Revisiting Ultraproducts in Fuzzy Predicate Logics

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***Abstract*—In this paper we examine different possibilities of defining reduced products and ultraproducts in fuzzy predicate logics. We present analogues to the Łos Theorem for these notions and discuss the advantages and drawbacks of each definition introduced. Following the work in [9], we show that these constructions are adequate for working in a reduced semantics.**

***Index Terms*—Ultraproducts, reduced products, fuzzy predicate logics, reduced semantics**

I. Introduction

Ultraproducts are a powerful tool in classical model theory. Its applications range from an ultraproduct version of the compactness theorem to algebraic characterizations of elementary classes.The method originated with Skolem in the 1930's, and has been used extensively since the work of Łoś (for a survey on the subject I refer the reader to [5]).

Being one of the basic methods of constructing models in classical mathematical logic, it is a natural question to ask for the fuzzy predicate case. Do ultraproducts play also such a relevant role? What are the conditions for their existence? What role they play in the proof of compactness and elementary equivalence results? In this paper we present the first step to answer these questions. Here we examine different possibilities of defining reduced products and ultraproducts in fuzzy predicate logics. We prove analogues to the Łos Theorem for these notions and discuss the advantages and drawbacks of each definition introduced. Following the work in [9], we show that these constructions are adequate for working in a reduced semantics.

We have tried to encompass the most commonly used definitions of ultraproduct and reduced product of the mathematical fuzzy logic literature (for a reference see [10], [11], [17], [16], [12] and [1]). We extend, when available, their results to work in arbitrary fuzzy predicate logics and equality-free languages. The paper is structured as follows: we start with some preliminaries on fuzzy predicate logics, then in section III, we study ultrafilters over a fixed L-algebra and in section IV, reduced products defined from pairs of filters (*d-filters* in Gerla's terms). Finally, in section V, we study reduced products with respect to Leibniz congruences. We conclude with a section devoted to future work.

II. Preliminaries

Our study of the model theory of fuzzy predicate logics is focused on the basic fuzzy predicate logic MTL∀ and some of its expansions based on propositional *core fuzzy logics*. For a thorough treatment of core fuzzy logics we refer to [14], [7] and [8].

Now we introduce the syntax of fuzzy predicate logics. A *predicate language* Γ is a triple (**P**,**F**,**A**) where **P** is a non-empty set of predicate symbols, **F** is a set of function symbols and **A** is a mapping assigning to each predicate and function symbol a natural number called the *arity of the symbol*. The function symbols F for which $\mathbf{A}(F) = 0$ are called the *object constants*. Formulas of the predicate language Γ are built up from the symbols in (**P**,**F**,**A**), the connectives and truth constants of a fixed core fuzzy logic L, the logical symbols $\forall$ and $\exists$, variables and punctuation. The formulas of a predicate language Γ will be called Γ-*formulas*. A Γ-*sentence* is a Γ-formula without free variables. Throughout the paper we consider the equality symbol as a binary predicate symbol not as a logical symbol, we work in equality-free fuzzy predicate logics. That is, the equality symbol is not necessarily present in all the languages and its interpretation is not fixed. Given a propositional core fuzzy logic L we denote by L∀ the corresponding fuzzy predicate logic. An axiomatic system for $L\forall$ can be found in [13] and [14].

Now, we introduce the semantics for the logic $L\forall$. A $\mathbf{B}$-*structure* for predicate language Γ is a tuple

$$\mathbf{M} = (M, (P_{\mathbf{M}})_{P\in\Gamma}, (F_{\mathbf{M}})_{F\in\Gamma})$$

where:

1) M is a non-empty set.
2) For each n-ary predicate $P \in \Gamma$, $P_{\mathbf{M}}$ is a $\mathbf{B}$-fuzzy relation $P_{\mathbf{M}} : M^n \to \mathbf{B}$.
3) For each n-ary function symbol $F \in \Gamma$, if $n > 0$, $F_{\mathbf{M}} : M^n \to M$ is a crisp function. If $n = 0$, $F_{\mathbf{M}}$ is an element of $\mathbf{M}$.

Given a $\mathbf{B}$-structure $\mathbf{M}$, an $\mathbf{M}$-*evaluation* of the object variables is a mapping v which assigns to each variable an element from M. By $\phi(x_1, \ldots, x_k)$ we mean that all the free variables of ϕ are among $x_1, \ldots, x_k$. Let v be an $\mathbf{M}$-evaluation, x a variable, and $d \in M$, we denote by $v[x \to d]$ the $\mathbf{M}$-evaluation such that $v[x \to d](x) = d$ and for each variable y different from x, $v[x \to d](y) = v(y)$. Let $\mathbf{M}$ be a $\mathbf{B}$-structure and v an $\mathbf{M}$-evaluation, we define the values of the terms and *truth values* of the formulas as follows:

$$\|c\|^{\mathbf{B}}_{\mathbf{M},v} = c_{\mathbf{M}},\ \|x\|^{\mathbf{B}}_{\mathbf{M},v} = v(x)$$

$$\|F(t_1, \ldots, t_n)\|^{\mathbf{B}}_{\mathbf{M},v} = F_{\mathbf{M}}(\|t_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each variable x, each object constant $c \in \Gamma$, each n-ary function symbol $F \in \Gamma$ for $n > 0$ and Γ-terms $t_1, \ldots, t_n$, respectively.

$$\|P(t_1, \ldots, t_n)\|^{\mathbf{B}}_{\mathbf{M},v} = P_{\mathbf{M}}(\|t_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each n-ary predicate $P \in \Gamma$,

$$\|\delta(\phi_1, \ldots, \phi_n)\|^{\mathbf{B}}_{\mathbf{M},v} = \delta_{\mathbf{B}}(\|\phi_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|\phi_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each n-ary connective $\delta \in$L and Γ-formulas $\phi_1, \ldots, \phi_n$. Finally, for the quantifiers,

$$\|\forall x\phi\|^{\mathbf{B}}_{\mathbf{M},v} = \inf\{\|\phi\|^{\mathbf{B}}_{\mathbf{M},v[x\to d]} : d \in M\}$$

$$\|\exists x\phi\|^{\mathbf{B}}_{\mathbf{M},v} = \sup\{\|\phi\|^{\mathbf{B}}_{\mathbf{M},v[x\to d]} : d \in M\}$$

It is said that a $\mathbf{B}$-structure is *safe* if a truth value is defined for each formula and evaluation. From now on we assume that all our structures are safe. If v is an evaluation such that for each $0 < i \leq n$, $v(x_i) = d_i$, and λ is either a Γ-term or a Γ-formula, we abbreviate by $\|\lambda(d_1, \ldots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}$ the expression $\|\lambda(x_1, \ldots, x_n)\|^{\mathbf{B}}_{\mathbf{M},v}$.

Now let ϕ be a Γ-sentence, given a $\mathbf{B}$-*structure* $\mathbf{M}$ for a predicate language Γ, it is said that $\mathbf{M}$ is a *model* of ϕ iff $\|\phi\|^{\mathbf{B}}_{\mathbf{M}} = 1$. And that $\mathbf{M}$ is a model of a set of Γ-sentences Σ iff for all $\phi \in \Sigma$, $\mathbf{M}$ is a model of ϕ. Let $T \cup \{\phi\}$ be a set of Γ-sentences. We say that ϕ is a *semantical consequence* of T (denoted by $T \models \phi$) iff for every $\mathbf{B}$-*structure* $\mathbf{M}$, if $\mathbf{M}$ is a model of T, then $\mathbf{M}$ is also a model of ϕ. From now on, we say that $(\mathbf{M}, \mathbf{B})$ is a Γ-*structure* instead of saying that $\mathbf{M}$ is a $\mathbf{B}$-structure for a predicate language Γ. In this section we have presented only a few definitions and notation, a detailed introduction to the syntax and semantics of fuzzy predicate logics can be found in [13] and [7].

III. Ultraproducts over an L-Algebra

The first notion of ultraproduct we study is defined over a fixed L-algebra. See for instance [17] for Rational Pavelka's logic (RPL) and [16] in the case of first-order fuzzy logic with graded syntax. Here we work with ultraproducts over a fixed L-algebra, but for arbitrary fuzzy predicate languages, using κ-complete ultrafilters.

Definition 1: Let I be a non-empty set and κ an infinite cardinal. A filter H over I is said to be κ-*complete* iff the intersection of any non-empty set of fewer than κ elements of H belongs to H.

Definition 2: Let I be a non-empty set and for each $i \in I$, let $(\mathbf{M}_i, \mathbf{B})$ be a Γ-structure. Assume that U is a κ-complete ultrafilter over I such that $|\mathbf{B}| < \kappa$. The *ultraproduct* $(\prod \mathbf{M_i}/U, \mathbf{B})$ of the structures $\{(\mathbf{M}_i, \mathbf{B}) : i \in I\}$ has as algebraic part (regarded as a classical first-order structure) the usual ultraproduct construction, that is, the direct product quotient modulo the congruence θ_U defined as usual: for every $\overline{d}, \overline{e} \in \prod M_i$, $(\overline{d}, \overline{e}) \in \theta_U$ iff $\{i \in I : \overline{d}(i) = \overline{e}(i)\} \in U$. And for each n-ary predicate $P \in \Gamma$, and every $\overline{d_1}, \ldots, \overline{d_n} \in \prod M_i$,

$$\|P([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U} = b \text{ iff}$$
$$\{i \in I : \|P(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}} = b\} \in U$$

We can see the first limitation of this definition in the assumption of κ-completeness of the ultrafilters. Observe that the ultraproduct is well-defined because, by Lemma 4.2.3. of [5], a proper ultrafilter U over a nonempty set I is κ-complete iff for every partition of I into fewer that κ parts, exactly one of the parts belongs to U. Then we consider the partition $(X_b : b \in B)$ where for each $b \in B$, $X_b = \{i \in I : \|P(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}} = b\}$. Since $|\mathbf{B}| < \kappa$, by Lemma 4.2.3. of [5], for some $b \in B$, $X_b \in U$.

It is a well-known fact (see Proposition 4.2.1. of [5]) that a filter H over a nonempty set I is κ-complete for every cardinal κ iff H is principal, but clearly we are not interested in principal ultrafilters (for instance when dealing with ultrapowers). Moreover, If κ is a singular cardinal, there are no nonprincipal κ-complete filters over κ (for a reference see [15]). Measurable cardinals κ are those for which there exists a nonprincipal κ-complete ultrafilter over κ. Clearly, ω is measurable. However κ-complete nonprincipal ultrafilters are much harder to come by for uncountable cardinals: the existence of uncountable measurable regular cardinals is a big cardinal axiom independent of ZFC. Thus, apart form set-theoretical considerations, it is precisely the case in which $\mathbf{B}$ is a finite algebra that makes this definition interesting.

Now, regarding the advantages, remark that, so defined, ultraproducts of classical first-order structures are two-valued and thus, our definition is an extension of the classical notion of ultraproduct. Now we present an analogue to the Łoś Theorem for ultraproducts in fuzzy predicate logics:

Theorem 3: Let I be a non-empty set and for each $i \in I$, let $(\mathbf{M}_i, \mathbf{B})$ be a Γ-structure. Assume that U is a κ-complete ultrafilter over I such that $|\mathbf{B}| < \kappa$. Then for every Γ-formula $\phi(x_1, \ldots, x_n)$ and elements $\overline{d_1}, \ldots, \overline{d_n} \in \prod M_i$, $b \in B$,

$$\|\phi([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U} = b \text{ iff}$$
$$\{i \in I : \|\phi(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}} = b\} \in U$$

Proof: By induction on the complexity of ϕ. For ϕ atomic it is clear, because U is an ultrafilter and by definition of the ultraproduct. Assume that for the Γ-formulas $\phi_1, \ldots, \phi_k$ the property holds. Let $\delta \in$L be a k-ary connective and for every $0 < j \leq k$, $\|\phi_j([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U} = a_j$.

($\Rightarrow$) If $\|\delta(\phi_1, \ldots, \phi_k)([\overline{d_1}]_{\theta_U}, \ldots, [\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U} = b$, then $\delta(a_1, \ldots, a_k) = b$. Now for every $0 < j \leq k$, let $\Psi_j = \{i \in I : \|\phi_j(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}} = a_j\}$. By inductive hypothesis, $\Psi_1 \cap \ldots \cap \Psi_k \in U$ and since $\Psi_1 \cap \ldots \cap \Psi_k$ is included in $\{i \in I : \|\delta(\phi_1, \ldots, \phi_k)(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}} = b\}$ and U is an ultrafilter we have

$$\{i \in I : \|\delta(\phi_1, \ldots, \phi_k)(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}} = b\} \in U.$$

($\Leftarrow$) Use the result we have just obtained to see that if $\{i \in I : \|\delta(\phi_1, \ldots, \phi_k)(\overline{d_1}(i), \ldots, \overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}} = b\} \in U$, since U is an ultrafilter, necessarily we have $\delta(a_1,\ldots,a_k)=b$. Therefore $\|\delta(\phi_1,\ldots,\phi_k)([\overline{d_1}]_{\theta_U},\ldots,[\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U}=b$.

Finally we prove the universal quantifier step. Assume inductively that the property holds for the Γ-formula $\phi(y,x_1\ldots x_n)$.

($\Rightarrow$) If $\|\forall x\phi([\overline{d_1}]_{\theta_U},\ldots,[\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U}=b$ we define

$$\Theta=\{\|\phi([\overline{e}]_{\theta_U},[\overline{d_1}]_{\theta_U},\ldots,[\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U}:\overline{e}\in\prod M_i\}.$$

We have that $b = inf\Theta$. Now we choose, for every $a\in\Theta$, $\overline{e}_a\in\prod M_i$ such that $a=\|\phi([\overline{e}_a]_{\theta_U},[\overline{d_1}]_{\theta_U},\ldots,[\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U}$. Let us denote by X_a the set $\{i\in I:\|\phi(\overline{e}_a(i),\overline{d_1}(i),\ldots,\overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}}=a\}$. By inductive hypothesis, for every $a\in\Theta$, $X_a\in U$ and since U is a κ-complete ultrafilter over I such that $|\mathbf{B}|<\kappa$, $\bigcap_{a\in\Theta}X_a\in U$. Let now $b_0\in B$ be such that

$$\{i\in I:\|\forall x\phi(\overline{d_1}(i),\ldots,\overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}}=b_0\}\in U.$$

Such a b_0 exists because the collection

$$(\{i\in I:\|\forall x\phi(\overline{d_1}(i),\ldots,\overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}}=b'\}:b'\in B)$$

is a partition of I into fewer than κ parts. Thus exactly one of the parts belongs to U, by κ-completeness. Observe that this implies that $b_0\le b$, because $\bigcap_{a\in\Theta}X_a\in U$.

We show now that $b_0=b$. Let us assume the contrary, that $b_0<b$. Then for each element

$$j\in\{i\in I:\|\forall x\phi(\overline{d_1}(i),\ldots,\overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}}=b_0\}$$

we could choose $k_j\in M_j$ such that

$$\|\phi(k_j,\overline{d_1}(j),\ldots,\overline{d_n}(j))\|^{\mathbf{B}}_{\mathbf{M_j}}<b.$$

Now we define an element of the product $\overline{k}\in\prod M_i$ in the following way: for every $i\in I$,

$$\overline{k}(i)=\begin{cases}k_j, & \text{if } i=j\\ \textit{arbitrary,} & \text{otherwise.}\end{cases}$$

And then we set $b_1=\|\phi([\overline{k}]_{\theta_U},[\overline{d_1}]_{\theta_U},\ldots,[\overline{d_n}]_{\theta_U})\|^{\mathbf{B}}_{\prod \mathbf{M_i}/U}$. Hence we have that $b_1\in\Theta$ and thus $b\le b_1$, which is a contradiction. To see that remark that this would imply that

$$\{i\in I:\|\phi(\overline{k}(i),\overline{d_1}(i),\ldots,\overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}}=b_1\}\in U$$

by inductive hypothesis. And at the same time,

$$\{i\in I:\|\phi(\overline{k}(i),\overline{d_1}(i),\ldots,\overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}}<b\}\in U$$

by definition of $\overline{k}$. Therefore we conclude that $b_0=b$ and consequently,

$$\{i\in I:\|\forall x\phi(\overline{d_1}(i),\ldots,\overline{d_n}(i))\|^{\mathbf{B}}_{\mathbf{M_i}}=b\}\in U.$$

The ($\Leftarrow$) direction follows from the same kind of argument than the ($\Leftarrow$) direction in the quantifier-free step. For the existential quantifier the proof is analogous. □

Continuity of Łukasiewicz connectives gives us a better notion of ultraproduct for RPL when working over the canonical algebra $[0,1]_{Ł}$ (in [6] it was shown that continuity guarantees that the limit with respect to an ultrafilter always exists). In this case, the assumption of κ-complete ultrafilters is not needed, the proof of the corresponding Fundamental Ultrafilter Theorem for RPL could be found in [17] and topological results leading to the proof of compactness are stated in [4].

IV. D-Filters and Reduced Products

The second definition we will examine is taken from [11]. The use of this notion would help us to overcome difficulties coming from the assumption of κ-completeness of the ultrafilters. First we recall some basic definitions and facts on homomorphisms and direct products.

Definition 4: Let $(\mathbf{M}_1,\mathbf{B}_1)$ be a Γ_1-structure and $(\mathbf{M}_2,\mathbf{B}_2)$ be a Γ_2-structure with $\Gamma_1\subseteq\Gamma_2$. We say that the pair (f,g) is a *homomorphism* of $(\mathbf{M}_1,\mathbf{B}_1)$ into $(\mathbf{M}_2,\mathbf{B}_2)$ iff

1) $g:\mathbf{B}_1\to\mathbf{B}_2$ is an L-algebra homomorphism of $\mathbf{B}_1$ into $\mathbf{B}_2$.
2) $f:M_1\to M_2$ is a mapping of M_1 into M_2.
3) For each n-ary function symbol $F\in\Gamma_1$ and elements $d_1,\ldots,d_n\in M_1$,
$$f(F_{\mathbf{M}_1}(d_1,\ldots,d_n))=F_{\mathbf{M}_2}(f(d_1),\ldots,f(d_n))$$
4) For each n-ary predicate $P\in\Gamma_1$ and elements $d_1,\ldots,d_n\in M_1$,
$$g(P_{\mathbf{M}_1}(d_1,\ldots,d_n))=P_{\mathbf{M}_2}(f(d_1),\ldots,f(d_n))$$

We say that (f,g) is a *σ-homomorphism* if g preserves the existing infima and suprema.

It is denoted by $(\mathbf{M},\mathbf{B})\cong(\mathbf{N},\mathbf{A})$ when these two structures are isomorphic (that is, there is a homomorphism (f,g) from $(\mathbf{M},\mathbf{B})$ into $(\mathbf{N},\mathbf{A})$ with f and g onto and one-to-one). If (f,g) is a σ-homomorphism of $(\mathbf{M}_1,\mathbf{B}_1)$ into $(\mathbf{M}_2,\mathbf{B}_2)$ such that f is onto, then for each Γ_1-formula $\phi(x_1,\ldots,x_n)$ and elements $d_1,\ldots,d_n\in M_1$,

$$g(\|\phi(d_1,\ldots,d_n)\|^{\mathbf{B}_1}_{\mathbf{M}_1})=\|\phi(f(d_1),\ldots,f(d_n))\|^{\mathbf{B}_2}_{\mathbf{M}_2}\quad(1)$$

We will refer to homomorphisms satisfying condition (1) as *elementary homomorphisms*. And we say that a congruence (E,θ) is *elementary* (σ-congruence, respectively) if the canonical mapping (f_E,g_θ) is an elementary homomorphism (σ-homomorphism, respectively). For a reference of congruences in fuzzy predicate logics see [9].

A. di Nola and G. Gerla introduced in [10] the notions of valuation structure and fuzzy model of a given first-order language in a categorial setting. There they show that certain operations such as direct products preserve first-order properties of this kind of models. In Proposition 2.1 of [10], they prove that the category of the valuation structures of a given type has direct products. Direct products for fuzzy structures were also studied by Bělohlávek in [1], but restricted to structures over complete residuated lattices and languages with an equality symbol interpreted as a similarity. The notion of *direct product of fuzzy algebras* is introduced in [7] and a kind of Birkhoff variety theorem for fuzzy algebras is presented (unpublished result of P. Hájek).

Definition 5: Let I be a non-empty set and for each $i\in I$, $(\mathbf{M}_i,\mathbf{B}_i)$ be a Γ-structure. The *direct product* $(\prod\mathbf{M_i},\prod\mathbf{B_i})$ of the structures $\{(\mathbf{M}_i,\mathbf{B}_i):i\in I\}$ is defined as follows:

- The domain is the cartesian product $\prod M_i$.
- $\prod \mathbf{B_i}$ is the direct product of the L-algebras $\{\mathbf{B}_i : i \in I\}$.
- For each n-ary function symbol $F \in \Gamma$, and every $\overline{d_1}, \dots, \overline{d_n} \in \prod M_i$,
$$F_{\prod \mathbf{M_i}}(\overline{d_1}, \dots, \overline{d_n}) = (F_{\mathbf{M}_i}(\overline{d_1}(i), \dots, \overline{d_n}(i)) : i \in I)$$
- For each n-ary predicate $P \in \Gamma$, and every $\overline{d_1}, \dots, \overline{d_n} \in \prod M_i$,
$$P_{\prod \mathbf{M_i}}(\overline{d_1}, \dots, \overline{d_n}) = (P_{\mathbf{M}_i}(\overline{d_1}(i), \dots, \overline{d_n}(i)) : i \in I)$$

Remark that the direct product is well-defined because the class of L-algebras is a variety and thus is closed under direct products. Since elements of the direct product are sequences, we use the notation $\overline{d} = (d(i) : i \in I)$ to refer to them.

G. Gerla introduced in [11] the notions of d-filter, of reduced product and of ultraproduct of a family of fuzzy models with definable quantifiers. That is, models such that for each quantifier there is a formula of the classical first-order language with equality with a unique monadic predicate A that defines it (for a reference see Definition 8.1 of [11]). He proved that these operations preserve first-order properties of fuzzy models with definable quantifiers. Our definition is based in that of G. Gerla, we use also pairs of filters (d-filters in Gerla's terms). We left for future work the study of the relationship between our research and the results of Bělohlávek on fuzzy Horn logic in [2] obtained used the notion of *safe reduced product*.

Definition 6: Let I be a non-empty set and for each $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i)$ be a Γ-structure. Let G and H be proper filters over I with $G \subseteq H$. The *reduced product* $(\prod \mathbf{M_i}/G, \prod \mathbf{B_i}/H)$ of the structures $\{(\mathbf{M}_i, \mathbf{B}_i) : i \in I\}$ is the quotient structure of $(\prod \mathbf{M_i}, \prod \mathbf{B_i})$ modulo the congruence (θ_G, θ_H), where θ_G, θ_H are defined as follows:

$$(\overline{d}, \overline{e}) \in \theta_G \text{ iff } \{i \in I : \overline{d}(i) = \overline{e}(i)\} \in G$$
$$(\overline{a}, \overline{b}) \in \theta_H \text{ iff } \{i \in I : \overline{a}(i) = \overline{b}(i)\} \in H$$

for every $\overline{d}, \overline{e} \in \prod M_i$ and $\overline{a}, \overline{b} \in \prod B_i$.

Observe that the reduced product is well-defined because the class of L-algebras is a variety and thus is closed under reduced products. Special cases of reduced products are the direct products, when $G = H = \{I\}$; *ultraproducts*, when H is an ultrafilter; and *reduced powers*, when for each $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i) = (\mathbf{M}, \mathbf{B})$ for the same structure. Remark that, when G is an ultrafilter, then H is also an ultrafilter, because $G \subseteq H$. Now we present an analogue to the Łoś Theorem for reduced products in fuzzy predicate logics:

Theorem 7: Let I be a non-empty set and for each $i \in I$, let $(\mathbf{M}_i, \mathbf{B}_i)$ be a Γ-structure, G and H proper filters over I with $G \subseteq H$ and such that (θ_G, θ_H) is a σ-congruence. Then for every Γ-formula $\phi(x_1, \dots, x_n)$ and elements $\overline{d_1}, \dots, \overline{d_n} \in \prod M_i$, $\overline{b} \in \prod B_i$,

$$\|\phi([\overline{d_1}]_{\theta_G}, \dots, [\overline{d_n}]_{\theta_G})\|^{\prod \mathbf{B_i}/H}_{\prod \mathbf{M_i}/G} = [\overline{b}]_{\theta_H} \text{ iff}$$
$$\{i \in I : \|\phi(\overline{d_1}, \dots, \overline{d_n})\|^{\prod \mathbf{B_i}}_{\prod \mathbf{M_i}}(i) = \overline{b}(i)\} \in H \text{ iff}$$
$$\{i \in I : \|\phi(\overline{d_1}(i), \dots, \overline{d_n}(i))\|^{\mathbf{B_i}}_{\mathbf{M_i}} = \overline{b}(i)\} \in H$$

Proof: Let $(\mathbf{B}_i : i \in I)$ be L-algebras and $(\overline{b_j} : j \in J)$ be a sequence of elements of its direct product $\prod \mathbf{B_i}$. Then we have that, if $sup_{j \in J}\overline{b_j}(i)$ exists then $[sup_{j \in J}\overline{b_j}](i) = sup_{j \in J}\overline{b_j}(i)$ (analogously for the infimum of a sequence of elements). As a direct consequence we obtain that the i-projection (f_i, g_i) defined in the natural way, for each coordinate $i \in I$ is a σ-homomorphism with f_i and g_i onto. Thus, for every Γ-formula ϕ and every $\overline{d_1}, \dots, \overline{d_n} \in \prod M_i$,

$$\|\phi(\overline{d_1}, \dots, \overline{d_n})\|^{\prod \mathbf{B_i}}_{\prod \mathbf{M_i}} = (\|\phi(\overline{d_1}(i), \dots, \overline{d_n}(i))\|^{\mathbf{B_i}}_{\mathbf{M_i}} : i \in I)$$

Using this result and the fact that σ-congruences are elementary congruences (for a reference see [9]) we obtain that

$$\|\phi([\overline{d_1}]_{\theta_G}, \dots, [\overline{d_n}]_{\theta_G})\|^{\prod \mathbf{B_i}/H}_{\prod \mathbf{M_i}/G} = [\|\phi(\overline{d_1}, \dots, \overline{d_n})\|^{\prod \mathbf{B_i}}_{\prod \mathbf{M_i}}]_{\theta_H}$$

and consequently, the desired result. □

The advantage of working with reduced products in general (instead that working only with ultraproducts) is that we can guarantee the existence of such σ-congruences. Therefore we can use Theorem 7 to obtain the desired structures preserving first-order properties. σ-congruences can be obtained using κ-complete filters for κ big enough (depending of the chosen $(\mathbf{M}_i, \mathbf{B}_i)$ structures and the cardinality of the language). Examples of κ-complete filters are the following:

- If κ is a regular cardinal, then the set of all $X \subseteq \kappa$ for which the cardinality of its complement $(\kappa - X)$ is smaller than κ, is a κ-complete nonprincipal filter over κ.
- Let $I = [0, 1]$ be the real unit interval and μ the Lebesgue measure. Then the set $H = \{X \subseteq [0, 1] : \mu(X) = 1\}$ is a countably complete filter.

It is easy to check that one of the important applications of ultraproducts holds in equality-free languages for reduced products: the reduced power of one structure is an elementary extension of this structure. On the side of the drawbacks we have that, in general, reduced products or ultraproducts of classical first-order structures are not necessarily two-valued.

V. Reduced Products and Leibniz congruences

Reduced semantics for fuzzy predicate logics were introduced in [9], where completeness results of these logics with respect to the semantics were presented. Now we show that we can use also reduced products when working with reduced models. In order to do so we begin by proving that the class of reduced models is closed under direct products. First let us remember some definitions and basic facts on reduced structures.

Definition 8: Let $(\mathbf{M}, \mathbf{B})$ be a Γ-structure and θ an L-congruence on $\mathbf{B}$. We define the relation $\Omega(\mathbf{M}, \mathbf{B}, \theta) \subseteq M \times M$ as follows: for every $d, e \in M$, $(d, e) \in \Omega(\mathbf{M}, \mathbf{B}, \theta)$ iff for every atomic Γ-formula, $\phi(y, x_1, \dots, x_n)$ and elements $d_1, \dots, d_n \in M$,

$$(\|\phi(d, d_1, \dots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}, \|\phi(e, d_1, \dots, d_n)\|^{\mathbf{B}}_{\mathbf{M}}) \in \theta$$

Fixed an L-congruence θ on the L-algebra $\mathbf{B}$, we showed in [9] that $\Omega(\mathbf{M}, \mathbf{B}, \theta)$ is the greatest E such that (E, θ) is a congruence on the model $(\mathbf{M}, \mathbf{B})$. Let I be a non-empty set and for each $i \in I$, let $(\mathbf{M}_i, \mathbf{B}_i)$ be a Γ-structure, G and H be proper filters over I with $G \subseteq H$ and (θ_G, θ_H)

a σ-congruence. Then it is easy to check that both, $(\{1\}, \theta_H)$ and $(\Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \theta_H), \theta_H)$ are σ-congruences. Thus, we can work with two limit cases, the smallest $(\{1\}, \theta_H)$ and the greatest $(\Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \theta_H), \theta_H)$. In the first case, the resulting reduced product is a model whose first-order algebraic structure is the same direct product. This kind of structures are called *filter products*. In equality-free logic, the filter-products and ultrafilter-products play the same role that reduced products and ultraproducts play in logic with equality. They have been considered before by W. Blok and D. Pigozzi in [3], for the special case of logical matrices.

Definition 9: It is said that a Γ-structure $(\mathbf{M}, \mathbf{B})$ is *reduced* iff $\Omega(\mathbf{M}, \mathbf{B}, Id_{\mathbf{B}})$ is the identity relation.

From now on we denote $(\Omega(\mathbf{M}, \mathbf{B}, Id_{\mathbf{B}}), Id_{\mathbf{B}})$ simply by $\Omega(\mathbf{M}, \mathbf{B})$ and we call it the *Leibniz congruence* of $(\mathbf{M}, \mathbf{B})$. Since the identity map clearly preserves infima and suprema, $\Omega(\mathbf{M}, \mathbf{B})$ is always a σ-congruence. The Leibniz congruence of a model identifies the elements that are indistinguishable using equality-free atomic formulas and parameters from the model. A reduced structure is the quotient of a model modulo this congruence. We will denote by $(\mathbf{M}, \mathbf{B})^r$ the quotient structure modulo the Leibniz congruence $\Omega(\mathbf{M}, \mathbf{B})$ and call it the *reduction* of $(\mathbf{M}, \mathbf{B})$.

Notation: For the sake of clarity from now on we will use both notations $(\prod \mathbf{M_i}, \prod \mathbf{B_i})$ and $\prod(\mathbf{M_i}, \mathbf{B_i})$ to refer to the direct product of the structures $\{(\mathbf{M}_i, \mathbf{B}_i) : i \in I\}$.

Lemma 10: Let I be a non-empty set and for each $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i)$ be a Γ-structure. If for every $i \in I$, (E_i, θ_i) is a congruence on $(\mathbf{M}_i, \mathbf{B}_i)$ we define $(\prod E_i, \prod \theta_i)$ by: for every $\overline{d}, \overline{e} \in \prod M_i$,

$(\overline{d}, \overline{e}) \in \prod E_i$ iff for every $i \in I$ $(\overline{d}(i), \overline{e}(i)) \in E_i$

and for every $\overline{a}, \overline{b} \in \prod B_i$,

$(\overline{a}, \overline{b}) \in \prod \theta_i$ iff for every $i \in I$ $(\overline{a}(i), \overline{b}(i)) \in \theta_i$

So defined $(\prod E_i, \prod \theta_i)$ is a congruence on $(\prod \mathbf{M_i}, \prod \mathbf{B_i})$. Moreover, $(\prod E_i, \prod \theta_i)$ is a σ-congruence iff for every $i \in I$, (E_i, θ_i) are σ-congruences.

Next proposition shows that the greatest congruence in a direct product is precisely the direct product of the greatest congruences of each model of the family. Consequently, we obtain that the Leibniz congruence of a direct product is the direct product of the Leibniz congruences.

Proposition 11: Let I be a non-empty set and for each $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i)$ be a Γ-structure. If for every $i \in I$, $(\Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i), \theta_i)$ is a congruence on $(\mathbf{M}_i, \mathbf{B}_i)$, then $(\prod \Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i), \prod \theta_i) = (\Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i), \prod \theta_i)$.

Proof: By Lemma 10, since Ω is the greatest congruence we have that $\prod \Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i) \subseteq \Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i)$. Conversely, assume that $\overline{d}, \overline{e} \in \prod M_i$ and $(\overline{d}, \overline{e}) \in \Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i)$, we need to show that, for every $i \in I$, $(\overline{d}(i), \overline{e}(i)) \in \Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i)$. By Definition 8, it is enough to prove that for every $i \in I$ the following holds: for every atomic Γ-formula, $\phi(y, x_1, \ldots, x_n)$ and elements $k_1^i, \ldots, k_n^i \in M_i$,

$$(\|\phi(\overline{d}(i), k_1^i, \ldots, k_n^i)\|_{\mathbf{M}_i}^{\mathbf{B}_i}, \|\phi(\overline{e}(i), k_1^i, \ldots, k_n^i)\|_{\mathbf{M}_i}^{\mathbf{B}_i}) \in \theta_i$$

We fix $i_0 \in I$, an atomic Γ-formula $\phi(y, x_1, \ldots, x_n)$ and elements $k_1^{i_0}, \ldots, k_n^{i_0} \in M_i$. Now we define $\overline{k}_1, \ldots, \overline{k}_n \in \prod M_{i_0}$ as follows:

$$\overline{k_j}(i) = \begin{cases} k_j^{i_0}, & \text{if } i = i_0 \\ 1, & \text{otherwise.} \end{cases}$$

for every $0 < j \leq n$. Since $(\overline{d}, \overline{e}) \in \Omega((\prod \mathbf{M_i}, \prod \mathbf{B_i}), \prod \theta_i)$, by Definition 8,

$$(\|\phi(\overline{d}, \overline{k}_1, \ldots, \overline{k}_n)\|_{\prod \mathbf{M_i}}^{\prod \mathbf{B_i}}, \|\phi(\overline{e}, \overline{k}_1, \ldots, \overline{k}_n)\|_{\prod \mathbf{M_i}}^{\prod \mathbf{B_i}}) \in \prod \theta_i$$

thus, by definition of $\prod \theta_i$, we obtain the desired result:

$$(\|\phi(\overline{d}(i_0), k_1^{i_0}, \ldots, k_n^{i_0})\|_{\mathbf{M}_{i_0}}^{\mathbf{B}_{i_0}}, \|\phi(\overline{e}(i_0), k_1^{i_0}, \ldots, k_n^{i_0})\|_{\mathbf{M}_{i_0}}^{\mathbf{B}_{i_0}}) \in \theta_{i_0}. \qquad \square$$

Now we prove that the reduction of a direct product is isomorphic to the direct product of reductions.

Theorem 12: Let I be a non-empty set and for each $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i)$ be a Γ-structure, then

$$(\prod \mathbf{M_i}, \prod \mathbf{B_i})^r \cong \prod (\mathbf{M}_i, \mathbf{B}_i)^r$$

Proof: We prove in general the following fact: let I be a non-empty set and for each $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i)$ be a Γ-structure. If for every $i \in I$, $(\Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i), \theta_i)$ is a congruence on $(\mathbf{M}_i, \mathbf{B}_i)$, then

$$(\prod \mathbf{M_i}/\Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i), \prod \mathbf{B_i}/\prod \theta_i)$$

is isomorphic to

$$(\prod \mathbf{M}_i/\Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i), \prod \mathbf{B}_i/\theta_i)$$

First we define (f, g) as follows: for every $\overline{b} \in \prod B_i$, $g(\overline{b}) = ([\overline{b}(i)]_{\theta_i} : i \in I)$ and for every $\overline{d} \in \prod M_i$, $f(\overline{d}) = ([\overline{d}(i)]_{\Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i)} : i \in I)$. So defined, it is easy to check that (f, g) is a homomorphism from $(\prod \mathbf{M_i}, \prod \mathbf{B_i})$ into $(\prod \mathbf{M}_i/\Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i), \prod \mathbf{B}_i/\theta_i)$ with f and g onto. Now we show that $ker(g) = \prod \theta_i$. Let $\overline{a}, \overline{b} \in \prod B_i$, then by definition of g,

$$g(\overline{a}) = g(\overline{b}) \text{ iff } ([\overline{a}(i)]_{\theta_i} : i \in I) = ([\overline{b}(i)]_{\theta_i} : i \in I).$$

This happens iff for every $i \in I$, $(\overline{a}(i), \overline{b}(i)) \in \theta_i$ (by definition of the congruence relation and of the direct product) iff $(\overline{a}, \overline{b}) \in \prod \theta_i$ (by definition of $\prod \theta_i$). By an analogous proof it can be shown that $ker(f) = \prod \Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i)$. Therefore, by Proposition 11, we have $ker(f) = \Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i)$ and thus we obtain the desired isomorphism. To obtain the exact statement of the theorem, for every $i \in I$, take $\theta_i = Id_{\mathbf{B}_i}$. $\square$

In [9] we defended that the relative relation between two structures, denoted by $\sim$, was a good candidate to play the same role that the isomorphism relation plays in classical predicate languages with equality. In [9] we presented different characterizations of this relation one of the most interesting is the condition stated in Theorem 14 of [9]:

Theorem 13: Let $(\mathbf{M}_1, \mathbf{B}_1)$ and $(\mathbf{M}_2, \mathbf{B}_2)$ be two Γ-structures. The following are equivalent:

1) There is a relative relation $(R, T) : (\mathbf{M}_1, \mathbf{B}_1) \sim (\mathbf{M}_2, \mathbf{B}_2)$
2) There are congruences (E_1, θ_1) and (E_2, θ_2) such that

$$(\mathbf{M}_1/E_1, \mathbf{B}_1/\theta_1) \cong (\mathbf{M}_2/E_2, \mathbf{B}_2/\theta_2)$$

Now we show that $\sim$ satisfies another property that isomorphisms have, that is, the direct products of two families of relative structures are also relative.

Theorem 14: Let I be a non-empty set and for each $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i)$ and $(\mathbf{N}_i, \mathbf{A}_i)$ be Γ-structures. If for every $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i) \sim (\mathbf{N}_i, \mathbf{A}_i)$, then $(\prod \mathbf{M_i}, \prod \mathbf{B_i}) \sim (\prod \mathbf{N_i}, \prod \mathbf{A_i})$.

Proof: Assume that for every $i \in I$, $(\mathbf{M}_i, \mathbf{B}_i) \sim (\mathbf{N}_i, \mathbf{A}_i)$. By the proof of Theorem 13 in [9], for every $i \in I$, there are congruences $(\Omega(\mathbf{M_i}, \mathbf{B_i}, \theta_i), \theta_i)$ and $(\Omega(\mathbf{N_i}, \mathbf{A_i}, \tau_i), \tau_i)$ such that

$$(\mathbf{M}_i/\Omega(\mathbf{M_i}, \mathbf{B_i}, \theta_i), \mathbf{B}_i/\theta_i) \cong (\mathbf{N}_i/\Omega(\mathbf{N_i}, \mathbf{A_i}, \tau_i), \mathbf{A}_i/\tau_i)$$

Therefore there is an isomorphism between the direct products of this two families of structures,

$$(\prod \mathbf{M}_i/\Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i), \prod \mathbf{B}_i/\theta_i)$$

and $(\prod \mathbf{N}_i/\Omega(\mathbf{N}_i, \mathbf{A}_i, \tau_i), \prod \mathbf{A}_i/\tau_i)$. But then, by the proof of Theorem 12, the model

$$(\prod \mathbf{M_i}/\Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i), \prod \mathbf{B_i}/\prod \theta_i)$$

is isomorphic to

$$(\prod \mathbf{M}_i/\Omega(\mathbf{M}_i, \mathbf{B}_i, \theta_i), \prod \mathbf{B}_i/\theta_i).$$

And the model

$$(\prod \mathbf{N_i}/\Omega(\prod \mathbf{N_i}, \prod \mathbf{A_i}, \prod \tau_i), \prod \mathbf{A_i}/\prod \tau_i)$$

is isomorphic to

$$(\prod \mathbf{N}_i/\Omega(\mathbf{N}_i, \mathbf{A}_i, \tau_i), \prod \mathbf{A}_i/\tau_i).$$

Consequently,

$$(\prod \mathbf{M_i}/\Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i), \prod \mathbf{B_i}/\prod \theta_i)$$

and

$$(\prod \mathbf{N_i}/\Omega(\prod \mathbf{N_i}, \prod \mathbf{A_i}, \prod \tau_i), \prod \mathbf{A_i}/\prod \tau_i)$$

are also isomorphic and the congruences

$$(\Omega(\prod \mathbf{M_i}, \prod \mathbf{B_i}, \prod \theta_i), \prod \theta_i)$$

and

$$(\Omega(\prod \mathbf{N_i}, \prod \mathbf{A_i}, \prod \tau_i), \prod \tau_i)$$

satisfy condition 2 of Theorem 13, thus we can conclude that $(\prod \mathbf{M_i}, \prod \mathbf{B_i}) \sim (\prod \mathbf{N_i}, \prod \mathbf{A_i})$. $\square$

By using the same kind of arguments it can be shown that the class of reduced models of a fuzzy predicate logic is closed under reduced products in general. The same happens for the definition of ultraproduct over an L-algebra studied in section III. For the lack of space we can not present all these proofs here.

VI. Future Work

Work in progress includes applications of the fundamental theorem to the characterization of elementary classes. Future work will be devoted to the study of universal Horn classes and to give characterizations of the notion of elementary equivalence and to explore some strengthenings of this notion.

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