vee \varphi^F \vee \psi^F$
$\circ(\varphi \vee \psi) \leftrightarrow (\varphi^F \wedge \psi^F) \vee \varphi^T \vee \psi^T$
$\circ(\varphi \to \psi) \leftrightarrow (\varphi \wedge \psi^F) \vee \varphi^F \vee \psi^T$.

Actually, it can be proved that this system of axioms, together with Modus Ponens as inference rule, is sound and complete with respect to the 6-valued semantics defined by the matrix $\mathcal{M}_6$. That is, if we denote the corresponding notion of proof by $\vdash_{\mathrm{LET}^{+}_{\mathrm{K}}}$, we have that, for every set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathrm{LET}^{+}_{\mathrm{K}}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathcal{M}_6} \varphi \qquad \text{(see [3])}.$$

Example 1. As shown in [1], $\mathrm{LET}^{+}_{\mathrm{K}}$ has expressive power to express each of the six truth-values in terms of a formula. Indeed, for each valuation v over the matrix $\mathcal{M}_6$, $v(\Phi_a(\varphi)) \in D$ iff $v(\varphi) = a$, where $\Phi_a(\varphi)$ is defined as follows:

$\Phi_T(\varphi) = \varphi \wedge \circ\varphi$
$\Phi_{T_0}(\varphi) = \varphi \wedge {\sim}(\neg\varphi \vee \circ\varphi)$
$\Phi_b(\varphi) = \varphi \wedge \neg\varphi$
$\Phi_F(\varphi) = \neg\varphi \wedge \circ\varphi$
$\Phi_{F_0}(\varphi) = \neg\varphi \wedge {\sim}(\varphi \vee \circ\varphi)$
$\Phi_n(\varphi) = {\sim}(\varphi \vee \neg\varphi)$.

Observe that $\varphi \approx \Phi_T(\varphi) \vee \Phi_{T_0}(\varphi) \vee \Phi_b(\varphi)$; $\neg\varphi \approx \Phi_F(\varphi) \vee \Phi_{F_0}(\varphi) \vee \Phi_b(\varphi)$; and $\circ\varphi \approx \Phi_T(\varphi) \vee \Phi_F(\varphi)$. To fix ideas, let φ be "A votes for candidate C". Then, we can express the following propositions:

[1] Namely, the following axioms: $\varphi \to (\psi \to \varphi)$; $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$; $(\varphi \wedge \psi) \to \varphi$; $(\varphi \wedge \psi) \to \psi$; $(\varphi \to (\psi \to \varphi \wedge \psi)$; $\varphi \to (\varphi \vee \psi)$; $\psi \to (\varphi \vee \psi)$; $(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$; and $\varphi \vee (\varphi \to \psi)$.

$\neg\varphi$: "A does not vote for candidate C"

$\circ\varphi$: "A has a firm and irrevocable decision regarding whether to vote for C or not"

$\Phi_T(\varphi)$: "A decided to vote for C

$\Phi_{T_0}(\varphi)$: "A has reasons to vote for C, and no reason not to vote for C"

$\Phi_b(\varphi)$: "A has reasons both to vote and not to vote for C"

$\Phi_n(\varphi)$: "A has no reason to vote C and no reason not to vote for C"

$\Phi_{F_0}(\varphi)$: "A has reasons not to vote for C, and no reason to vote for C"

$\Phi_F(\varphi)$: "A decided not to vote for C"

Of the six cases of the form $\psi_a(\varphi)$, the four truth-values other than T and F correspond to FDE. In those four cases, A can change its mind without trivializing the decision, namely: voting for C (or not voting for C) may be additionally considered in the future, contrary to A's previous opinion, while still being logically coherent.

3 FP(LET^+_K): a Two-Layer Logic for Reasoning About the Probability of LET^+_K Formulas

In this section, we will define a two-layer modal logic on top of LET^+_K. Following previous approaches [5,6], the idea is to introduce a modal operator P such that, for every formula φ of the logic LET^+_K, $P\varphi$ is a fuzzy (i.e. $[0,1]$-valued) formula whose truth-value is to be interpreted as the probability of φ. The logic we will define next is built over a two-layer language: non-modal formulas are LET^+_K formulas, used to describe and paraconsistently reason about possibly conflicting events, and modal formulas (without nesting of the operator P) to represent and reason about the probability of the non-modal formulas. Note that a compound modal formula $P\varphi \to_\text{Ł} P\psi$, where $\to_\text{Ł}$ is Łukasiewicz implication, expresses the qualitative comparative statement "ψ is at least as probable as φ".

3.1 On Probabilities for LET^+_K

Recall that $\Omega = \mathbf{6}^{Var}$, and that the logical equivalence relation $\varphi \approx \psi$ is defined by the condition $Mod(\varphi) = Mod(\psi)$. Since the set Var is finite, we have that the quotient set $\mathcal{L}/_\approx$ is finite. Indeed, it has $2^{(6^k)}$, where k is the number of elements of Var. We will denote by $[\varphi]$ the equivalence class of φ.

Next, based on [10], we can define the notion of a probability on the logic LET^+_K: *a probability on* LET^+_K*-formulas* is a mapping $p : \mathcal{L} \to [0,1]$ such that the following properties are satisfied:

(P1) If $\vdash_{\text{LET}^+_\text{K}} \varphi$ then $p(\varphi) = 1$ (P2) $\varphi \vdash_{\text{LET}^+_\text{K}} \psi$ for all ψ, implies $p(\varphi) = 0$

(P3) If $\varphi \vdash_{\text{LET}^+_\text{K}} \psi$ then $p(\varphi) \leq p(\psi)$ (P4) $p(\varphi \vee \psi) = p(\varphi) + p(\psi) - p(\varphi \wedge \psi)$

Proposition 1 (See [1]). *For each probability $p : \mathcal{L} \to [0,1]$ there is a probability measure $\mu : 2^\Omega \to [0,1]$ such that $p(\varphi) = \mu(Mod(\varphi))$. The converse also holds: given a measure μ over Ω, the map $p(\varphi) = \mu(Mod(\varphi))$ is a probability.*

An immediate consequence is that probabilities respect logical equivalence.

Corollary 1. *Let* $p : \mathcal{L} \to [0,1]$ *be a probability for* $\mathrm{LET}^+_{\mathrm{K}}$. *If* $\varphi \approx \psi$ *then* $p(\varphi) = p(\psi)$.

3.2 Syntax and Semantics of the Logic FP($\mathbf{LET^+_K}$)

The language of FP($\mathrm{LET}^+_{\mathrm{K}}$) logic contains two types of formulas:

(i) non-modal formulas: they consist of the set $\mathcal{L}$ of $\mathrm{LET}^+_{\mathrm{K}}$-formulas built from the set *Var* of propositional variables and connectives $\wedge, \vee, \neg, \to, \circ$. They will be denoted by lower-case greek letters φ, ψ, etc.

(ii) modal formulas: atomic modal formulas are of the form $P\varphi$, where φ is a non-modal formula, i.e. a $\mathrm{LET}^+_{\mathrm{K}}$-formula; compound modal formulas are built from atomic modal formulas using the connectives of Łukasiewicz logic: $\to_{\text{Ł}}$ and $\neg_{\text{Ł}}$. They will be denoted by upper-case greek letters Φ, Ψ, etc.

Thus defined, the language has a two-layered structure: $\mathcal{L}$ of purely propositional formulas, and another of non-nested modal formulas, denoted $M(\mathcal{L})$.

As for the semantics, recall from [1] that a *probabilistic model* is a pair $\mathsf{M} = (\Omega, \mu)$ where $\mu : 2^{\Omega} \to [0,1]$ is a probability measure on the set $\Omega = \mathbf{6}^{Var}$ of $\mathcal{M}_6$-valuations. In [1] the models are more general, however, the ones considered here are enough to our purposes.

Now, given an arbitrary modal formula Φ, the truth value of Φ in $\mathsf{M} = (\Omega, \mu)$, denoted by $\|\Phi\|_{\mathsf{M}}$, is inductively defined as follows:

- If Φ is an atomic modal formula $P\varphi$, then $\|P\varphi\|_{\mathsf{M}} = \mu(Mod(\varphi))$.
- If Φ is a compound modal formula, then its truth value $\|\Phi\|_{\mathsf{M}}$ is computed by evaluating its subformulas occurring in Φ, and then combined by using Łukasiewicz truth functions.

Note that the evaluation of modal formulas does not depend on the particular world but on the whole model.

As for the notion of logical consequence, a formula Φ logically follows from a set of modal formulas Γ, written $\Gamma \models_{Prob} \Phi$, if for any probabilistic model $\mathsf{M} = (\Omega, \mu)$, $\|\Phi\|_{\mathsf{M}} = 1$ whenever $\|\Psi\|_{\mathsf{M}} = 1$ for all $\Psi \in \Gamma$.

3.3 An Axiomatization for FP($\mathbf{LET^+_K}$)

We axiomatically define here the probabilistic logic FP($\mathrm{LET}^+_{\mathrm{K}}$). The axioms of FP($\mathrm{LET}^+_{\mathrm{K}}$) are the following (where $\leftrightarrow_{\text{Ł}}$, $\oplus$ and $\ominus$ are definable connectives in Ł with truth functions $1 - |x - y|$, $\min(1, x + y)$ and $\max(0, x - y)$, respectively):

($\mathrm{LET}^+_{\mathrm{K}}$) axioms of $\mathrm{LET}^+_{\mathrm{K}}$ (for non-modal formulas).

(Ł) axioms ofŁukasiewicz logic Ł (for modal formulas).

(FP1) $P(\varphi \vee \psi) \leftrightarrow_{\text{Ł}} P\varphi \oplus (P\psi \ominus P(\varphi \wedge \psi))$. [2]

(FP2) $P(\top)$, where $\top = \circ\circ p$ for a fixed $p \in Var$.

(FP3) $P(\sim\varphi) \leftrightarrow_{\text{Ł}} \neg_{\text{Ł}} P\varphi$, with $\sim\varphi := \varphi \to \neg\top$ a classical negation in $\mathrm{LET}^+_{\mathrm{K}}$.

[2] Łukasiewicz standard semantics of connectives $\leftrightarrow_{\text{Ł}}$, $\oplus$ and $\ominus$ allows this axiom to faithfully represent condition (P4) for probabilities in LET^+_{K}, see Sect. 3.1.

The rules of inference of $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$ are:

(MP) *modus ponens* (for both $\mathrm{LET}^+_{\mathrm{K}}$ and Ł implications).
(Mon) *monotonicity* for P: if $\varphi \to \psi$ is a theorem of $\mathrm{LET}^+_{\mathrm{K}}$, then $P\varphi \to_{\text{Ł}} P\psi$ is a theorem of $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$.

The above axioms and rules define in the usual way a notion of proof, denoted $\vdash_{\mathrm{FPLET}}$. Note that, thanks to (FP2) and the rule (Mon), the following necessitation rule for P is derivable:

(Nec) *necessity for* P: if $\vdash_{\mathrm{LET}^+_{\mathrm{K}}} \varphi$, then $\vdash_{\mathrm{FPLET}} P\varphi$.

Also, by the (Mon) rule, it easily follows that if $\varphi \approx \psi$ then $\vdash_{\mathrm{FPLET}} P\varphi \leftrightarrow_{\text{Ł}} P\psi$.

Following the same approach of [5] with the necessary modifications, one can prove the following completeness result for $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$.

Theorem 1. $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$ *is sound and complete for deductions from modal theories, that is, for any finite set of modal formulas* $\Gamma \cup \{\Phi\}$, $\Gamma \models_{Prob} \Phi$ *iff* $\Gamma \vdash_{\mathrm{FPLET}} \Phi$.

Proof. Soundness it is easy to verify. As for completeness, assume $\Gamma \nvdash_{\mathrm{FPLET}} \Phi$. We will prove that $\Gamma \not\models_{Prob} \Phi$, i.e. that there exists a probabilistic model M satisfying Γ but not Φ. Following the strategy in [5] we translate modal theories over $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$ into theories over Ł. For each equivalence class $[\varphi]$ in the language of $\mathrm{LET}^+_{\mathrm{K}}$, choose one formula as a representative of the class, for instance the DNF form of φ, denoted $\overline{\varphi}$ (see [1]). As observed in Subsect. 3.1, we only have finitely-many different formulas $\overline{\varphi}$'s.

Now we define two transformations of atomic modal formulas into new propositional variables in the language of Łukasiewicz logic:

$$(P\varphi)^\star = v_\varphi \qquad\qquad (P\varphi)^\blacktriangledown = v_{\overline{\varphi}}$$

For $\triangleleft \in \{\star, \blacktriangledown\}$, for each modal formula Ψ, let us denote by $\Psi^\triangleleft$ the corresponding Ł-formula obtained from Ψ by considering any atomic modal subformula of the form $P\varphi$ as a new propositional variable $(P\varphi)^\triangleleft$. Define, therefore: $\Gamma^\triangleleft = \{\Psi^\triangleleft \mid \Psi \in \Gamma\}$ and $\mathrm{Ax}^\triangleleft = \{\Upsilon^\triangleleft \mid \Upsilon$ is an instance of axiom (FPi), $i = 1, 3\} \cup \{(P\varphi)^\triangleleft \to_{\text{Ł}} (P\psi)^\triangleleft \mid \vdash_{\mathrm{LET}^+_{\mathrm{K}}} \varphi \to \psi\}$.

The important fact to notice here is that $Ax^\blacktriangledown$ is finite since there are only finitely-many propositional variables $v_{\overline{\varphi}}$, and we want to prove that $\Gamma \vdash_{\mathrm{FPLET}} \Phi$ iff $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \vdash_{\text{Ł}} \Phi^\blacktriangledown$. Then, using the same technique used in [5] in a more general uncertainty reasoning context, we prove it in two steps.

Claim 1: $\Gamma \vdash_{\mathrm{FPLET}} \Phi$ iff $\Gamma^\star \cup \mathrm{Ax}^\star \vdash_{\text{Ł}} \Phi^\star$.

This is clear, since, by construction, there is a one-to-one correspondence between proofs from Γ in $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$ and proofs from $\Gamma^\star \cup \mathrm{Ax}^\star$ in Ł.
Claim 2: $\Gamma^\star \cup \mathrm{Ax}^\star \vdash_{\text{Ł}} \Phi^\star$ iff $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \vdash_{\text{Ł}} \Phi^\blacktriangledown$.

Proof. (of Claim 2) One direction, if $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \vdash_{\text{Ł}} \Phi^\blacktriangledown$ then $\Gamma^\star \cup \mathrm{Ax}^\star \vdash_{\text{Ł}} \Phi^\star$, is also direct. Let us prove the converse direction, that is, if $\Gamma^\star \cup \mathrm{Ax}^\star \vdash_{\text{Ł}} \Phi^\star$ then $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \vdash_{\text{Ł}} \Phi^\blacktriangledown$. Suppose $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \nvdash_{\text{Ł}} \Phi^\blacktriangledown$. Since $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown$ is finite then, by the strong finite completeness of Ł, $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \not\models_{\text{Ł}} \Phi^\blacktriangledown$, hence there is an Ł-evaluation e such that $e(\Gamma^\blacktriangledown) \subseteq e(Ax^\blacktriangledown) = \{1\}$ and $e(\Phi^\blacktriangledown) < 1$. We need to extend the evaluation e to an evaluation e^+ for all propositional variables v_φ. The natural way is to define, for every φ in the language of $\mathrm{LET}^+_{\mathrm{K}}$, $e^+(v_\varphi) = e(v_{\overline{\varphi}})$. In this way, it is easy to check that $e^+(\Psi^\star) = e(\Psi^\blacktriangledown)$ for every modal Ψ, and hence e^+ becomes also a model of $\Gamma^\star \cup \mathrm{Ax}^\star$ but not of $\Phi^\star$. Hence $\Gamma^\star \cup \mathrm{Ax}^\star \not\models_{\text{Ł}} \Phi^\star$, and by soundness of Ł, we finally have $\Gamma^\star \cup \mathrm{Ax}^\star \nvdash_{\text{Ł}} \Phi^\star$. □

As a consequence of both claims we have the following equivalence:

$$\Gamma \nvdash_{\mathrm{FPLET}} \Phi \text{ iff } \Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \nvdash_{\text{Ł}} \Phi^\blacktriangledown. \tag{1}$$

Since are assuming $\Gamma \nvdash_{\mathrm{FPLET}} \Phi$, it follows that $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown \nvdash_{\text{Ł}} \Phi^\blacktriangledown$ as well. Now, again, since $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown$ is finite and Ł is strongly standard complete for finite theories, we know there exists an Ł-evaluation $\bar{e}$ that is a model of $\Gamma^\blacktriangledown \cup \mathrm{Ax}^\blacktriangledown$ such that $\bar{e}(\Phi^\blacktriangledown) < 1$.

Define, for any non-modal formula φ, $p(\varphi) = \bar{e}((P\varphi)^\blacktriangledown)$. We can check that p is a probability for $\mathrm{LET}^+_{\mathrm{K}}$-formulas:

- If $\vdash_{\mathrm{LET}^+_{\mathrm{K}}} \varphi$ then, by the (Nec) rule, $\vdash_{\mathrm{FPLET}} P\varphi$, hence $\vdash_{\text{Ł}} (P\varphi)^\blacktriangledown$, hence $\bar{e}((P\varphi)^\blacktriangledown) = 1$, thus $p(\varphi) = 1$.
- If $\varphi \vdash_{\mathrm{LET}^+_{\mathrm{K}}} \psi$ for every ψ, then $\vdash_{\mathrm{LET}^+_{\mathrm{K}}} {\sim}\varphi$, hence again by (Nec) $p({\sim}\varphi) = 1$, and by (FP3), $p(\varphi) = 1 - p({\sim}\varphi) = 0$.
- If $\varphi \vdash_{\mathrm{LET}^+_{\mathrm{K}}} \psi$ then $\vdash_{\mathrm{LET}^+_{\mathrm{K}}} \varphi \to \psi$, hence by (Nec) $P\varphi \to_{\text{Ł}} P\psi$ is a theorem of $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$, and hence $(P\varphi \to_{\text{Ł}} P\psi)^\blacktriangledown = (P\varphi)^\blacktriangledown \to_{\text{Ł}} (P\psi)^\blacktriangledown$ is a theorem of Ł, and thus $\bar{e}((P\varphi)^\blacktriangledown \to_{\text{Ł}} (P\psi)^\blacktriangledown) = 1$, that is, $p(\varphi) = \bar{e}((P\varphi)^\blacktriangledown) \leq \bar{e}((P\psi)^\blacktriangledown) = p(\psi)$.
- By axiom (FP1), $(P(\varphi \vee \psi))^\blacktriangledown \leftrightarrow_{\text{Ł}} (P\varphi)^\blacktriangledown \oplus ((P\psi)^\blacktriangledown \ominus (P(\varphi \wedge \psi)^\blacktriangledown) \in Ax^\blacktriangledown$, thus, the value of this Ł-formula by $\bar{e}$ is 1, and this exactly means $p(\varphi \vee \psi) = p(\varphi) + p(\psi) - p(\varphi \wedge \psi)$.

Now, by Proposition 1, there is a probability measure μ on 2^Ω such that $p(\varphi) = \mu(Mod(\varphi))$ for any $\mathrm{LET}^+_{\mathrm{K}}$-formula φ. Therefore, $\mathsf{M} = (\Omega, \mu)$ is a probabilistic model such that $\|\Psi\|_{\mathsf{M}} = \bar{e}(\Psi^\blacktriangledown) = 1$ for all $\Psi \in \Gamma$ but $\|\Phi\|_{\mathsf{M}} = \bar{e}(\Phi^\blacktriangledown) < 1$. In other words, $\Gamma \not\models_{Prob} \Phi$. □

By means of atomic sentences of the form $P\varphi$, for any sentence φ in $\mathrm{LET}^+_{\mathrm{K}}$, one can make inferences in $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$ in order to describe the probabilistic interrelations of these and other related sentences in $\mathrm{LET}^+_{\mathrm{K}}$.

Example 2. Continuing with Example 1, if χ stands for B votes for candidate C, then if a theory Γ in $\mathrm{FP}(\mathrm{LET}^+_{\mathrm{K}})$ proves the formula like $P\varphi \to_{\text{Ł}} P(\Phi_b(\chi))$ it means that Γ derives that the probability that A votes for C is less or equal than the probability that B has reasons both to vote and not vote for C. On the other hand, if Γ derives the formula $P\varphi \oplus P\varphi$, it means that from Γ it follows that the probability that A votes for C is at least 1/2.

4 Conclusions

In this paper we have introduced a two-layer fuzzy probabilistic logic to reason about the probability of formulas of the logic of evidence and truth $\mathrm{LET}^{+}_{\mathrm{K}}$, based on extensive results of [1] about probabilities on LET^{+}_{K}, which in turn extends a similar study on Belnap-Dunn four-valued logic in [8]. Soundnesss and completeness results have been proved. As for future work, we plan to investigate a logic to reason with LET^{+}_{K} conditional probabilities, also considered in [1].

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Approximate Unification in Possibilistic Logic

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Abstract. This short paper discusses how semantic similarity between words or sentences, computed from word embeddings, can be processed in the setting of possibilistic logic, and used in approximate unification mechanisms. The approach is compared to a recent proposal by A. Hunter who uses similarity between sentence embeddings for identifying semantic conflicts between atoms in propositional logic when each atom is associated with a sentence.

1 Introduction

This short paper is motivated by a recent article by Anthony Hunter [8]. In this work, the author defines a quasi-equality relation between two sentences, based on a numerical similarity measure between sentence embeddings [11]. When the similarity exceeds some chosen threshold, the two sentences are considered as quasi-equal. This is the basis for extending the concept of contradiction in classical logic to situations involving an atom and the negation of a quasi-equivalent atom, without being limited to strict equality. This opens the possibility of developing logical reasoning mechanisms closer to natural languages.

Possibilistic logic is known as handling classical logic formulae associated with certainty levels [3,5]. In possibilistic logic, the validity of a proof path is the one of its weakest link. The validity of a conclusion is then the validity of its best proof. A conclusion is thus obtained with a certainty level reflecting the confidence in the formulas used in its (best) proof. Moreover, extensions of possibilistic logic enable us to handle fuzzy set or fuzzy relation membership degrees as certainty levels, as well as fuzzy constants and fuzzily restricted quantifiers [7].

Fuzzy sets have been advocated as tool for representing the semantics of natural languages [12]. For now more than ten years, another kind of representation is available with word and sentence embeddings [10]. This article examines how vector representations can also be used in possibilistic logic to address the problem discussed in [8] without the need to define specific consequence relations or to choose thresholds.

The paper is organized as follows. Section 2 summarizes the main features and issues raised by the approach proposed by Anthony Hunter [8]. Then Sect. 3, after a short refresher on possibilistic logic, discusses how semantic similarity between atoms can be turned into uncertain equivalences, which can be used in

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