Exploring extensions of possibilistic logic over Gödel logic

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Abstract. In this paper we present completeness results of several fuzzy logics trying to capture different notions of necessity (in the sense of Possibility theory) for Gödel logic formulas. In a first attempt, based on different characterizations of necessity measures on fuzzy sets, a group of logics, with Kripke style semantics, is built over a restricted language, indeed a two-level language composed of non-modal and modal formulas, the latter moreover not allowing nested applications of the modal operator N. Besides, a full fuzzy modal logic for graded necessity over Gödel logic is also introduced together with an algebraic semantics, the class of NG-algebras.

1 Introduction

The most general notion of uncertainty is captured by monotone set functions with two natural boundary conditions. In the literature, these functions have received several names, like *Sugeno measures* [24] or *plausibility measures* [20]. Many popular uncertainty measures, like probabilities, upper and lower probabilities, Dempster-Shafer plausibility and belief functions, or possibility and necessity measures, can be therefore seen as particular classes of Sugeno measures.

In this paper, we specially focus on possibilistic models of uncertainty. A *possibility measure* on a complete Boolean algebra of events $\mathcal{U} = (U, \wedge, \vee, \neg, \overline{0}^{\mathcal{U}}, \overline{1}^{\mathcal{U}})$ is a Sugeno measure μ^* satisfiying the following $\vee$-decomposition property for any countable set of indices I

$$\mu^*(\vee_{i\in I}\, u_i) = \sup_{i\in I} \mu^*(u_i),$$

while a *necessity measure* is a Sugeno measure μ_* satisfying the $\wedge$-decomposition property

$$\mu_*(\wedge_{i\in I}\, u_i) = \inf_{i\in I} \mu_*(u_i).$$

Possibility and necessity are *dual* classes of measures, in the sense that if μ^* is a possibility measure, then the mapping $\mu_*(u) = 1 - \mu^*(\neg u)$ is a necessity measure, and vice versa. If $\mathcal{U}$ is the power set of a set X, then any dual pair of measures (μ^*, μ_*) on $\mathcal{U}$ is induced by a *normalized possibility distribution*, i.e. a mapping $\pi : X \to [0,1]$ such that, $\sup_{x \in X} \pi(x) = 1$, and, for any $A \subseteq X$,

$$\mu^*(A) = \sup\{\pi(x) \mid x \in A\} \text{ and } \mu_*(A) = \inf\{1 - \pi(x) \mid x \notin A\}.$$

Appropriate extensions of uncertainty measures on algebras of events more general than Boolean algebras need to be considered in order to represent and reason about the uncertainty of non-classical events. For instance, the notion of (finitely additive) probability has been generalized in the setting of MV-algebras by means of the notion of *state* [22]. In particular, the well-known Zadeh's notion of probability for fuzzy sets (as the expected value of the membership function) defines a state over an MV-algebra of fuzzy sets. States on MV-algebras have been used in [12] to provide a logical framework for reasoning about the probability of (finitely-valued) fuzzy events. Another generalization of the notion of probability measure has been recently studied in depth by defining probabilistic states over Gödel algebras [1].

On the other hand, extensions of the notions of possibility and necessity measures for fuzzy sets have been proposed under different forms and used in different logical systems extending the well-known Dubois-Lang-Prade's possibilistic logic to fuzzy events, see e.g. [7, 9, 16, 3, 2, 4]. All the notions of necessity for fuzzy sets considered in the literature turn out to be of the form

$$N(A) = \inf_{x \in U} \pi(x) \Rightarrow A(x) \qquad (*)$$

where A is a fuzzy set in some domain U, $\pi : U \to [0,1]$ is a possibility distribution on U and $\Rightarrow$ is some suitable many-valued implication function. In particular, the following notions of necessity have been discussed:

(1) $x \Rightarrow_{KD} y = \max(1-x, y)$ (Kleene-Dienes);
(2) $x \Rightarrow_{RG} y = 1$ if $x \leq y$, and $x \Rightarrow_{RG} y = 1-x$ otherwise (reciprocal of Gödel);
(3) $x \Rightarrow_{Ł} y = \min(1, 1-x+y)$ (Łukasiewicz).

All these definitions actually extend the above definition over classical sets or events.

In the literature different logical formalizations to reason about such extensions of the necessity of fuzzy events can be found. In [19], and later in [17], a full many-valued modal approach is developed over the finitely-valued Łukasiewicz logic in order to capture the notion of necessity defined using $\Rightarrow_{KD}$. A logic programming approach over Gödel logic is investigated in [3] and in [2] by relying on $\Rightarrow_{KD}$ and $\Rightarrow_{RG}$, respectively. More recently, following the approach of [12], modal-like logics to reason about the necessity of fuzzy events in the framework of MV-algebras have been defined in [13], in order to capture the notion of necessity defined by $\Rightarrow_{KD}$ and $\Rightarrow_{Ł}$.

The purpose of this paper is to explore different logical approaches to reason about the necessity of fuzzy events over Gödel algebras. In more concrete terms, our ultimate aim is to study a full modal expansion of the $[0,1]$-valued Gödel logic with a modality N such that the truth-value of a formula $N\varphi$ (in $[0,1]$) can be interpreted as the degree of necessity of φ, according to some suitable semantics. In this context, although this does not extend the classical possibilistic logic, it seems also interesting to investigate the notion of necessity definable from Gödel implication, which is the standard fuzzy interpretation of the implication connective in Gödel logic:

(4) $x \Rightarrow_G y = 1$ if $x \leq y$, and $x \Rightarrow_G y = y$ otherwise (Gödel);

This work is structured as follows. After this introduction, in Section 2 we recall a characterization of necessity measures on fuzzy sets defined by implications $\Rightarrow_{KD}$ and $\Rightarrow_{RG}$ and provide a (new) characterization of those defined by $\Rightarrow_G$. These characterizations are the basis for the completeness results of several logics introduced in Section 3 capturing the corresponding notions of necessity for Gödel logic formulas. These logics, with Kripke style semantics, are built over a two-level language composed of modal and non-modal formulas, the latter not allowing nested applications of the modal operator. In Section 4 a full fuzzy modal logic for graded necessity over Gödel logic is introduced together with an algebraic semantics. Finally, in Section 5 we mention some open problems and new research goals we plan to address in the near future.

Due to lack of space, we cannot include preliminaries on basic notions regarding Gödel logic and its expansions with truth-constants, with Monteiro-Baaz's operator Δ and with an involutive negation, that will be used throughout the paper. Instead, the reader is referred to [17, 10, 11] for the necessary background.

2 Some necessity measures over Gödel algebras of fuzzy sets and their characterizations

Let X be a (finite) set and let $F(X) = [0,1]^X$ be the set of fuzzy sets over X, i.e. the set of functions $f : X \to [0,1]$. $F(X)$ can be regarded as a Gödel algebra equipped with the pointwise extensions of the operations of the standard Gödel algebra $[0,1]_G$. In the following, for each $r \in [0,1]$, we will denote by $\overline{r}$ the constant function $\overline{r}(x) = r$ for all $x \in X$.

Definition 1 *A mapping $N : F(X) \to [0,1]$ satisfying*

$(N1)$ $N(\wedge_{i\in I} f_i) = \inf_{i\in I} N(f_i)$
$(N2)$ $N(\overline{r}) = r$, *for all* $r \in [0,1]$

is called a basic necessity*.*

If $N : F(X) \to [0,1]$ is a basic necessity then it is easy to check that it also satisfies the following properties:

(i) $\min(N(f), N(\neg_G f)) = 0$
(ii) $N(f \Rightarrow_G g) \leq N(f) \Rightarrow_G N(g)$

The classes of necessity measures based on the Kleene-Dienes implication and the reciprocal of Gödel implication have been already characterized in the literature. We do not consider here the one based on Łukasiewicz implication.

Lemma 2 ([3, 2]) *Let $N : F(X) \to [0,1]$ be a basic necessity. Consider the following properties:*

(N_{KD}) $\quad N(\overline{r} \Rightarrow_{KD} f) = r \Rightarrow_{KD} N(f)$, *for all* $r \in [0,1]$
(N_{RG}) $\quad N(\overline{r} \Rightarrow_{RG} f) = r \Rightarrow_{RG} N(f)$, *for all* $r \in [0,1]$

Then, we have:

(1) $\quad N$ *satisfies* (N_{KD}) *iff* $N(f) = \inf_{x \in X} \pi(x) \Rightarrow_{KD} f(x)$
(2) $\quad N$ *satisfies* (N_{RG}) *iff* $N(f) = \inf_{x \in X} \pi(x) \Rightarrow_{RG} f(x)$

for some possibility distribution $\pi : X \to [0,1]$ *such that* $\sup_{x \in X} \pi(x) = 1$.

The characterization of the necessity measures based on Gödel implication is somewhat more complex since it needs to consider also an associated class of possibility measures which are not dual in the usual strong sense.

Definition 3 *A mapping $\Pi : F(X) \to [0,1]$ satisfying*

$(\Pi 1)$ $\quad \Pi(\vee_{i \in I} f_i) = \sup_{i \in I} \Pi(f_i)$
$(\Pi 2)$ $\quad \Pi(\overline{r}) = r$, *for all* $r \in [0,1]$

is called a basic possibility.

Note that if $\Pi : F(X) \to [0,1]$ is a basic possibility then it also satisfies $\max(\Pi(\neg f), \Pi(\neg\neg f)) = 1$.

For each $x \in X$, let us denote by $\mathbf{x}$ its characteristic function, i.e. the function from $F(X)$ such that $\mathbf{x}(y) = 1$ if $y = x$ and $\mathbf{x}(y) = 0$ otherwise. Observe that each $f \in F(X)$ can be written as

$$f = \bigwedge_{x \in X} \mathbf{x} \Rightarrow_G \overline{f(x)} = \bigvee_{x \in X} \mathbf{x} \wedge \overline{f(x)}.$$

Therefore, if N and Π are a pair of basic necessity and possibility on $F(X)$ respectively, by $(N1)$ and $(\Pi 1)$ we have

$$N(f) = \inf_{x \in X} N(\mathbf{x} \Rightarrow_G \overline{f(x)}) \text{ and } \Pi(f) = \sup_{x \in X} \Pi(\mathbf{x} \wedge \overline{f(x)}).$$

Then we obtain the following characterizations.

Proposition 4 *Let $\Pi : F(X) \to [0,1]$ be a basic possibility. Π further satisfies*

$(\Pi 3)$ $\quad \Pi(f \wedge \overline{r}) = \min(\Pi(f), r)$, *for all* $r \in [0,1]$

iff there exists $\pi : X \to [0,1]$ *such that* $\sup_{x \in X} \pi(x) = 1$ *and, for all* $f \in F(X)$, $\Pi(f) = \sup_{x \in X} \min(\pi(x), f(x))$.

Proof: One direction is easy. Conversely, assume that $\Pi : F(X) \to [0,1]$ satisfies $(\Pi 1)$ and $(\Pi 3)$. Then, taking into account the above observations, we have

$$\Pi(f) = \sup_{x \in X} \Pi(\mathbf{x} \wedge \overline{f(x)}) = \sup_{x \in X} \min(\Pi(\mathbf{x}), f(x)).$$

Hence, the claim easily follows by defining $\pi(x) = \Pi(\mathbf{x})$. $\square$

Proposition 5 *Let $N : F(X) \to [0,1]$ be a basic necessity and $\Pi : F(X) \to [0,1]$ be a basic possibility satisfying $(\Pi 3)$. N and Π further satisfy*

(ΠN) $\quad N(f \Rightarrow_G \overline{r}) = \Pi(f) \Rightarrow_G r$*, for all* $r \in [0,1]$

iff there exists $\pi : X \to [0,1]$ such that $\sup_{x \in X} \pi(x) = 1$ and

$$N(f) = \inf_{x \in X} \pi(x) \Rightarrow_G f(x) \text{ and } \Pi(f) = \sup_{x \in X} \min(\pi(x), f(x)).$$

Proof: As for the possibility Π, this is already shown above in Proposition 4. Let N be defined as $N(f) = \inf_{x \in X} \pi(x) \Rightarrow_G f(x)$ for the possibility distribution $\pi : F(X) \to [0,1]$ determined by Π. We have $N(f \Rightarrow_G \overline{r}) = \inf_{x \in X}(\pi(x) \Rightarrow_G (f(x) \Rightarrow_G r)) = \inf_{x \in X}((\pi(x) \wedge f(x)) \Rightarrow_G r) = (\sup_{x \in X} \pi(x) \wedge f(x)) \Rightarrow_G r = \Pi(f) \Rightarrow_G r$. Hence, Π and N satisfy (ΠN).

Conversely, suppose that N and Π satisfy (ΠN). Then, using the fact that $\Pi(\mathbf{x}) = \pi(x)$ for each $x \in X$, we have $N(f) = \inf_{x \in X} N(\mathbf{x} \Rightarrow_G \overline{f(x)}) = \inf_{x \in X} \Pi(\mathbf{x}) \Rightarrow_G f(x) = \inf_{x \in X} \pi(x) \Rightarrow_G f(x)$. $\square$

3 Four complete logics: the two-level language approach

The language of the logics we are going to consider in this section consists of two classes of formulas:

(i) the set $Fm(V)$ of non-modal formulas $\varphi, \psi \ldots$, which are formulas of $G_\Delta(\mathbb{Q})$ (Gödel logic G expanded with Baaz's projection connective Δ and truth constants $\overline{r}$ for each rational $r \in [0,1]$) built from the set of propositional variables $V = \{p_1, p_2, \ldots\}$;

(ii) and the set $MFm(V)$ of modal formulas $\Phi, \Psi \ldots$, built from atomic modal formulas $N\varphi$, with $\varphi \in Fm(V)$, where N denotes the modality *necessity*, using the connectives from G_Δ and truth constants $\overline{r}$ for each rational $r \in [0,1]$. Notice that nested modalities are not allowed.

The axioms of the logic NG^0 of basic necessity are the axioms of $G_\Delta(\mathbb{Q})$ for non-modal and modal formulas plus the following necessity related modal axioms:

$(N1)$ $N(\varphi \to \psi) \to (N\varphi \to N\psi)$
$(N2)$ $N(\overline{r}) \leftrightarrow \overline{r}$, for each $r \in [0,1] \cap \mathbb{Q}$.

The rules of inference of NG^0 are *modus ponens* (for modal and non-modal formulas) and *necessitation*: from $\vdash \varphi$ infer $\vdash N\varphi$.

It is worth noting that NG^0 proves the formula $N(\varphi \wedge \psi) \leftrightarrow (N\varphi \wedge N\psi)$, which encodes a characteristic property of necessity measures.

As for the semantics we consider several classes of *possibilistic* Kripke models. A *basic necessity Kripke model* is a system $\mathcal{M} = \langle W, e, I\rangle$ where:

- W is a non-empty set whose elements are called *nodes* or *worlds*,
- $e : W \times V \to [0,1]$ is such that, for each $w \in W$, $e(w, \cdot) : V \to [0,1]$ is an evaluation of propositional variables which is extended to a $G_\Delta(\mathbb{Q})$-evaluation of (non-modal) formulas of $Fm(V)$ in the usual way.
- For each $\varphi \in Fm(V)$ we define its associated function $\hat{\varphi}_W : W \to [0,1]$, where $\hat{\varphi}_W(w) = e(w,\varphi)$. Let $\widehat{Fm} = \{\hat{\varphi} \mid \varphi \in Fm(V)\}$
- $I : \widehat{Fm} \to [0,1]$ is a basic necessity over $\widehat{Fm}$ (as a G-algebra), i.e. it satisfies (i) $I(\hat{\overline{r}}_W) = r$, for all $r \in [0,1] \cap \mathbb{Q}$
(ii) $I(\wedge_{i\in I}\hat{\varphi}_{iW}) = \inf_{i\in I} I(\hat{\varphi}_{iW})$.

Now, given a modal formula Φ, the truth value of Φ in $\mathcal{M} = \langle W, e, I\rangle$, denoted $\|\Phi\|_{\mathcal{M}}$, is inductively defined as follows:

- If Φ is an atomic modal formula $N\varphi$, then $\|N\varphi\|_{\mathcal{M}} = I(\hat{\varphi}_W)$
- If Φ is a non-atomic modal formula, then its truth value is computed by evaluating its atomic modal subformulas, and then by using the truth functions associated to the $G_\Delta(\mathbb{Q})$ connectives occurring in Φ.

We will denote by $\mathcal{N}$ the class of basic necessity Kripke models. Then, taking into account that $G_\Delta(Q)$-algebras are locally finite, following the same approach of [13] with the necessary modifications, one can prove the following result.

Theorem 6 *NG^0 is sound and complete for modal theories w.r.t. the class $\mathcal{N}$ of basic necessity structures.*

Now our aim is to consider extensions of NG^0 which faithfully capture the three different notions of necessity measure considered in the previous section. We start by considering the following additional axiom:

$$(N_{KD})\quad N(\overline{r} \vee \varphi) \leftrightarrow (\overline{r} \vee N\varphi),\ \text{ for each } r \in [0,1] \cap \mathbb{Q}.$$

Let NG_{KD} be the axiomatic extension of NG^0 with (N_{KD}). Then, using Lemma 2, it is easy to prove that indeed NG_{KD} captures the reasoning about KD-necessity measures.

Theorem 7 *N_{KD} is sound and complete for modal theories w.r.t. the subclass $\mathcal{N_{KD}}$ of necessity structures $\mathcal{M} = \langle W, e, I\rangle$ such that the necessity measure I is defined as, for every $\varphi \in Fm(V)$, $I(\hat{\varphi}_W) = \inf_{w\in W} \pi(w) \Rightarrow_{KD} \hat{\varphi}_W(w)$ for some possibility distribution $\pi : W \to [0,1]$ on the set of possible worlds W.*

To capture RG-necessities, we need to expand the base logic $G_\Delta(\mathbb{Q})$ with an involutive negation $\sim$. This corresponds to the logic $G_\sim(\mathbb{Q})$, as defined in [10].

So we define NG_{RG} as the axiomatic extension of NG^0 over $G_\sim(\mathbb{Q})$ (instead of over $G_\Delta(\mathbb{Q})$) with the following axiom:

$$(N_{RG})\quad N({\sim}\varphi \to \overline{1-r}) \leftrightarrow ({\sim}N\varphi \to \overline{1-r}),\ \text{ for each } r \in [0,1]\cap\mathbb{Q}.$$

Then, using again Lemma 2 and the fact that also $G_\sim(\mathbb{Q})$-algebras are locally finite, one can also prove the following result.

Theorem 8 *NG_{RG} is sound and complete for modal theories w.r.t. the subclass $\mathcal{N}_{\mathcal{RG}}$ of necessity structures*[1] *$\mathcal{M} = \langle W, e, I\rangle$ such that the necessity measure I is defined as, for every $\varphi \in Fm(V)$, $I(\hat{\varphi}_W) = \inf_{w\in W} \pi(w) \Rightarrow_{RG} \hat{\varphi}_W(w)$ for some possibility distribution $\pi : W \to [0,1]$ on the set of possible worlds W.*

It is worth pointing out that if we added the Boolean axiom $\varphi \vee \neg\varphi$ to the logics N_{KD} and N_{RG}, both extensions would basically collapse into the classical possibilistic logic.

Finally, to define a logic capturing N_G-necessities, we need to expand the language of NG^0 with an additional operator Π to capture the associated possibility measures according to Proposition 5. Therefore we consider the extended set $MFm(V)^+$ of modal formulas $\Phi, \Psi \ldots$ as those built from atomic modal formulas $N\varphi$ and $\Pi\varphi$, with $\varphi \in Fm(V)$, truth-constants $\overline{r}$ for each $r \in [0,1]\cap\mathbb{Q}$ and G_Δ connectives. Then the axioms of the logic $N\Pi_G$ are those of $G_\Delta(\mathbb{Q})$ for non-modal and modal formulas, plus the following necessity related modal axioms:

$(N1)\quad N(\varphi \to \psi) \to (N\varphi \to N\psi)$
$(N2)\quad N(\overline{r}) \leftrightarrow \overline{r},$
$(\Pi 1)\quad \Pi(\varphi \vee \psi) \leftrightarrow (\Pi\varphi \vee \Pi\psi)$
$(\Pi 2)\quad \Pi(\overline{r}) \leftrightarrow \overline{r},$
$(\Pi 3)\quad \Pi(\varphi \wedge \overline{r}) \leftrightarrow (\Pi\varphi \wedge \overline{r})$
$(N\Pi)\quad N(\varphi \to \overline{r}) \leftrightarrow (\Pi\varphi \to \overline{r})$

where $(N2), (\Pi 2), (\Pi 3)$ and $(N\Pi)$ hold for each $r \in [0,1]\cap\mathbb{Q}$. Inference rules of $N\Pi_G$ are those of $G_\Delta(\mathbb{Q})$ and necessitation for N and Π.

Now, we also need to consider expanded Kripke structures of the form $\mathcal{M} = \langle W, e, I, P\rangle$, where W and e are as above and the mappings $I, P :\to [0,1]$ are such that, for every $\varphi \in Fm(V)$, $I(\hat{\varphi}_W) = \inf_{w\in W} \pi(w) \Rightarrow_G \hat{\varphi}_W(w)$ and $P(\hat{\varphi}_W) = \sup_{w\in W} \min(\pi(w), \hat{\varphi}_W(w))$, for some possibility distribution $\pi : W \to [0,1]$. Call $\mathcal{NP}_{\mathcal{G}}$ the class for such structures. Then, using Proposition 5 we get the following result.

Theorem 9 *$N\Pi_G$ is sound and complete for modal theories w.r.t. the class $\mathcal{NP}_{\mathcal{G}}$ of structures.*

[1] With the proviso that the evaluations e of propositional variables extend to $G_\sim(\mathbb{Q})$-evaluations for non-modal formulas and not simply to $G_\Delta(\mathbb{Q})$-evaluations.

4 Possibilistic Necessity Gödel Logic and its algebraic semantics: the full modal approach

The logics defined in the previous section are not proper modal logics since the notion of well-formed formula excludes those formulas with occurrences of nested modalities. Our aim in this section is then to explore a full (fuzzy) modal approach.

We start as simple as possible by defining a fuzzy modal logic over Gödel propositional logic G to reason about the necessity degree of G-propositions. The language of *Possibilistic Necessity Gödel logic*, PNG, is defined as follows: formulas of PNG are built from the set of G-formulas using G-connectives and the operator N. Axioms of PNG are those of Gödel logic plus the following modal axioms:

1. $N(\varphi \to \psi) \to (N\varphi \to N\psi)$.
2. $N\psi \leftrightarrow NN\psi$.
3. $\neg N\overline{0}$.

Deduction rules for PNG are Modus Ponens and Necessitation for N (from ψ derive $N\psi$). These axioms and rules define a notion of proof $\vdash_{PNG}$ in the usual way.

Notice that in PNG the *Congruence Rule* "from $\varphi \leftrightarrow \psi$ derive $N\varphi \leftrightarrow N\psi$" as well as the theorems $N\overline{1}$ and $N(\varphi \wedge \psi) \leftrightarrow (N\varphi \wedge N\psi)$ are derivable. Also observe that, if we had restricted the Necessitation Rule only to theorems, we would have obtained a local consequence relation (instead of the global one we have introduced here). For this weaker version of the logic, the Deduction Theorem in its usual form would hold, nevertheless, this logic would not be algebraizable.

Theorem 10 [Deduction Theorem] *If $T \cup \{\varphi, \phi\}$ is any set of* PNG*-formulas, then $T \cup \{\varphi\} \vdash_{PNG} \phi$ iff $T \vdash_{PNG} (\varphi \wedge N\varphi) \to \phi$.*

Kripke style semantics based on possibilistic structures (W, e, I) could be also defined as in Section 3, but now the situation is more complex due to the fact that we are dealing with a full modal language. Moreover, it seems even more complex to try to get some completeness results with respect to this semantics so this is left for future research. This is the reason why in the rest of the paper we will turn our attention to the study of an algebraic semantics, following the ideas developed in [15, 14] for the case of a probabilistic logic over Łukasiewicz logic, and see how far we can go. We start by defining a suitable class of algebras which are expansions of Gödel algebras with a new unary operator trying to capture the notion of necessity.

Definition 11 *A* NG-algebra *is a structure (A, N) where A is a G-algebra and $N : A \to A$ is a monadic operator such that:*

1. $N(x \Rightarrow y) \Rightarrow (Nx \Rightarrow Ny) = 1$
2. $Nx = NNx$

3. $N1 = 1$

The function N *is called an* internal possibilistic state *on the* G*-algebra* A.

Observe that, so defined, the class of NG-algebras is a variety. Examples of internal possibilistic states are the identity function Id, the Δ operator and the $\neg\neg$ operator. The variety of G-algebras can be considered as a subvariety of NG-algebras, namely the subvariety obtained by adding the equation $N(x) = x$. It is easy to check, using the definition of NG-algebra that, for every NG-algebra (A, N) such that $N(A) = A$ we have $N = Id$, and that, given $a, b \in A$, $a \leq b$ implies $Na \leq Nb$.

Definition 12 *A* NG-filter F *on a NG-algebra* (A, N) *is a filter on the G-algebra* A *with the following property: if* $a \in F$*, then* $Na \in F$*.*

By an argument analogous to the one in Lemma 2.3.14 of [17], if $\sim_F$ is the relation defined by: for every $a, b \in A$, $a \sim_F b$ iff $(a \Rightarrow b) \in F$ and $(b \Rightarrow a) \in F$, then $\sim_F$ is a congruence on (A, N) and the quotient algebra $(A, N)/\sim_F$ is an NG-algebra.

Lemma 13 *Let* F *be a NG-filter on a NG-algebra* (A, N)*. Then, the least NG-filter containing* F *as a subset and a given* $a \in A$ *is*

$$F' = \{u \in A : \exists v \in F \text{ such that } u \geq v * a * Na\}$$

By Corollary 4.8 of [5], it is easy to check that PNG is finitely algebraizable and that the equivalent algebraic semantics of PNG is the variety of the NG-algebras. As a corollary we obtain the following general completeness result.

Theorem 14 *The logic* PNG *is strongly complete with respect to the variety of NG-algebras. This means that for any set of formulas* $\Gamma \cup \{\Phi\}$, $\Gamma \vdash_{\mathrm{PNG}} \Phi$ *iff, for all NG-algebra* A *and for all evaluation* e *on* A*, if* $e(\Psi) = 1^A$ *for all* $\Psi \in \Gamma$*, then* $e(\Phi) = 1^A$.

Observe that it is not possible to prove completeness with respect to linearly ordered NG-algebras. Otherwise $N(\Phi \vee \Psi) \leftrightarrow (N\Phi \vee N\Psi)$ would be a theorem. Now we prove some satisfiability results of formulas of the logic *PNG*.

Formulas of the language of PNG can be seen also as terms of the language of the NG-algebras. Therefore for the sake of clarity, in the following proofs we work with first-order formulas of the language of NG-algebras proving that they are satisfiable, if the corresponding formulas of the language of PNG are satisfiable.

Proposition 15 *Let* $\phi(x_1, \ldots, x_n)$ *be a* PNG*-formula. If* ϕ *is satisfiable, then* $\phi = \overline{1}$ *is satisfiable in a* NG*-algebra* (B, Ω)*, by a sequence* $(b_1, \ldots, b_n)$ *of elements of* B *such that, for every* $0 < i \leq n$*, we have either* $b_i = 1$ *or* $\Omega(b_i) = 0$.

Proof: Let (A, N) be a NG-algebra such that $\phi = \overline{1}$ is satisfiable in (A, N) by $(a_1, \ldots, a_n)$. Without loss of generality we assume that there is $k \leq n$ such that for every $0 < i \leq k$, $N(a_i) \neq 0$ and for $i > k$, $N(a_i) = 0$.

Now we build a finite sequence of NG-algebras $(B_1, \ldots, B_k)$ and homomorphisms $(h_1, \ldots, h_k)$ such that for every $0 < i \leq k$, ϕ is satisfied in B_i by

$$(c_1, \ldots, c_{i-1}, h_i \circ h_{i-1} \circ \cdots \circ h_1(a_i), \ldots, h_i \circ h_{i-1} \circ \cdots \circ h_1(a_n))$$

where each $c_i \in \{0, 1\}$. We define only the first homomorphism, the others can be introduced analogously. Let $F = \{x \in A : Nx \geq Na_1\}$. So defined, it is easy to check that F is a NG-filter. And since, by a previous assumption, $Na_1 \neq 0$, the filter F is proper. Thus, $(A, N)/\sim_F$ is a NG-algebra. Now let h_1 be the canonical homomorphism from (A, N) to $(A, N)/\sim_F$, and let $B_1 = (A, N)/\sim_F$. It is easy to check that $\phi = \overline{1}$ is satisfied in B_1 by $(h_1(a_1), \ldots, h_1(a_n))$, that $h_1(a_1) = 1$ and that for $i > k$, $N(h_1(a_i)) = 0$. Finally, take $(B, \Omega) = (B_k, h_k \circ \cdots \circ h_1 \circ N)$. $\square$

Definition 16 *An* unnested atomic formula *of the language of the NG-algebras, is an atomic formula of one of the following four forms: $x = y$, $c = y$ (where c is a constant $c \in \{\overline{0}, \overline{1}\}$), $Nx = y$ or $F(\overline{x}) = y$ (for some function symbol F of the language of the Gödel algebras).*

Lemma 17 *Let ϕ be a term of the language of the NG-algebras. Then there is a set Γ^ϕ of unnested atomic formulas such that, for every NG-algebra (A, N):*

$$\phi = \overline{1} \text{ is satisfiable in } (A, N) \text{ iff } \Gamma^\phi \text{ is satisfiable in } (A, N).$$

Proof: It is a direct consequence of Theorem 2.6.1 of [21]. $\square$

Example: Let ϕ be the term $x_1 \vee N(x_2 \Rightarrow N(x_3 \Rightarrow \overline{0}))$, take Γ^ϕ to be the following set of unnested atomic formulas:

$$\{x_1 \vee y = z, \overline{1} = z, Nw = y, (x_2 \Rightarrow v) = w, Nq = v, (x_3 \Rightarrow p) = q, \overline{0} = p\}$$

Theorem 18 *Let $\phi(x_1, \ldots, x_n)$ be a* PNG*-formula. If ϕ is satisfiable, then $\phi = \overline{1}$ is satisfiable in the NG-algebra $([0,1]_G, \Delta)$ by a sequence of rational numbers.*

Proof: Let (A, Ω) be a NG-algebra in which $\phi(x_1, \ldots, x_n) = \overline{1}$ is satisfiable by a n-tuple $(a_1, \ldots, a_n)$. Without loss of generality we may assume that:

- ϕ is a conjunction of unnested atomic formulas (by using Lemma 17).
- for every $0 < i \leq n$, $a_i \neq 0$ and $a_i \neq 1$ (otherwise we can work with the formulas $\phi(x_i/\overline{1})$ or $\phi(x_i/\overline{0})$, by substituting the corresponding variables by the constants $\overline{0}$ or $\overline{1}$).
- for every i, we have $\Omega(a_i) = 0$ (by Proposition 15).

Now we consider the unnested conjuncts of ϕ. For the sake of simplicity, assume that there is $k \leq n$ such that only in case that $0 < i \leq k$, the variable x_i has an occurrence in an unnested atomic formula of the form $Nx_i = y$. We work now with the formula $\gamma = \phi(Nx_i/\overline{0})$, obtained by substituting in ϕ all the occurrences of Nx_i by the constant $\overline{0}$, for every $0 < i \leq k$.

Observe that, so defined, γ is a conjunction of unnested atomic formulas in the language of G-algebras which is satisfied in (A, Ω) by $(a_1, \ldots, a_n)$. Therefore, the conjunction $\gamma_0 = \gamma \wedge \bigwedge_{0<i\leq k}(x_i \neq 1)$ is also satisfied in (A, Ω) by $(a_1, \ldots, a_n)$ (by our assumptions at the beginning of this proof). Finally, since γ_0 is a formula in the language of the G-algebras, it is satisfied in $[0,1]_G$ by a sequence of rational numbers, and thus, by definition of γ_0, it is easy to check that ϕ is also satisfied in $([0,1]_G, \Delta)$. □

5 Future Work

Several issues related to the logic PNG deserve further investigation. A topic that is worth studying in depth is the relation between the algebraic semantics for the logic PNG (and of some meaningful axiomatic extensions) and the Kripke style semantics of the kind used in Section 3. This is crucial if one wants to keep as the intended graded semantics of the N operator one of the possibilistic necessities of the families described in Section 2. Actually, the PNG logic might only capture the logic of basic necessities, and so, different axioms (and possibly operators as well) must be added in order to capture other more specific families of necessities, somehow related to axioms (N_{KD}), (N_{RG}), $(\Pi 3)$ or the axiom $(N\Pi)$.

Also as a future task, we aim at studying the complexity of the sets of satisfiable formulas for both NG_{KD}, NG_{RG} and $N\Pi_G$. Given the results in [18], we conjecture that the problem of checking satisfiability for those logics is in PSPACE. As for PNG, notice that from the results in the above section and the fact that satisfiability for G_Δ is an NP-complete problem (easily derivable from [17]), we immediately obtain that the set of satisfiable PNG-formulas is in NP.

Acknowledgments The authors are grateful to the anonymous referees for their valuable comments that helped them improve the final version of this paper. They also acknowledge partial support from the Spanish projects MULOG2 (TIN2007-68005-C04) and *Agreement Technologies* (CONSOLIDER CSD2007-0022, INGENIO 2010), as well as the ESF Eurocores-LogICCC/MICINN project (FFI2008-03126-E/FILO). Marchioni also acknowledges partial support of the Juan de la Cierva Program of the Spanish MICINN.

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