First-order t-norm based fuzzy logics with truth-constants: distinguished semantics and completeness properties☆

Francesc Esteva, Lluís Godo

Institut d'Investigació en Intel·ligència Artificial (IIIA)
Spanish National Research Council (CSIC)
08193 Bellaterra, Catalonia, Spain

Carles Noguera

Department of Mathematics and Computer Science
University of Siena,
53100 Siena, Italy

Abstract

This paper aims at being a systematic investigation of different completeness properties of first-order predicate logics with truth-constants based on a large class of left-continuous t-norms (mainly continuous and weak nilpotent minimum t-norms). We consider standard semantics over the real unit interval but also we explore alternative semantics based on the rational unit interval and on finite chains. We prove that expansions with truth-constants are conservative and we study their real, rational and finite chain completeness properties. Particularly interesting is the case of considering canonical real and rational semantics provided by the algebras where the truth-constants are interpreted as the numbers they actually name. Finally, we study completeness properties restricted to evaluated formulae of the kind $\overline{r} \to \varphi$, where φ has not additional truth-constants.

Key words: Algebraic Logic, Evaluated formulae, Mathematical Fuzzy Logic, First-order Predicate Non-classical logics, Residuated lattices, T-norm based fuzzy logics, Truth-constants.

1. Introduction

T-norm based fuzzy logics are infinitely-valued logics and thus can be cast in the tradition of many-valued logics. The first infinitely-valued systems studied in the literature were the infinitely-valued Łukasiewicz and Gödel-Dummett logics, defined in the fifties and proved to be complete with respect to semantics over the real unit interval equipped with suitable truth-fucntions (Łukasiewicz and minimum t-norms and their residua respectively). As a continuation of this

☆Dedicated to Franco Montagna in the occasion of his 60th birthday.

tradition, t-norm based fuzzy logics appear as logical systems axiomatizing $[0,1]$-valued residuated calculi using t-norms as truth-functions for the conjunction connective and their residua as truth-functions for the implication connective. Indeed, the qualifier "*t-norm based*" comes from the use of t-norms[2] and related operations as truth-functions, operations which are in turn related to generalized set-theoretic operations in Zadeh's Fuzzy Set Theory.

T-norm based fuzzy logics can also be cast in the tradition of algebraic logic. Indeed, mainly after the work of Chang [4], the development of infinitely-valued logics became more algebraic. As a matter of fact, t-norm-based fuzzy logics are axiomatic extensions of Höhle's Monoidal logic [28], or FL_{ew} (Full Lambek calculus with exchange and weakening) as called by Ono and others (see e.g. [20]), which is an algebraizable logic in the sense of Blok and Pigozzi having the variety of residuated lattices[3] as its equivalent algebraic semantics. This algebraizability result extends to any axiomatic extension of Monoidal Logic and thus any t-norm based fuzzy logic is also algebraizable with equivalent algebraic semantics a corresponding subvariety of resiuated lattices. Moreover, t-norm based fuzzy logics can be seen as a part of the family of *substructural logics* since, as in FL_{ew}, they typically lack the structural rule of contraction and have two conjunctions (additive $\wedge$ and multiplicative &), an additive disjunction $\vee$, an implication $\rightarrow$ (which satisfies the residuation law with respect to &) and the truth-constant for falsum $\overline{0}$.

From a temporal perspective, the birth of t-norm based fuzzy logics represented a blossom of infinitely-valued logical systems, which were almost reduced until then to Łukasiewicz and Gödel-Dummett logics and related systems. In 1996 Product Logic (the logic of the product t-norm and its residuum) was defined in [26] and in 1998 Hájek published his influential monograph [21] where he introduces the Basic (fuzzy) Logic BL (both in a propositional and first order versions) as a basis to frame all the logics of continuous t-norms and their residua (finally proved to be so in [6]). This frame was fully enlarged in [13] by introducing the most general t-norm-based fuzzy logic (by observing that a t-norm has residuum if and only if it is left-continuous), the Monoidal T-norm-based Logic MTL, of which BL is an axiomatic extension, and proved in [29] to capture the common tautologies of all left-continuous t-norms and their residua. Following these initial developments, an increasing number of papers with deep and interesting results about (both propositional and first-order) t-norm based fuzzy logics have been published in the last decade. That includes works on completeness results, functional representation, proof theory, decidability, computational complexity, arithmetical hierarchy, expanded systems, and game semantics among others. Thus, it has been considered a specific branch of Mathematical Logic called *Mathematical Fuzzy Logic* (see e.g. [23]), as opposed to the so-called *Fuzzy Logic* simpliciter, which arose from Zadeh's Fuzzy Set Theory as a toolbox of mathematical methods, non-necessarily related to any logic in strict sense, used in applications.

However, most of the research on Mathematical Fuzzy Logic so far has been

[2] A t-norm $*$ is a binary operation on the real unit interval that is associative, commutative, non-decreasing and satisfying $x * 0 = 0$ and $x * 1 = x$.

[3] We will use the term residuated lattice to refer to bounded, commutative, integral, residuated lattice-ordered monoids.

made under two important constraints. First, since the intended semantics for fuzzy logic systems has been that of the algebras defined over the real unit interval, the so-called *standard semantics*, the emphasis has been put on completeness results with respect to these semantics. This limitation has already disappeared in a few papers (see [11, 19, 8, 18]) where alternative semantics and their associated completeness properties have been explored, in particular, algebras over the rational unit interval and finite chains. Second, the literature on fuzzy logic systems is usually related to what we may call *truth-preserving deductive systems*, i.e. many-valued logics where the semantical consequence relation is defined as preservation of 1 as the only designated value. It can be argued that these logics do not take a real advantage of being many-valued since, when it comes to consequence relation, they only use one truth value: the full truth, and they disregard somehow the notion of partial truth that seems strongly related to the idea of comparative truth and essential in applications. A convenient way to overcome this drawback consists in introducing truth-constants into the language, allowing an explicit syntactical treatment of partial truth. This approach actually goes back to Pavelka [34] who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz logic obtained by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in [0, 1]$, together with some additional axioms. Pavelka proved that his logic is strongly complete in a non-finitary sense, heavily relying on the continuity of Łukasiewicz truth-functions. Novák extended Pavelka's approach to a first-order logic [33]. Hájek showed in [21] that Pavelka's logic (both propositional and first order) could be significantly simplified while keeping the Pavelka-style completeness results. Indeed he showed it is enough to extend the language only by a countable number of truth-constants, one constant $\overline{r}$ for each *rational* $r \in (0, 1)$, and by adding to Łukasiewicz Logic the so-called *book-keeping axioms*.

Similar expansions with truth-constants for other propositional t-norm based fuzzy logics have been analogously defined in [12, 15, 16, 17, 36], but Pavelka-style completeness could not be obtained, as Łukasiewicz logic is the only t-norm based logic whose truth-functions are continuous. Thus, in these papers rather than Pavelka-style completeness the authors have focused on the usual notion of completeness of a logic. It is interesting to note that: (1) the logic to be expanded with truth-constants has to be complete with respect to the algebra given by a left-continuous t-norm and its residuum[4]; (2) the expanded logic is still a truth-preserving logic, but its richer language admits formulae of type $\overline{r} \to \varphi$ saying that, when evaluated at 1, the truth degree of φ is greater or equal than r; and (3) this logic is again algebraizable in the sense of Blok and Pigozzi. In the cited papers the expansions with truth-constants of propositional logics based on a big class of continuous t-norms and a well-behaved class of other left-continuous t-norms, and their completeness properties are studied.

But, except for some preliminary results in [17], the expanded first-order systems corresponding to the logics of a t-norm and its residuum with truth-constants have not been systematically investigated. This paper wants to be a systematic study of completeness properties of the expansions with truth-constants of first-order t-norm based fuzzy logics with respect to three distin-

[4]Since we need to know the operations between truth constants used in the book-keeping axioms.

guished semantics: real, rational and finite-chain semantics. We focus on logics based on the most relevant continuous and weak nilpotent minimum t-norms. To this end, the paper is structured as follows. First, Section 2 gives the basic notions and results on propositional t-norm based fuzzy logics and their expansions with truth-constants. After a brief overview of basic facts on first order t-norm based logics, Section 3 provides some results related to rational and finite-chain semantics for first-order t-norm based fuzzy logics (without truth-constants) that will be needed in the next section. Finally, Section 4 contains the study of the expansions with truth-constants focusing on (i) conservativeness of the expansion of first-order t-norm based fuzzy logics with truth-constants, (ii) real, rational and finite-chain completeness properties for these expanded logics, and (iii) real and rational completeness properties when restricting to the so-called *evaluated formulae*.

2. Preliminaries on underlying propositional logics

2.1. Propositional t-norm based fuzzy logics

The basic logic in this paper is the Monoidal T-norm based logic, MTL for short, that was defined in [13] by means of a Hilbert-style calculus in the language $\mathcal{L} = \{\&, \to, \wedge, \overline{0}\}$ of type $\langle 2, 2, 2, 0\rangle$. The only inference rule is *Modus Ponens* and the axiom schemata are the following (taking $\to$ as the least binding connective):

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$\varphi \& \psi \to \varphi$
(A3)	$\varphi \& \psi \to \psi \& \varphi$
(A4)	$\varphi \wedge \psi \to \varphi$
(A5)	$\varphi \wedge \psi \to \psi \wedge \varphi$
(A6)	$\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$
(A7a)	$(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$
(A7b)	$(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$
(A8)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A9)	$\overline{0} \to \varphi$

Other usual connectives are defined as follows:

$$\begin{aligned} \varphi \vee \psi &:= ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi); \\ \varphi \leftrightarrow \psi &:= (\varphi \to \psi) \& (\psi \to \varphi); \\ \neg \varphi &:= \varphi \to \overline{0}; \\ \overline{1} &:= \neg \overline{0}. \end{aligned}$$

MTL was defined as a generalization of the Basic fuzzy Logic BL introduced by Petr Hájek in [21], and in fact BL is actually the extension of MTL obtained by adding the divisibility axiom: $\varphi \wedge \psi \to \varphi \& (\varphi \to \psi)$. Most of well-known fuzzy logics, among them Łukasiewicz (Ł), Gödel (G) and Product (Π) logics, can be presented as axiomatic extensions of MTL. Tables 1 and 2 collect some axiom schemata and the axiomatic extensions of MTL they define.[5] For any logic L

[5] Of course, some of these logics were known well before MTL was introduced; we only want

axiomatic extension of MTL, the notion of proof, denoted by $\vdash_{\mathrm{L}}$, is defined as usual from the corresponding set of axioms and modus pones as the only rule of inference.

Axiom schema	Name
$\neg\neg\varphi \to \varphi$	Involution (Inv)
$\neg\varphi \vee ((\varphi \to \varphi\&\psi) \to \psi)$	Cancellation (C)
$\varphi \to \varphi\&\varphi$	Contraction (Con)
$\varphi \wedge \psi \to \varphi\&(\varphi \to \psi)$	Divisibility (Div)
$\varphi \wedge \neg\varphi \to \overline{0}$	Pseudo-complementation (PC)
$(\varphi\&\psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi\&\psi)$	Weak Nilpotent Minimum (WNM)

Table 1: Some usual axiom schemata in fuzzy logics.

Logic	Additional axiom schemata
SMTL	(PC)
IMTL	(Inv)
WNM	(WNM)
NM	(Inv) and (WNM)
BL	(Div)
SBL	(Div) and (PC)
Ł	(Div) and (Inv)
Π	(Div) and (C)
G	(Con)

Table 2: Some axiomatic extensions of MTL obtained by adding the corresponing additional axiom schemata.

The algebraic counterpart of these logics are the variety of MTL-algebras and its subvarieties. An *MTL-algebra* is a structure[6] $\mathcal{A} = \langle A, \&^{\mathcal{A}}, \to^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ such that:

1. $\langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a bounded lattice.
2. $\langle A, \&^{\mathcal{A}}, \overline{1}^{\mathcal{A}} \rangle$ is a commutative monoid.
3. The operations $\&^{\mathcal{A}}$ and $\to^{\mathcal{A}}$ form an adjoint pair:
$$\forall a, b, c \in A,\ a\&^{\mathcal{A}}b \leq c \text{ iff } b \leq a \to^{\mathcal{A}} c.$$
4. The prelinearity condition $(a \to^{\mathcal{A}} b) \vee^{\mathcal{A}} (b \to^{\mathcal{A}} a) = \overline{1}^{\mathcal{A}}$ is satisfied for all $a, b \in A$.

An additional negation operation is defined as $\neg^{\mathcal{A}}a := a \to^{\mathcal{A}} \overline{0}^{\mathcal{A}}$ for every $a \in A$.

The class of MTL-algebras forms a variety, denoted $\mathbb{MTL}$, and in fact MTL is an algebraizable logic in the strong sense of Blok and Pigozzi [3] whose equivalent algebraic semantics is $\mathbb{MTL}$. Hence, there is a one-to-one correspondence

to point out that it is possible to present them as the axiomatic extensions of MTL obtained by adding the corresponding axioms to the Hilbert style calculus for MTL given above.

[6] We will omit superscripts in the operations of the algebras when clear from the context.

between axiomatic extensions of MTL and subvarieties of $\mathbb{MTL}$. Given an axiomatic extension L of MTL, we call *L-algebras* the elements of its corresponding subvariety (Łukasiewicz and classical logics are exceptions to this rule, since their corresponding algebras are called *MV-algebras* and *Boolean algebras* respectively).

If the lattice order of an MTL-algebra $\mathcal{A}$ is total we say that it is an MTL-chain. In [21, 13] it is proved that all subvarieties of MTL are generated by their chains, since due to prelinearity condition any MTL-algebras can be decomposed as subdirect product of MTL-chains. Thus, any axiomatic extension of MTL is complete with respect to the class of chains of the corresponding subvariety.

Actually, MTL-algebras are a particular kind of the structures typically used as semantics for substructural logics (as studied e.g. in the monograph [20]); namely they are prelinear commutative integral bounded residuated lattice-ordered monoids.

However, both the logics BL and MTL were introduced in order to axiomatize the common tautologies of logical calculi defined by continuous and left-continuous t-norms, respectively. It turns out that a t-norm $*$ has a residuum if, and only if, is left-continuous, and in such a case the residuum is defined as $x \Rightarrow y = \max\{z \in [0,1] : x * z \leq y\}$, for all $x, y \in [0,1]$. As a consequence, MTL is the most general t-norm based logic. Moreover, a (left-continuous) t-norm $*$ is continuous if, and only if, it satisfies the following divisibility condition: for each $x, y \in [0,1]$, $x * (x \Rightarrow y) = \min\{x, y\}$.

MTL-chains on the real unit interval $[0,1]$, called *standard algebras*, are those determined by left-continuous t-norms. In fact if $*$ is a left-continuous t-norm and $\Rightarrow$ its residuum, then $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$ is an MTL-chain, and conversely, any MTL-chain on $[0,1]$ is of this form. Table 3 shows remarkable examples of t-norms and their residua. Namely, Łukasiewicz, Product and Minimum t-norms are the three basic examples of continuous t-norms since any other continuous t-norm can be obtained by a simple algebraic construction, called *ordinal sum*, from isomorphic copies of these three t-norms. Namely, given left-continuous t-norms $*_1$ and $*_2$ and $a \in (0,1)$, one considers isomorphic copies of the algebras $[0,1]_{*_1}$ and $[0,1]_{*_2}$ respectively over the intervals $[0,a]$ and $[a,1]$ and defines their ordinal sum, denoted as $[0,a]_{*_1} \oplus [a,1]_{*_2}$, as the standard MTL-chain whose t-norm $*$ is: $x * y = x *_1 y$ if $x, y \in [0,a]$, $x * y = x *_2 y$ if $x, y \in [a,1]$, and $x * y = \min\{x, y\}$ otherwise. The t-norm on the resulting algebra is denoted as $*_1 \oplus *_2$.[7] The decomposition of continuous t-norms (equivalently standard BL-chains) as ordinal sums of basic components has been generalized [22, 6] to a general theorem on the structure of all BL-chains: every saturated BL-chain is an ordinal sum of MV-chains, Π-chains and G-chains. Furthermore, a slightly different ordinal sum construction has been introduced in [1, 2] for the $\overline{0}$-free subreducts of BL-algebras, the so-called *basic hoops*.

The fourth example in Table 3 describes the class of the so-called *Weak Nilpotent Minimum t-norms* (WNM-t-norms for short); actually there is one such a left-continuous t-norm for each negation function[8] n. In particular, when

[7]Notice that we are slightly abusing the language to consider ordinal sums both as a construction to obtain t-norms and to obtain MTL-chains.

[8]A negation function is a mapping $n : [0,1] \to [0,1]$ such that $n(1) = 0$, it is order-reversing, and $x \leq n(n(x))$ for every $x \in [0,1]$.

$n(x) = 1 - x$, the t-norm is called *Nilpotent Minimum* and denoted $*_{\mathrm{NM}}$. The standard MTL-algebras defined by the mentioned four prominent t-norms will be respectively denoted as $[0,1]_{\mathrm{Ł}}$, $[0,1]_{\Pi}$, $[0,1]_{\mathrm{G}}$ and $[0,1]_{\mathrm{NM}}$. See e.g. [30] for further details on t-norms.

t-norm name	t-norm $x * y =$	residuum $x \Rightarrow y =$
Łukasiewicz: $*_{\mathrm{Ł}}$	$\max\{x+y-1, 0\}$	$\min\{1-x+y, 1\}$
Product: $*_{\Pi}$	$x \cdot y$	$\begin{cases} 1, & \text{if } x \leq y, \\ y/x, & \text{otherwise} \end{cases}$
Minimum: $*_{\mathrm{G}}$	$\min(x, y)$	$\begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise} \end{cases}$
WNM: $*_n$	$\begin{cases} 0, & \text{if } x \leq n(y), \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x \leq y, \\ \max\{n(x), y\}, & \text{otherwise.} \end{cases}$

Table 3: Some t-norms and their residua.

Due to the above mentioned relationship between axiomatic extensions of MTL and subvarieties of MTL-algebras, it is meaningful to consider for each left-continuous t-norm $*$ the logic L_*, which is the axiomatic extension of MTL whose equivalent algebraic semantics is the variety generated by $[0,1]_*$, denoted $\mathbf{V}([0,1]_*)$. Well-known examples of these logics L_* are Łukasiewicz, Product, Gödel and Nilpotent Minimum logics, denoted respectively as Ł, Π, G and NM, and which correspond to the cases when $*$ is the Łukasiewicz t-norm, Product t-norm, Minimum t-norm and the Nilpotent Minimum t-norm respectively. In this paper we mainly focus on logics L_* with $*$ belonging to the following classes of left-continuous t-norms:

CONT-fin: the set of continuous t-norms which are finite ordinal sum of the basic components.[9]

WNM-fin: the set of WNM-t-norms whose associated negation function[10] has finitely many discontinuity points.[11]

Among those t-norms in **CONT-fin** let us mention two that play a distinctive role in what follows: the ordinal sum of $*_{\mathrm{Ł}}$ and $*_{\mathrm{G}}$, whose corresponding logic will be denoted $\mathrm{Ł} \oplus \mathrm{G}$, and the ordinal sum of $*_{\mathrm{Ł}}$ and $*_{\Pi}$, whose corresponding logic will be denoted $\mathrm{Ł}\oplus\Pi$. Among those belonging to **WNM-fin**, two examples that we will also be relevant in this paper are the t-norms $\otimes = *_{n_1}$ and $\star = *_{n_2}$ defined respectively by the following negation functions:[12]

$$n_1(x) = \begin{cases} 1, & \text{if } x = 0 \\ 1/2, & \text{if } 0 < x \leq 1/2 \\ 0, & \text{otherwise} \end{cases}$$

[9]Recall that a basic component is one of the following BL-chains: $[0,1]_{\mathrm{Ł}}$, $[0,1]_{\Pi}$, $[0,1]_{\mathrm{G}}$.

[10]If $*$ is a left-continuous t-norm and $\Rightarrow$ its residdum, the associated negation function is $n_*(x) = x \Rightarrow 0$.

[11]This property is referred as the Finite Partition Property (FPP) in [16].

[12]Notice that there is nothing special in the choice of the numbers 1/2, 1/3 and 2/3 in their definitions as we might as well use in the first case any $c \in (0,1)$ instead of $\frac{1}{2}$, and in the second case any $c \in (\frac{1}{2}, 1)$ instead of $\frac{2}{3}$ (and $1-c$ instead of $\frac{1}{3}$) and we would obtain isomorphic t-norms and thus the same logics.

$$n_2(x) = \begin{cases} 1 - x, & \text{if } x \in [0, 1/3] \cup (2/3, 1] \\ 2/3, & \text{otherwise} \end{cases}$$

2.2. Completeness properties and their relationships

Completeness and many other properties have been proven for a larger class of logics, the so-called *core fuzzy logics* introduced by Hájek and Cintula in [7, 25]. Briefly stated, propositional core fuzzy logics are expansions (extensions with, possibly, some extra connectives) of MTL satisfying a local deduction theorem analogous to that of MTL and whose additional connectives satisfy a congruence condition with respect to the double implication. Any core fuzzy logic L is shown to be also algebraizable and its equivalent algebraic semantics is the variety of L-algebras, which are the natural expansions of the underlying MTL-algebras. L-algebras also decompose as subdirect products of L-chains. The results contained in this section are purposedly formulated for core fuzzy logics since the expansions with truth-constants of t-norm based fuzzy logics which we will deal with are indeed core fuzzy logics.

For any class $\mathbb{K}$ of L-algebras, we will denote by $\models_{\mathbb{K}}$ the consequence relation defined in the following way: for every set of L-formulae Γ and every L-formula φ, $\Gamma \models_{\mathbb{K}} \varphi$ iff for each $\mathcal{A} \in \mathbb{K}$ and $\mathcal{A}$-evaluation e, $e(\varphi) = \overline{1}^{\mathcal{A}}$ whenever $e[\Gamma] \subseteq \{\overline{1}^{\mathcal{A}}\}$. As usual, we will write $\models_{\mathbb{K}} \varphi$ instead of $\emptyset \models_{\mathbb{K}} \varphi$ and $\Gamma \models_{\mathcal{A}} \varphi$ instead of $\Gamma \models_{\{\mathcal{A}\}} \varphi$. We will use the following terminology and notation to refer to the usual three notions of completeness for propositional core fuzzy logics,

Definition 2.1. *Let* L *be a propositional core fuzzy logic and let* $\mathbb{K}$ *be a class of* L*-chains. We say that* L *has the property of* strong $\mathbb{K}$-completeness, S$\mathbb{K}$C *for short, when for every set of* L*-formulae* Γ *and every* L*-formula* φ, $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbb{K}} \varphi$. L *has the property of* finite strong $\mathbb{K}$-completeness, FS$\mathbb{K}$C *for short, if the equivalence holds for every* finite Γ, *and* L *has the property of* $\mathbb{K}$-completeness, $\mathbb{K}$C *for short, if it holds for* $\Gamma = \emptyset$.

Important particular cases of these properties appear when $\mathbb{K}$ is the class $\mathcal{R}$ of standard L-chains, the class $\mathcal{Q}$ of L-chains defined over the rational unit interval or the class $\mathcal{F}$ of finite L-chains.[13] All these completeness properties, their relationship and algebraic equivalent (or sufficient) conditions have been studied in [8]. In particular, the following general results for the FS$\mathbb{K}$C and the S$\mathbb{K}$C have been proved.

Theorem 2.2 ([8])**.** *Let* L *be a propositional core fuzzy in a finite language and let* $\mathbb{K}$ *be a class of* L*-chains. Then the following are equivalent:*

(i) L *has the* FS$\mathbb{K}$C*.*

(ii) Every countable L*-chain is partially embeddable into* $\mathbb{K}$*.*

The implication from (ii) to (i) holds also for infinite languages.

[13]In [8] semantics given by the class $\mathcal{R}^\star$ of hyperreal L-chains has been also considered. However, since rational and hyperreal completeness properties turn out to be completely equivalent, both in propositional and first-order logics, the hyperreal semantics will not be considered in this paper.

Theorem 2.3 ([8]). *Let* L *be a propositional core fuzzy logic and let* $\mathbb{K}$ *be a class of* L*-algebras. Then* L *has the* S$\mathbb{K}$C *iff every non-trivial countable* L*-chain can be embedded into some chain from* $\mathbb{K}$.

We recall also the following relationships between standard and rational completeness properties and a useful characterization of the S$\mathcal{F}$C:

Theorem 2.4 ([8]). *Let* L *be a propositional core fuzzy logic. Then:*

(i) If L *has a real completeness property (*$\mathcal{R}$C, FS$\mathcal{R}$C *or* S$\mathcal{R}$C*), then* L *also has the corresponding rational completeness property (*$\mathcal{Q}$C, FS$\mathcal{Q}$C *or* S$\mathcal{Q}$C*).*

(ii) L *has the* S$\mathcal{Q}$C *if, and only if, it has the* FS$\mathcal{Q}$C.

(iii) L *has the* S$\mathcal{F}$C *if, and only if, all* L*-chains are finite.*

Completeness properties for the above mentioned particular logics[14] are collected in Table 4.

Logic	FS$\mathcal{R}$C	S$\mathcal{R}$C	S$\mathcal{Q}$C	$\mathcal{F}$C, FS$\mathcal{F}$C	S$\mathcal{F}$C
MTL, SMTL, G, WNM, NM, $L_\otimes$, $L_\star$	Yes	Yes	Yes	Yes	No
BL, SBL, Ł, Ł$\oplus$G	Yes	No	Yes	Yes	No
Π, Ł$\oplus\Pi$	Yes	No	Yes	No	No

Table 4: Completeness properties for some prominent propositional t-norm based logics.

2.3. Expansions with truth-constants

Now, given a left-continuous t-norm $*$ and its corresponding logic L_*, let $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1\rangle$ be a *countable* subalgebra of $[0,1]_*$. Then the expansion of L_* with truth-constants from $\mathcal{C}$ is the propositional logic $L_*(\mathcal{C})$ defined as follows:

(i) the language of $L_*(\mathcal{C})$ is the one of L_* expanded with a new propositional constant $\overline{r}$ for each $r \in C \setminus \{0, 1\}$,

(ii) the axioms and rules of $L_*(\mathcal{C})$ are those of L_* plus the book-keeping axioms for each $r, s \in C$:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$(\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow s}$$

It is straightforward to check that $L_*(\mathcal{C})$ is a core fuzzy logic and hence algebraizable, its algebraic counterpart consisting of the class of $L_*(\mathcal{C})$-algebras, defined as structures $\mathcal{A} = \langle A, \&, \to, \wedge, \vee, \langle \overline{r}^{\mathcal{A}} : r \in C\rangle\rangle$ such that:

1. $\langle A, \&, \to, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ is an L_*-algebra, and

2. for every $r, s \in C$ the following identities hold:

$$\overline{r}^{\mathcal{A}}\&\overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$$

[14]For their proofs see [8] and references thereof, except for the cases of the t-norms $\otimes$ and $\star$ whose completeness properties are derivable from the results in [32].

$$\overline{r}^{\mathcal{A}} \to \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}.$$

$L_*(\mathcal{C})$-chains defined over the real unit interval $[0,1]$ are called *standard*. Among them, there is one which reflects the intended standard semantics, the so-called *canonical standard* $L_*(\mathcal{C})$-chain

$$[0,1]_{L_*(\mathcal{C})} = \langle [0,1], *, \Rightarrow, \min, \max, \langle r : r \in C \rangle \rangle,$$

which is the standard chain over $[0,1]_*$ where the truth-constants are interpreted by their defining values. It is worth to point out that for a logic $L_*(\mathcal{C})$ there may exist multiple standard chains as soon as there exist different ways of interpreting the truth-constants on $[0,1]$ respecting the book-keeping axioms. It has been shown [12, 16] that if $*$ is either a continuous or a WNM-t-norm, then for every filter F of $\mathcal{C}$ there exists a standard $L_*(\mathcal{C})$-chain $\mathcal{A}$ such that $\{r \in C : \overline{r}^{\mathcal{A}} = 1\} = F$.

Since $L_*(\mathcal{C})$ is a core fuzzy logic, it is complete with respect to the class of all $L_*(\mathcal{C})$-chains. As an easy consequence of this fact one obtains that $L_*(\mathcal{C})$ is a conservative expansion of L_* [16]. The issue of which kinds of real completeness properties hold for the logics $L_*(\mathcal{C})$, both with respect to all standard chains or only the canonical one, has been addressed in the literature [21, 5, 15, 36, 12, 16]. The results obtained in these papers are collected in Table 5, where we use the notations Can$\mathcal{R}$C, CanFS$\mathcal{R}$C and CanS$\mathcal{R}$C to denote respectively the three kinds of completeness properties with respect to the canonical standard algebra.

Logic	FS$\mathcal{R}$C	S$\mathcal{R}$C	Can$\mathcal{R}$C	CanFS$\mathcal{R}$C	CanS$\mathcal{R}$C
Ł($\mathcal{C}$)	Yes	No	Yes	Yes	No
$\Pi(\mathcal{C})$	Yes	No	Yes	No	No
G($\mathcal{C}$)	Yes	Yes	Yes	No	No
(Ł$\oplus$G)($\mathcal{C}$), $a \notin C$	Yes	No	Yes	No	No
(Ł$\oplus\Pi$)($\mathcal{C}$), $a \notin C$	Yes	No	Yes	No	No
Other $L_*(\mathcal{C})$ $* \in$ **CONT-fin**	Yes	No	No	No	No
NM($\mathcal{C}$), $L_\otimes(\mathcal{C})$, $L_\star(\mathcal{C})$	Yes	Yes	Yes	No	No
Other $L_*(\mathcal{C})$, $* \in$ **WNM-fin**	Yes	Yes	No	No	No

Table 5: Standard completeness properties for propositional t-norm based logics with truth-constants (a denotes the element separating both components in the ordinal sum).

As we can see in the table only expansions of Łukasiewicz logic enjoy the CanFS$\mathcal{R}$C. Nevertheless, this situation can be improved when the language is restricted to the so-called *evaluated formulae*, i.e. formulae $\overline{r} \to \varphi$, where φ has no new truth-constants. For logics based on WNM-t-norms $*_n$ one must further require that $r > n(r)$ and the resulting formulae are called *positively evaluated formulae*. The completeness results obtained in the papers [15, 36, 12, 16] show that the fragment of (positively) evaluated formulae of a logic in Table 5 enjoys the CanFS$\mathcal{R}$C (denoted as CanFS$\mathcal{R}$C$_{ev}$) if, and only if, the full logic enjoys the Can$\mathcal{R}$C.

Finally, we briefly report on the issue of rational completeness for the expanded logics $L_*(\mathcal{C})$. The power of rational semantics for core fuzzy logics has been shown in [11] and [8] (recall e.g. Theorem 2.4). Assume that the rational unit interval $[0,1]^{\mathbb{Q}}$ is closed under $*$ and its residuum $\Rightarrow$. Let $[0,1]^{\mathbb{Q}}_*$ be the L_*-chain defined by the restriction of $*$ and $\Rightarrow$ to $[0,1]^{\mathbb{Q}}$. Let $\mathcal{C}$ be a subalgebra of $[0,1]^{\mathbb{Q}}_*$ and consider the logic $\mathrm{L}_*(\mathcal{C})$. Now $\mathrm{L}_*(\mathcal{C})$-chains defined over the rational unit interval are called *rational* and among them, the one which reflects the intended rational semantics is the so-called *canonical rational* chain $[0,1]^{\mathbb{Q}}_{\mathrm{L}_*(\mathcal{C})}$, defined as in the real case. The rational completeness properties have been studied in [18]. The results for general formulae are summarized in Table 6.

Logic	S$\mathcal{Q}$C	Can$\mathcal{Q}$C	CanFS$\mathcal{Q}$C	CanS$\mathcal{Q}$C
Ł$(\mathcal{C})$	Yes	Yes	Yes	No
$\Pi(\mathcal{C})$	Yes	Yes	No	No
G$(\mathcal{C})$	Yes	Yes	No	No
(Ł$\oplus$G)$(\mathcal{C})$, $a \notin C$	Yes	Yes	No	No
(Ł$\oplus\Pi$)$(\mathcal{C})$, $a \notin C$	Yes	Yes	No	No
Other $\mathrm{L}_*(\mathcal{C})$, $* \in$ **CONT-fin**	Yes	No	No	No
NM$(\mathcal{C})$, $\mathrm{L}_\otimes(\mathcal{C})$, $\mathrm{L}_\star(\mathcal{C})$	Yes	Yes	No	No
Other $\mathrm{L}_*(\mathcal{C})$, $* \in$ **WNM-fin**	Yes	No	No	No

Table 6: Rational completeness properties for propositional t-norm based logics with truth-constants (a denotes the element separating both components in the ordinal sum).

When we restrict to positively evaluated formulae, the situation for the logics under consideration is analogous to the case of real semantics, that is, those logics that already enjoy the Can$\mathcal{Q}$C they also enjoy the CanFS$\mathcal{Q}$C$_{ev}$, and viceversa.

As it regards to canonical strong completeness for evaluated formulae, both for the real and rational semantics, in [18] it is shown that it fails almost always. Indeed, it is only true for expansions of G, NM, $\mathrm{L}_\otimes$ and $\mathrm{L}_\star$ when the algebra $\mathcal{C}$ of truth-constants satisfies the following topological property: it has no positive sup-*accessible* points, i.e. an accumulation point which is the supremum of a strictly increasing sequence from C.

3. On first-order t-norm based fuzzy logics and their semantics

After recalling some basic definitions and results about predicate t-norm based fuzzy logics, in this section we investigate and solve two problems related to rational and finite-chain semantics for first-order t-norm based fuzzy logics (without truth-constants) that will be needed in the next section when studying their expansions with truth-constants. Nevertheless, since we consider the results interesting by themselves in the frame of first-order fuzzy logics, we present them separately in this section (not dealing with truth-constants).

3.1. Basic notions

The notion of t-norm based fuzzy logic has been extended to the first-order case following the definition given by Hájek for the predicate version of BL. Given a propositional core fuzzy logic L, the language $\mathcal{PL}$ of L$\forall$ is built in the standard classical way from the propositional language $\mathcal{L}$ of L by enlarging it with a set of predicate symbols *Pred* and a set of function symbols *Funct* and a set of object variables Var, together with the two classical quantifiers $\forall$ and $\exists$. The set of terms $Term$ is the minimum set containing the elements of Var and closed under the functions. The atomic formulae are expressions of the form $P(t^1, \ldots, t^n)$, where $P \in Pred$ and $t^1, \ldots, t^n \in Term$. The set of all formulae is obtained by closing the set of atomic formulae under combination by propositional connectives and quantification, i.e. if φ is a formula and x is an object variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulae as well.

In first-order core fuzzy logics it is usual to restrict the semantics to chains only. Given an L-chain $\mathcal{A}$, an $\mathcal{A}$-structure is $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in Pred}, \langle f_{\mathbf{M}} \rangle_{f \in Funct} \rangle$ where $M \neq \emptyset$, $f_{\mathbf{M}} : M^{ar(f)} \to M$, and $P_{\mathbf{M}} : M^{ar(P)} \to A$ for each $f \in Funct$ and $P \in Pred$ (where ar is the function that gives the arity of function and predicate symbols). For each evaluation of variables $v : Var \to M$, the interpretation of a $t \in Term$, denoted $t_{\mathbf{M},v}$, is defined as in classical first-order logic. The truth-value $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ of a formula is defined inductively from

$$\|P(t^1, \ldots, t^n)\|^{\mathcal{A}}_{\mathbf{M},v} = P_{\mathbf{M}}(t^1_{\mathbf{M},v}, \ldots, t^n_{\mathbf{M},v}),$$

taking into account that the value commutes with connectives, and defining

$$\begin{aligned}\|(\forall x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} &= \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y, \text{ except } x\} \\ \|(\exists x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} &= \sup\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables } y, \text{ except } x\}\end{aligned}$$

if the infimum and supremum exist in $\mathcal{A}$, otherwise the truth-value(s) remain undefined. An $\mathcal{A}$-structure $\mathbf{M}$ is called *safe* if all infs and sups needed for the definition of the truth-value of any formula exist in $\mathcal{A}$. Then the truth-value of a formula φ in a safe $\mathcal{A}$-structure $\mathbf{M}$ is just

$$\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} \mid v : Var \to M\}.$$

When $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = 1$ for a safe $\mathcal{A}$-structure $\mathbf{M}$, the pair $\langle \mathbf{M}, \mathcal{A} \rangle$ is said to be a *model* for φ, written $\langle \mathbf{M}, \mathcal{A} \rangle \models \varphi$. Sometimes we will call the pair $\langle \mathbf{M}, \mathcal{A} \rangle$ an L$\forall$-*structure*. We will say that φ is an $\mathcal{A}$-tautology when, for each safe $\mathcal{A}$-structure $\mathbf{M}$, $\langle \mathbf{M}, \mathcal{A} \rangle$ is a model of φ.

The axioms for L$\forall$ are obtained from those of L by substitution of propositional variables with formulae of $\mathcal{PL}$ plus the following axioms for quantifiers (the same used in [21] when defining BL$\forall$):

$(\forall 1)$	$(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
$(\exists 1)$	$\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
$(\forall 2)$	$(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
$(\exists 2)$	$(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
$(\forall 3)$	$(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

The rules of inference of L$\forall$ are *modus ponens* and *generalization*: from φ infer $(\forall x)\varphi$.

A completeness theorem for BL$\forall$ was proven in [21] and the completeness theorems of other first-order t-norm based fuzzy logics defined in the literature have been proven in the corresponding papers where the propositional logics are introduced.

Theorem 3.1 ([24]). *For any first-order core fuzzy logic* L$\forall$, *it holds that*

$T \vdash_{\mathrm{L}\forall} \varphi$ *iff* $\langle \mathbf{M}, \mathcal{A} \rangle \models \varphi$ *for each model* $\langle \mathbf{M}, \mathcal{A} \rangle$ *of* T *with* $\mathcal{A}$ *being a countable chain,* *for any set of sentences T and any formula φ.*

The properties S$\mathbb{K}$C, FS$\mathbb{K}$C and $\mathbb{K}$C are defined analogously as in the propositional case. Observe that the previous theorem says that every first-order core fuzzy logic enjoys the S$\mathbb{K}$C when $\mathbb{K}$ is the class of all countable chains. A usual way to prove S$\mathbb{K}$C consists on showing that every non-trivial countable L-chain can be σ-embedded (i.e. with an embedding which preserves existing suprema and infima) into some chain from $\mathbb{K}$. As proved in [8] this is a sufficient, but in general not necessary, condition for the S$\mathbb{K}$C. This method has been used to prove strong real completeness for a number of important logics. Others have been shown to lack any real completeness property as a consequence of the studies on the arithmetical complexity of the set of standard tautologies (showing that it is not recursively enumerable; see e.g. [24]). See some results in Table 7.

Logic	$\mathcal{R}$C	FS$\mathcal{R}$C	S$\mathcal{R}$C
MTL∀, SMTL∀, G∀, WNM∀, NM∀, $\mathrm{L}_{\otimes}$∀, $\mathrm{L}_{\star}$∀	Yes	Yes	Yes
BL∀, Ł∀, Π∀, (Ł⊕G)∀, (Ł⊕Π)∀	No	No	No

Table 7: Standard completeness properties for some prominent first-order t-norm based logics.

Other distinguished semantics are discussed in the following subsections. Related to this, an almost complete analog of Theorem 2.4 for first-order logics has been proved:

Theorem 3.2 ([8]). *Let* L *be a propositional core fuzzy logic. Then:*

(i) If L∀ *has a real completeness property, then* L∀ *also has the corresponding rational completeness property.*

(ii) L∀ *has the* S$\mathcal{F}$C *if, and only if, all* L*-chains are finite.*

The next two propositions yield necessary conditions for the completeness properties of first-order fuzzy logics that will be useful to refute some completeness results in this paper:

Proposition 3.3 ([8]). *If for some family* $\mathbb{K}$ *of* L*-chains the logic* L∀ *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C, S$\mathbb{K}$C *resp.), then* L *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C, S$\mathbb{K}$C *resp.) as well.*

Proposition 3.4 ([17]). *Let* L *and* L′ *be two propositional core fuzzy logics such that* L′∀ *is a conservative expansion of* L∀*. Let* $\mathbb{K}'$ *be a class of* L′*-chains and let* $\mathbb{K}$ *be the class of their* L*-reducts. If* L′∀ *enjoys the* $\mathbb{K}'$C *(*FS$\mathbb{K}'$C, S$\mathbb{K}'$C *resp.), then* L∀ *enjoys the* $\mathbb{K}$C *(*FS$\mathbb{K}$C, S$\mathbb{K}$C *resp.) as well.*

3.2. On rational completeness of first-order continuous t-norm based logics

The semantics based on rational chains has been considered in [8] for first-order core fuzzy logics in general. In particular, by Theorem 3.2 all the logics that enjoy the S$\mathcal{R}$C in Table 7 enjoy the S$\mathcal{Q}$C as well (this is also the case for the logics L_*∀ for any $* \in$ **WNM-fin**). In the same paper it has been shown that the S$\mathcal{Q}$C property can be proved by using the σ-embedding method for two prominent first-order logics that do not enjoy the S$\mathcal{R}$C, Ł∀ and Π∀, while it fails for BL∀ and SBL∀. The reason for that failure in the latter two logics is the formula $\Phi = (\forall x)(\chi \& \psi) \to (\chi \& (\forall x)\psi)$, where x is not free in χ, which in [21, page 102] was shown to be satisfied by every model on a densely ordered BL-chain, so in particular it is a rational tautology, but it has coutermodels in BL$\forall$ and SBL$\forall$. However, in [8] it is also shown how that problem can be overcome: namely, if BL$\forall$ and SBL$\forall$ are extended with Φ as an axiom schema, then the resulting logics do enjoy the S$\mathcal{Q}$C.

In this subsection we will continue the study of rational completeness properties of first-order continuous t-norm based logics. To this end, we need to consider two families of standard BL-chains defined by t-norms in **CONT-fin**.

Definition 3.5. *Let* $[0,1]_* = \bigoplus_{i=1,\dots n} \mathcal{C}_i$ *be a standard* BL*-chain with its decomposition in ordinal sum of basic components. We say that* $[0,1]_*$ *is of* type I *if there is* $i \in \{1, \dots, n-1\}$ *such that one of the following holds:*

- $\mathcal{C}_i \cong [0,1]_{\text{Ł}}$ *and* $\mathcal{C}_{i+1} \cong [0,1]_{\text{G}}$
- $\mathcal{C}_i \cong [0,1]_{\text{Ł}}$ *and* $\mathcal{C}_{i+1} \cong [0,1]_{\Pi}$
- $\mathcal{C}_i \cong [0,1]_{\Pi}$ *and* $\mathcal{C}_{i+1} \cong [0,1]_{\text{G}}$
- $\mathcal{C}_i \cong \mathcal{C}_{i+1} \cong [0,1]_{\Pi}$
- $\mathcal{C}_i \cong \mathcal{C}_{i+1} \cong [0,1]_{\text{Ł}}$
- $\mathcal{C}_i \cong [0,1]_{\Pi}$ *and* $\mathcal{C}_{i+1} \cong [0,1]_{\text{Ł}}$

Otherwise we say that $[0,1]_*$ *is of* type II*, that is, when* $[0,1]_*$ *is of one of the following forms:* $[0,1]_{\text{Ł}}$*,* $[0,1]_{\Pi}$*,* $[0,1]_{\text{G}}$*,* $[0,1]_{\text{G}} \oplus [0,1]_{\Pi}$ *or* $[0,1]_{\text{G}} \oplus [0,1]_{\text{Ł}}$*.*

Next theorem gives a characterization of the logics Ł$_*\forall$ with $* \in$ **CONT-fin** that enjoy rational completeness in terms of provability of the formula Φ and in terms of type II chains. Moreover, interestingly enough, it shows that in this context the three rational completeness properties turn out to be equivalent with the σ-embeddability.

Theorem 3.6. *Given a continuous t-norm* $* \in$ **CONT-fin***, the following are equivalent:*

1. *Every non-trivial countable* Ł$_*$*-chain is* σ*-embeddable into a rational* Ł$_*$*-chain.*
2. Ł$_*\forall$ *has the* S$\mathcal{Q}$C*.*
3. Ł$_*\forall$ *has the* $\mathcal{Q}$C*.*
4. $\vdash_{\text{Ł}_*\forall} (\forall x)(\chi \& \psi) \to (\chi \& (\forall x)\psi)$*.*
5. $[0,1]_*$ *is of type II.*

Proof. $1 \Rightarrow 2$ is a particular case of the general fact that σ-embeddability into a class $\mathbb{K}$ is a sufficient condition for the S$\mathbb{K}$C, and $2 \Rightarrow 3$ is trivial.

$3 \Rightarrow 4$: Assume that Ł$_*\forall$ has the $\mathcal{Q}$C. Since, as we have already mentioned, Φ is satisfied in any model over a densely ordered BL-chain, Φ is in particular a tautology for models over rational Ł$_*$-chains and hence, by $\mathcal{Q}$C, we obtain $\vdash_{\text{Ł}_*\forall} \Phi$.

$4 \Rightarrow 5$: Suppose that $[0,1]_*$ is of type I. Thus, it has two consecutive components, say $\mathcal{C}_1$ and $\mathcal{C}_2$, in its ordinal sum decomposition of one the six forms listed in Definition 3.5. We divide the proof in two cases:

(i) Let $\mathcal{C}_2$ be isomorphic either to $[0,1]_{\mathrm{G}}$ or to $[0,1]_{\Pi}$. If $\mathcal{C}_1$ is isomorphic to $[0,1]_{\Pi}$, let a be any element in the interior of $\mathcal{C}_1$ and let $\mathcal{A}$ be the subalgebra generated by $([0,1]\setminus C_1)\cup\{a\}$. If $\mathcal{C}_1$ is isomorphic to $[0,1]_{\mathrm{L}}$, let b be a rational in the interior of $\mathcal{C}_1$, let $\mathcal{A}$ be the subalgebra generated by $([0,1]\setminus C_1)\cup\{b\}$ and let a be the maximum of $\mathcal{A}\cap C_1$. Observe that in both cases $a = \inf C_2$, which is an element in the interior of C_1 and hence non-idempotent. Let X be a countable subset of C_2 such that in $\mathcal{A}$ $\inf X = \inf C_2$. Taking a convenient model over $\mathcal{A}$ and evaluating Φ we obtain $(\inf\{a * x \mid x \in X\}) \Rightarrow (a * \inf X) = a \Rightarrow a * a \neq 1$. Therefore Φ is not a theorem of $\mathrm{L}_*\forall$.

(ii) Let $\mathcal{C}_2$ be isomorphic to $[0,1]_{\mathrm{L}}$. In such a case, consider the chain $\mathcal{B}$ obtained from $[0,1]_*$ by replacing the component $\mathcal{C}_2$ by an isomorphic copy $\mathcal{C}_2'$ of the cancellative hoop $(0,1]_{\Pi}$. It is clear that $\mathcal{B}$ is an L_*-chain by noticing that $(0,1]_{\Pi} \in ISP_U([0,1]_{\mathrm{L}})$. Now, notice that $\mathcal{B}$ has no idempotent element separating the components $\mathcal{C}_2'$ and $\mathcal{C}_1$. Then one can use the same construction as in (i) to obtain a subalgebra such that the infimum of $\mathcal{C}_2'$ is not idempotent, and a countermodel of the formula Φ over this subalgebra.

$5 \Rightarrow 1$: We know from the results in [8] that all non-trivial countable MV-chains, Π-chains and G-chains are σ-embeddable into a rational chain of the same kind. This already proves the implication when $[0,1]_*$ is $[0,1]_{\mathrm{G}}$, $[0,1]_{\Pi}$ or $[0,1]_{\mathrm{L}}$. If $[0,1]_* = \mathcal{C}_1 \oplus \mathcal{C}_2$, then the chains of the variety generated by $[0,1]_*$ belong to $HSP_U(\mathcal{C}_1) \cup (ISP_U(\mathcal{C}_1) \oplus HSP_U(\mathcal{C}_2))$ (see [9]). Therefore, if $[0,1]_* = [0,1]_{\mathrm{G}} \oplus [0,1]_{\Pi}$ or $[0,1]_* = [0,1]_{\mathrm{G}} \oplus [0,1]_{\mathrm{L}}$, any L_*-chain can be σ-embedded into the rational L_*-chain component-wise since, in these two cases, all the elements in the lower component are idempotent. $\square$

3.3. On the relationship of first-order tautologies over the standard algebra and over finite chains

As stated in Theorem 3.2 a first-order core fuzzy logic enjoys the strong finite-chain completeness S$\mathcal{F}$C if, and only if, all the chains in its corresponding variety are finite. Obviously, this is not the case in any t-norm based logic and hence we can forget about this kind of completeness. Moreover, we will show in this subsection how the weaker properties of completeness w.r.t. the finite-chain semantics (FS$\mathcal{F}$C and $\mathcal{F}$C) for many first-order t-norm based logics can be also refuted by considering the relation between the set of tautologies of the standard algebra and the set of tautologies of finite chains.

As we already mentioned, in many first-order logics (e.g. Ł$\forall$, $\Pi\forall$ and in general for every $\mathrm{L}_*\forall$ for $* \in \textbf{CONT-fin}\setminus\{*_{\mathrm{G}}\}$) not only the standard completeness fails, but the set of standard tautologies is not even recursively enumerable. On the other hand, Hájek shows that in the case of Łukasiewicz logic the standard tautologies coincide with the common tautologies of finite MV-chains.

Theorem 3.7 ([21]). *A first-order formula φ is a $[0,1]_{\mathrm{L}}$-tautology if, and only if, φ is a tautology over all finite* MV*-chains.*

For first-order Gödel logic the situation is different.

Theorem 3.8 ([35]). *The first-order $[0,1]_{\mathrm{G}}$-tautologies do not coincide with the common first-order tautologies over finite* G*-chains.*

Actually, the failure expressed in this last theorem is much more general since it is related to the lack of continuity of the truth value functions as the following results show.

Theorem 3.9. *Let $[0,1]_*$ be the standard chain defined by a left-continuous t-norm $*$ non-isomorphic to the Łukasiewicz t-norm. Then:*

(i) If $*$ is continuous, then the formula[15]

$$(C\forall) \quad (\exists x)(P(x) \to (\forall y)P(y))$$

is an $\mathcal{A}$-tautology for any finite L_* chain $\mathcal{A}$, but it is not a $[0,1]_*$-tautology.

(ii) If $*$ is not continuous, then the formula Φ

$$(\forall x)(\chi \& \psi) \to (\chi \& (\forall x)\psi), \text{ where } x \text{ is not free in } \chi$$

is an $\mathcal{A}$-tautology for any finite L_* chain $\mathcal{A}$, but it is not a $[0,1]_*$-tautology.

Proof. (i) It is well known that, for any continuous t-norm $*$ which is not isomorphic to Łukasiewicz t-norm, the corresponding negation $n_*(x) = x \Rightarrow_* 0$ is not (right) continuous at $x = 0$ (see e.g. the Annex in [13]). Let $b = lim_{x \to 0^+} n_*(x)$. We know that $b < 1$. Take an infinite decreasing sequence $1 > a_1 > a_2 > \ldots > a_n \ldots > 0$ with limit 0. Consider the $[0,1]_*$-model $\mathbf{M} = \langle \mathbb{N}, P_\mathbf{M} \rangle$ where $P_\mathbf{M}(n) = a_n$. Then $\|(\exists x)(P(x) \to (\forall y)P(y))\|^{[0,1]_*}_{\mathbf{M},e} = \sup_n\{a_n \Rightarrow_* (\inf_n a_n)\} = \sup_n\{a_n \Rightarrow_* 0\} = \sup_n n_*(a_n) = b < 1$. On the other hand, it is clear that the formula has value $\overline{1}^{\mathcal{A}}$ in any structure over a finite L_*-chain $\mathcal{A}$.

(ii) For simplicity, let us take the following instance of Φ:

$$(\forall x)(P(c) \& Q(x)) \to P(c) \& (\forall x)Q(x)$$

where c is a 0-ary functional symbol. If the t-norm is not right-continuous there must exist a sequence $\langle a_n : n \geq 1 \rangle$ and an element b such that $b * \inf\{a_n, n \geq 1\} < \inf\{b * a_n, n \geq 1\}$. Consider the $[0,1]_*$-model $\mathbf{M} = \langle \mathbb{N}, P_\mathbf{M}, Q_\mathbf{M} \rangle$ and an evaluation of variables e such that $P_\mathbf{M}(c_\mathbf{M}) = b$ and $Q_\mathbf{M}(n) = a_n$ for every n. Then $\|(\forall x)(P(c) \& Q(x)) \to P(c) \& (\forall x)Q(x)\|^{[0,1]_*}_{\mathbf{M},e} = \inf\{b * a_n, n \geq 1\} \Rightarrow_* (b * \inf\{a_n, n \geq 1\}) < 1$. But an easy computation shows that for any finite chain the formula is a tautology (take into account that the inf becomes a min). □

As (i) and (ii) of Theorem 3.9 cover all logics of a t-norm and its residuum different from Łukasiewicz, it is clear that only in case that all truth functions are continuous (Łukasiewicz) the result holds.

Corollary 3.10. *Let $*$ be a left-continuous t-norm. Then the first-order $[0,1]_*$-tautologies coincide with the common $\mathcal{A}$-tautologies for all finite L_*-chains $\mathcal{A}$ if, and only if, $[0,1]_* \cong [0,1]_\mathrm{Ł}$.*

And, interestingly enough, the above gives also as a consequence the failure of finite-chain completeness properties for all of these logics.

Corollary 3.11. *For any left-continuous t-norm $*$, the $\mathrm{L}_*\forall$ does not have the* $\mathcal{F}$C.

Proof. If $*$ is isomorphic to Łukasiewicz t-norm, the result easily follows from Theorem 3.7 and the fact that the standard tautologies of Ł$\forall$ form a non-recursively enumerable set. For the rest of the cases, the result is a consequence of the last corollary and the general completeness result of $\mathrm{L}_*\forall$ with respect to the common tautologies of L_*-chains [25, Th. 5]. □

---

[15] This formula is in fact an instance of the *witnessing axioms* studied in [25].

4. First-order t-norm based logics with truth-constants

First-order t-norm based fuzzy logics with truth-constants, in principle, could be introduced in two different ways:

- Given a left-continuous t-norm $*$ and a countable subalgebra $\mathcal{C} \subseteq [0,1]_*$, consider the logic $\mathrm{L}_*(\mathcal{C})$ and take its first-order extension $\mathrm{L}_*(\mathcal{C})\forall$.
- Given a left-continuous t-norm $*$ consider its associated propositional logic L_*, take its first-order extension $\mathrm{L}_*\forall$ and now (by enhancing the language with the constants and adding the book-keeping axioms) define its expansion $\mathrm{L}_*\forall(\mathcal{C})$ with truth-constants from a countable algebra $\mathcal{C} \subseteq [0,1]_*$.

However, these two methods turn out to define the same logic. To fix the notation, we will use the second one: $\mathrm{L}_*\forall(\mathcal{C})$.

As in the propositional case, we are interested in completeness properties of these logics with respect to distinguished semantics. We will find again some positive and some negative results. For the negative ones two propositions are very useful: 3.3 and 3.4, in the sense that the failure of a completeness property in a weaker logic implies the failure in the stronger one (and the same propositions hold as well when the language is restricted to (positively) evaluated formulae).

In order to apply Proposition 3.4 we need to show that adding truth-constants to a first-order logic $\mathrm{L}_*\forall$ results into a conservative expansion. This is the aim of the next subsection.

4.1. Conservativeness results

In the case of Łukasiewicz t-norm Hájek *et al.* already proved in [27] that RPL$\forall$ (Rational Pavelka predicate logic[16]) is a conservative expansion of Ł$\forall$. Actually, from the proofs in [27] we can extract the following result:

Lemma 4.1 (cf. [27]). *Let $\mathcal{C}$ be a subalgebra of $[0,1]_{\text{Ł}}^{\mathbb{Q}}$, $\mathcal{A}$ a countable* MV-*chain and* $\mathbf{M}$ *an $\mathcal{A}$-safe structure in a predicate language for Ł$\forall$. Then there is a divisible*[17] MV-*chain $\mathcal{A}'$ such that $\mathcal{A}$ is σ-embeddable into $\mathcal{A}'$, and the truth-constants from $\mathcal{C}$ are interpretable in $\mathcal{A}'$ in such a way that* $\mathbf{M}$ *is also an $\mathcal{A}'$-safe structure for Ł$\forall(\mathcal{C})$.*

The conservativeness result is also valid for the remaining basic continuous t-norms and also for the family of SMTL t-norms. These are left-continuous t-norms whose associated negation is the so-called Gödel negation: $n(x) = 0$ for all $x \neq 0$ and $n(0) = 1$.

Lemma 4.2. *Let $*$ be an* SMTL-*t-norm, $\mathcal{C}$ a countable subalgebra of* $[0,1]_*$ *and* $\mathbf{M}$ *an $\mathcal{A}$-safe structure in a predicate language for* $\mathrm{L}_*\forall$. *Then the truth-constants from $\mathcal{C}$ are interpretable in $\mathcal{A}$ in such a way that* $\mathbf{M}$ *is also an $\mathcal{A}$-safe structure for* $\mathrm{L}_*\forall(\mathcal{C})$.

[16]In our notation RPL$\forall$ corresponds to Ł$\forall(\mathcal{C})$ when $C = [0,1] \cap \mathbb{Q}$.

[17]A MV-chain $\mathcal{A}$ is called *divisible* if for every natural m and every $x \in A$ there exists $y \in A$ such that $y \oplus \overset{m}{\ldots} \oplus y = x$, where $\oplus$ is defined as $x \oplus y = \neg x \Rightarrow y$.

Proof. For every $r \in C \setminus \{0\}$ interpret $\overline{r}$ as $\overline{1}^{\mathcal{A}}$, and $\overline{0}$ as $\overline{0}^{\mathcal{A}}$. This turns $\mathcal{A}$ into a chain for the expanded language. It is clear that $\mathbf{M}$ is also $\mathcal{A}$-safe in this language since the interpretation of the constants does not give any new value. $\square$

From these lemmata we obtain conservativeness results for logics based on continuous and WNM-t-norms. In the proof of next theorem we use the fact that an SBL-t-norm is an ordinal sum either with a first component which is not a Łukasiewicz component or without a first component (cf. [9]).

Theorem 4.3. *Let $*$ be a continuous t-norm and $\mathcal{C}$ a countable subalgebra of $[0,1]_*$ such that, if $*$ is not an* SBL*-t-norm, the truth-constants in the Łukasiewicz first component of the decomposition correspond to rational numbers. Then* $\mathrm{L}_*\forall(\mathcal{C})$ *is a conservative expansion of* $\mathrm{L}_*\forall$.

Proof. Let $\Gamma \cup \{\varphi\}$ be $\mathrm{L}_*\forall$-formulae such that $\Gamma \not\vdash_{\mathrm{L}_*\forall} \varphi$. We must show that $\Gamma \not\vdash_{\mathrm{L}_*\forall(\mathcal{C})} \varphi$. By hypothesis, there is some safe $\mathrm{L}_*\forall$-structure $\langle \mathbf{M}, \mathcal{A}\rangle$ such that $\langle \mathbf{M}, \mathcal{A}\rangle \models \Gamma$ and $\langle \mathbf{M}, \mathcal{A}\rangle \not\models \varphi$, where $\mathcal{A}$ is a countable L_*-chain. If $*$ is an SBL-t-norm, then $\mathcal{A}$ is an SBL-chain and applying Lemma 4.2 the problem is solved. If $*$ is not an SBL-t-norm, then $*$ is the ordinal sum of a Łukasiewicz component and a hoop $\mathcal{B}$. Then, by [9, Prop. 3], $\mathcal{A}$ has to be a chain of $HSP_U([0,1]_{\mathrm{L}}) \cup (ISP_U([0,1]_{\mathrm{L}}) \oplus HSP_U(\mathcal{B}))$. Then $\mathcal{A}$ is either an MV-chain or the ordinal sum (in the sense of hoops) of an MV-chain $\mathcal{A}_1$ and a hoop $\mathcal{A}_2$ of $HSP_U(\mathcal{B})$. Take $\mathcal{A}'$ as the ordinal sum of the divisible hull $\mathcal{A}'_1$ of $\mathcal{A}$ (as done in Lemma 4.1) and $\mathcal{A}_2$. Thus, we have obtained a BL-chain $\mathcal{A}'$ belonging to $\mathbf{V}([0,1]_*)$ (again by Proposition 3 of [9]). Then we define an $\mathrm{L}_*(\mathcal{C})$-chain over $\mathcal{A}'$ interpreting the truth-constants from the Łukasiewicz first component of $\mathcal{C}$ as the corresponding truth-values of $\mathcal{A}'_1$ and the remaining truth-constants as $\overline{1}^{\mathcal{A}'}$. By the previous lemmata $\mathcal{A}$ is σ-embeddable into $\mathcal{A}'$ as $\mathrm{L}_*(\mathcal{C})$-chains. Thus we have obtained a chain $\mathcal{A}'$ in the expanded language such that $\mathbf{M}$ is an $\mathcal{A}'$-safe structure and therefore $\langle \mathbf{M}, \mathcal{A}'\rangle \models \Gamma$ while $\langle \mathbf{M}, \mathcal{A}'\rangle \not\models \varphi$, and the theorem is proved. $\square$

Theorem 4.4. *For every $* \in$ **WNM-fin** and every countable $\mathcal{C} \subseteq [0,1]_*$, the logic* $\mathrm{L}_*\forall(\mathcal{C})$ *is a conservative expansion of* $\mathrm{L}_*\forall$.

Proof. Let φ be an $\mathrm{L}_*\forall$-formula such that $\not\vdash_{\mathrm{L}_*\forall} \varphi$. Since $\mathrm{L}_*\forall$ has the S$\mathbb{K}$C with respect to the class $\mathbb{K} = \{[0,1]_*\}$ (it easily follows from the results in [32]), there is an $\mathrm{L}_*\forall$-structure $\langle \mathbf{M}, [0,1]_*\rangle$ such that $\langle \mathbf{M}, [0,1]_*\rangle \not\models \varphi$. Then it is clear that we also have $\langle \mathbf{M}, [0,1]_{\mathrm{L}_*(\mathcal{C})}\rangle \not\models \varphi$, and hence $\not\vdash_{\mathrm{L}_*\forall(\mathcal{C})} \varphi$. $\square$

4.2. Real, rational and finite-chain completeness results

First of all, let us comment about finite-chain completeness. It is clear that, from Corollary 3.11 and the conservativeness results of last subsection, the logics $\mathrm{L}_*\forall(\mathcal{C})$, for $*$ being a continuous (non-isomorphic to Łukasiewicz), an SMTL or a **WNM-fin** t-norm, do not enjoy the $\mathcal{F}$C. Nevertheless, the relation between standard tautologies and finite-chain tautologies in the case of first order Łukasiewicz logic recalled in Theorem 3.7 easily generalizes to the logics $\text{Ł}\forall(\mathcal{C})$:

Theorem 4.5. [18] *Let $\mathcal{C}$ be a subalgebra of $[0,1]_{\text{Ł}}^{\mathbb{Q}}$. A first-order formula φ is a tautology of $[0,1]_{\text{Ł}(\mathcal{C})}$ if, and only if, φ is a tautology of $L_n(\mathcal{C} \cap L_n)$ for each $n \geq 2$ such that the truth values associated to the truth-constants appearing in φ belong to L_n.*

Notice the peculiarity of this result, as it describes the set of tautologies of a chain in a given language with truth-constants in terms of tautologies of finite chains in different languages, i.e. different sets of truth-constants.

The rest of the section is devoted to the completeness properties with respect to the (canonical) rational and real semantics. First, taking into account that for every continuous t-norm $*$ different from Gödel t-norm the $\mathcal{R}$C fails for $\text{L}_*\forall$ (recall Table 3) and using Proposition 4.3 it also fails for their expansions with truth-constants. However, for WNM-t-norm based logics we can give positive answers to some completeness problems.

Theorem 4.6. *For every $* \in$ **WNM-fin**, the logic $\text{L}_*\forall(\mathcal{C})$ enjoys the S$\mathcal{R}$C.*

Proof. In [32] it has been proved that every countable L_*-chain $\mathcal{A}$ is embeddable into an L_*-chain $\mathcal{B}$ over $[0,1]$. It is not difficult to observe that in fact we can assume it to be a σ-embedding; call it f. Assume, in addition, that $\mathcal{A}$ is an $\text{L}_*(\mathcal{C})$-chain. For every $r \in C$, interpret $\overline{r}$ in $\mathcal{B}$ as $f(\overline{r}^{\mathcal{A}})$, which gives a standard $\text{L}_*(\mathcal{C})$-chain. Thus, we obtain the S$\mathcal{R}$C for $\text{L}_*\forall(\mathcal{C})$. □

Moreover, some of them enjoy the Can$\mathcal{R}$C:

Theorem 4.7. *The logics $\text{G}\forall(\mathcal{C})$, $\text{NM}\forall(\mathcal{C})$, $\text{L}_{\otimes}\forall(\mathcal{C})$ and $\text{L}_{\star}\forall(\mathcal{C})$ enjoy the Can$\mathcal{R}$C.*

Proof. Let $*$ denote any of these four t-norms. We will prove that if $\nvdash_{\text{L}_*\forall(\mathcal{C})} \varphi$ for some formula φ, then there is an $\text{L}_*\forall(\mathcal{C})$-structure $\langle \mathbf{M}, [0,1]_{\text{L}_*(\mathcal{C})} \rangle$ such that $\langle \mathbf{M}, [0,1]_{\text{L}_*(\mathcal{C})} \rangle \not\models \varphi$.

If $\nvdash_{\text{L}_*\forall(\mathcal{C})} \varphi$, then there exists an $\text{L}_*\forall(\mathcal{C})$-structure $\langle \mathbf{M}, \mathcal{A} \rangle$ over a countable L_*-chain $\mathcal{A}$ and an evaluation v such that $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} < \overline{1}^{\mathcal{A}}$. Take $s = \min(\{r \in C \mid \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}, r \text{ appears in } \varphi\} \cup \{1\})$. It is not difficult to see that $s > \neg s$ (if $s = 1$ it is obvious; if $s = r$ such that $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$, it is enforced by the book-keeping axioms). Define $g : A \to [0,1]$ such that $g(\overline{1}^{\mathcal{A}}) = 1$, $g(\overline{0}^{\mathcal{A}}) = 0$ and injects $A \setminus \{\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\}$ into $(\neg s, s)$ preserving existing suprema and infima on $(0,1)$, and such that $g(\overline{r}^{\mathcal{A}}) = r$ for every truth-constant appearing in φ such that $\overline{r}^{\mathcal{A}} \neq \overline{1}^{\mathcal{A}}, \overline{0}^{\mathcal{A}}$. If $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in Pred}, \langle f_{\mathbf{M}} \rangle_{f \in Funct} \rangle$, using the mapping g we can produce a structure $\langle \mathbf{M}', [0,1]_{\text{L}_*(\mathcal{C})} \rangle$, where $\mathbf{M}' = \langle M, \langle P_{\mathbf{M}'} \rangle_{P \in Pred}, \langle f_{\mathbf{M}} \rangle_{f \in Funct} \rangle$, with $P^{\mathbf{M}'} : M^{ar(P)} \to [0,1]$ is defined as $P_{\mathbf{M}'} = g \circ P_{\mathbf{M}}$, and hence for every evaluation of variables e on M one has

$$\|P(t_1, t_2, \ldots, t_n)\|^{[0,1]_{\text{L}_*(\mathcal{C})}}_{\mathbf{M}',e} = g(\|P(t_1, t_2, \ldots, t_n)\|^{\mathcal{A}}_{\mathbf{M},e})$$

for each predicate symbol P and terms $t_1, t_2, \ldots, t_n$. Now we can prove that the following statements hold for any subformula ψ of φ and every evaluation of variables e on M:

[18] In this theorem Ł_n denotes the finite MV-chain defined over $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ and given $\mathcal{C} \subseteq [0,1]_{\text{Ł}}^{\mathbb{Q}}$, $\text{Ł}_n(\mathcal{C} \cap \text{Ł}_n)$ denotes the expansion of Ł_n with the truth-constants from $C \cap \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ (which, obviously, must be canonically interpreted).

a) If $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$, then $\|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e} \geq s$, [19]

b) If $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}$, then $\|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e} = g(\|\psi\|^{\mathcal{A}}_{\mathbf{M},e})$,

c) If $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{0}^{\mathcal{A}}$, then $\|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e} \leq \neg s$.

The proof is by induction on the complexity of ψ and we provide some details next:

- If $\psi = \overline{r}$, then $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{r}^{\mathcal{A}}$. Then either $\overline{r}^{A} = \overline{1}^{\mathcal{A}}$ and hence $r \geq s$, or $\overline{r}^{A} = \overline{0}^{\mathcal{A}}$ and hence $r \leq \neg s$, or $\overline{r}^{A} \neq \overline{1}^{\mathcal{A}}, \overline{0}^{\mathcal{A}}$, and hence $g(\overline{r}^{A}) = r = \|\overline{r}\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e}$.
- If $\psi = P(t_1, t_2, \ldots, t_n)$, then it holds by definition.
- Suppose $\psi = \alpha\&\beta$. If $\|\alpha\&\beta\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$, then $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \|\beta\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$ and thus by induction hypothesis $\|\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e}, \|\beta\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e} \geq s$, and hence $\|\alpha\&\beta\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e} \geq s$. If $\|\alpha\&\beta\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{0}^{\mathcal{A}}$, then $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},e} \leq \neg\|\beta\|^{\mathcal{A}}_{\mathbf{M},e}$ and one can check that the result holds by separating cases for the values of α and β and using the induction hypothesis. If $\|\alpha\&\beta\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{1}^{\mathcal{A}}, \overline{0}^{\mathcal{A}}$ and again by separating cases and using the induction hypothesis the result easily follows.
- If $\psi = \alpha \rightarrow \beta$, to check the result is again a matter of routinary proof by cases and usage of the induction hypothesis.
- Suppose $\psi = (\forall x)\alpha$, and let $V(e)$ denote the set of evaluations of variables v such that $e(y) = v(y)$ for all variables y, except x. Recall that $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \mid v \in V(e)\}$.

  If $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$, then for every such $v \in V(e)$ we have $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} = \overline{1}^{\mathcal{A}}$, and hence $\|\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',v} \geq s$, which implies that $\|(\forall x)\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e} \geq s$.

  If $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} \neq \overline{1}^{\mathcal{A}}, \overline{0}^{\mathcal{A}}$, then there is no evaluation such that $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} = \overline{0}^{\mathcal{A}}$ and it is enough to consider the infimum over the subset $V^+(e) \subseteq V(e)$ of those evaluations v such that $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \neq \overline{1}^{\mathcal{A}}$, i.e. $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \mid v \in V^+(e)\} \neq \overline{1}^{\mathcal{A}}, \overline{0}^{\mathcal{A}}$. Then, since g preserves the existing infima, we have the following equalities: $g(\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e}) = g(\inf\{\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \mid v \in V^+(e)\}) = \inf\{g(\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v}) \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',v} \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',v} \mid v \in V(e)\} = \|(\forall x)\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e}$.

  If $\|(\forall x)\alpha\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{0}^{\mathcal{A}}$, then we distinguish two cases. First, assume that for some $v \in V(e)$ $\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} = \overline{0}^{\mathcal{A}}$. Then, by induction hypothesis, $\|\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',v} \leq \neg s$, and hence $\|(\forall x)\alpha\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e} \leq \neg s$. Assume now that for every $v \in V(e)$,

[19] Take into account that if $\|(\exists x)\varphi(x)\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$ then $\|(\exists x)\varphi(x)\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M}',e}$ could be 1 (if there is an individual $a \in M$ such that $\varphi(a) = 1$) or s (if there is not an individual $a \in M$ such that $\varphi(a) = 1$).

$\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v} \neq \overline{0}^{\mathcal{A}}$. Restrict again to $V^{+}(e)$. Then for every $v \in V^{+}(e)$, $\|\alpha\|^{[0,1]_{\mathrm{L}_*(\mathcal{C})}}_{\mathbf{M}',v} = g(\|\alpha\|^{\mathcal{A}}_{\mathbf{M},v})$ and hence $\|(\forall x)\alpha\|^{[0,1]_{\mathrm{L}_*(\mathcal{C})}}_{\mathbf{M}',e} = \neg s$, since g is one-to-one in $A \setminus \{\overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\}$.

- If $\psi = (\exists x)\alpha$, the reasoning is similar to the previous one (now it uses that g preserves existing suprema).

Finally, from the above statements the theorem easily follows since $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} < \overline{1}^{\mathcal{A}}$, and thus $\|\varphi\|^{[0,1]_{\mathrm{L}_*(\mathcal{C})}}_{\mathbf{M}',v} \leq s < 1$. □

Therefore, we have solved all the standard completeness problems for first-order logics under our scope, since in the remaining cases the properties obviously do not hold as they already fail for the corresponding propositional logics with truth-constants (i.e. CanFS$\mathcal{R}$C for every $* \in$ **WNM-fin**, and Can$\mathcal{R}$C for all $* \in$ **WNM-fin** except the four distinguished ones). Table 8 collects the results.

Logic	$\mathcal{R}$C, FS$\mathcal{R}$C, S$\mathcal{R}$C	Can$\mathcal{R}$C	CanFS$\mathcal{R}$C
$\mathrm{L}_*\forall(\mathcal{C})$, $* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$	No	No	No
$\mathrm{G}\forall(\mathcal{C})$, $\mathrm{NM}\forall(\mathcal{C})$, $\mathrm{L}_{\otimes}\forall(\mathcal{C})$, $\mathrm{L}_{\star}\forall(\mathcal{C})$	Yes	Yes	No
Other $\mathrm{L}_*\forall(\mathcal{C})$, $* \in$ **WNM-fin**	Yes	No	No

Table 8: Standard completeness properties for first-order t-norm based logics with truth-constants.

As regards to rational completeness properties, we have again S$\mathcal{Q}$C for all those logics satisfying S$\mathcal{R}$C (i.e. those based on WNM-t-norms), by virtue of Theorem 3.2. Next theorem considers this property for logics based on continuous t-norms (by using the distinction in Definition 3.5).

Theorem 4.8. *Let $* \in$ **CONT-fin** and $\mathcal{C}$ be a subalgebra of $[0,1]^{\mathbb{Q}}_*$. Then:*

- *If $*$ is of type I, then $\mathrm{L}_*\forall(\mathcal{C})$ does not enjoy the $\mathcal{Q}$C.*
- *If $*$ is of type II, then $\mathrm{L}_*\forall(\mathcal{C})$ enjoys the S$\mathcal{Q}$C.*

Proof. If $*$ is of type I, by Theorem 3.6 $\mathrm{L}_*\forall$ does not enjoy the $\mathcal{Q}$C, so it also fails for $\mathrm{L}_*\forall(\mathcal{C})$. Otherwise, $\mathrm{L}_*\forall$ enjoys the rational σ-embeddability property. Therefore, for every non-trivial countable $\mathrm{L}_*(\mathcal{C})$-chain $\mathcal{A}$, we know that as an L_*-algebra it is σ-embeddable into a rational L_*-chain. So, by considering in the rational chain the interpretations of the constants given by the embedding, we have that $\mathcal{A}$ is σ-embeddable into a rational $\mathrm{L}_*(\mathcal{C})$-chain, and hence we have the S$\mathcal{Q}$C. □

However, except for the Gödel case, they never satisfy the Can$\mathcal{Q}$C. Indeed, for every $* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$ since the logic $\mathrm{L}_*\forall(\mathcal{C})$ does not enjoy the Can$\mathcal{R}$C, there is a formula φ such that $\nvdash_{\mathrm{L}_*\forall(\mathcal{C})} \varphi$ and $\models_{[0,1]_{\mathrm{L}_*\forall(\mathcal{C})}} \varphi$. But $[0,1]^{\mathbb{Q}}_{\mathrm{L}_*\forall(\mathcal{C})}$ is a σ-subalgebra of $[0,1]_{\mathrm{L}_*\forall(\mathcal{C})}$ (i.e. such that inclusion is a σ-embedding), so we also have $\models_{[0,1]^{\mathbb{Q}}_{\mathrm{L}_*\forall(\mathcal{C})}} \varphi$ and hence a counterexample to the Can$\mathcal{Q}$C. On the contrary, the Can$\mathcal{Q}$C is true for Gödel, nilpotent minimum and the two special cases of WNM ($\star$ and $\otimes$) predicate logics with truth-constants. To prove it we need the following lemma.

Lemma 4.9. *Let $\mathcal{A} = \langle A, *, \Rightarrow_*, n, 0, 1\rangle$ be a chain of either $\mathrm{L}_\star$ or $\mathrm{L}_\otimes$. Then $\mathcal{A}$ is σ-embeddable into a chain over $[0,1]^{\mathbb{Q}}$ of the same variety.*

Proof. We give the proof for $\mathrm{L}_\otimes$ as the remaining case is completely analogous. Take the chain $\mathcal{A}$ and apply the same construction as in the proof of [[21], Lemma 5.3.1] which consists on putting a copy of the rationals from $(0,1)$ into each "hole" (a pair of elements $\langle x, y\rangle$ such that y is the successor of x). Formally, let $A' = \bigcup\{C_x \mid x \in A\}$, where for each $x \in A$,

$$C_x = \begin{cases} \{\langle x, 0\rangle\}, & \text{if } x \text{ has no successor in } A \\ \{\langle x, r\rangle \mid r \in [0,1)^{\mathbb{Q}}\},, & \text{if } x \text{ has a successor in } A \end{cases}$$

Define $n'\langle x, 0\rangle = \langle n(x), 0\rangle$ and for $r \neq 0$,

$$n\langle x, r\rangle = \begin{cases} \langle n(x^+), 0\rangle, & \text{if } x \text{ belongs to a constant interval in } A \\ \langle n(x^+), 1-r\rangle, & \text{if } x \text{ belongs to an involutive interval in } A \end{cases}$$

where x^+ is the successor of x in A. Take into account that by the symmetry of any negation over a chain (see annex of [13]) if r is in an involutive interval, then $n(r)$ have to be in an involutive interval and if x^+ is the successor of x, then $n(x)$ is the successor of $n(x^+)$. An easy computation shows that A' with the WNM operation and its residuum associated to n' is a chain of $\mathrm{L}_\otimes$ and, since the order is dense, it is isomorphic to an $\mathrm{L}_\otimes$-chain over $[0,1]^{\mathbb{Q}}$. Moreover the mapping $r \to \langle r, 0\rangle$ is a σ-embedding of $\mathcal{A}$ into the $\mathrm{L}_\otimes$-chain over $[0,1]^{\mathbb{Q}}$. $\square$

Theorem 4.10. *The logics $\mathrm{G}\forall(\mathcal{C})$, $\mathrm{NM}\forall(\mathcal{C})$, $\mathrm{L}_\otimes\forall(\mathcal{C})$ and $\mathrm{L}_\star\forall(\mathcal{C})$ enjoy the* Can$\mathcal{Q}$C.

Proof. The proof for any t-norm $* \in \{*_{\mathrm{G}}, *_{\mathrm{NM}}, \star, \otimes\}$ is a combination of the method followed in Lemma 5.3.1 of [21] (extended in the previous lemma) that gives a σ-embedding of any countable L_*-chain into the L_*-chain over $[0,1]^{\mathbb{Q}}$ and the proof of Theorem 4.7. If a formula is not provable there exist a structure over a countable chain $\mathcal{A}$ such that the value of the formula is not $\overline{1}^{\mathcal{A}}$. By the embedding there exists a structure over $[0,1]^{\mathbb{Q}}$ and an evaluation of the variables such that the value of the formula is less than 1. Then, by following the same reasoning as in the proof of Theorem 4.7, the proof is completed. $\square$

Again, the rational completeness properties are completely solved since in the remaining cases the properties fail because they were already false in the propositional level. See all the obtained results in Table 9.

Logic $\mathrm{L}_*\forall(\mathcal{C})$	$\mathcal{Q}$C, FS$\mathcal{Q}$C, S$\mathcal{Q}$C	Can$\mathcal{Q}$C	CanFS$\mathcal{Q}$C
$* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$ of type I	No	No	No
$* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$ of type II	Yes	No	No
$* = *_{\mathrm{G}}, *_{\mathrm{NM}}, \otimes, \star$	Yes	Yes	No
Other $* \in$ **WNM-fin**	Yes	No	No

Table 9: Rational completeness properties for first-order t-norm based logics with truth-constants.

4.3. The case of evaluated formulae

In this section we restrict the completeness properties of our first-order logics to (positively) evaluated formulae in the hope of improving the completeness results we have obtained in general. We denote by $\mathcal{R}\mathrm{C}_{ev}$, $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$, $\mathrm{S}\mathcal{R}\mathrm{C}_{ev}$, $\mathcal{Q}\mathrm{C}_{ev}$, $\mathrm{FS}\mathcal{Q}\mathrm{C}_{ev}$, $\mathrm{S}\mathcal{Q}\mathrm{C}_{ev}$ the restriction of the properties we have been studying in the previous section to evaluated formulae (in the case of continuous t-norm based logics), or to positively evaluated formulae (in the case of WNM-t-norm based logics).[20] These completeness properties are straightforwardly refuted in many cases. Namely, for each $* \in \mathbf{CONT\text{-}fin} \setminus \{*_{\mathrm{G}}\}$ there is a constant-free formula φ such that $\nvdash_{\mathrm{L}_*\forall} \varphi$ and $\models_{[0,1]_*} \varphi$, and hence, since φ is equivalent to the evaluated formula $\overline{1} \to \varphi$ and $\mathrm{L}_*\forall(\mathcal{C})$ is a conservative expansion of $\mathrm{L}_*\forall$, we also have a counterexample to the $\mathcal{R}\mathrm{C}_{ev}$ of $\mathrm{L}_*\forall(\mathcal{C})$. Observe now that the same formula φ is a tautology of the rational algebra $[0,1]_*^{\mathbb{Q}}$ and hence, considering again $\overline{1} \to \varphi$ we obtain a counterexample to the $\mathrm{Can}\mathcal{Q}\mathrm{C}_{ev}$ of $\mathrm{L}_*\forall(\mathcal{C})$.

In addition, the completeness properties for (positively) evaluated formulae are also refuted in those cases where they already fail at the propositional level (and hence also including the failure of $\mathrm{CanS}\mathcal{R}\mathrm{C}$ and $\mathrm{CanS}\mathcal{Q}\mathrm{C}$ for those cases mentioned in the last paragraph of Section 2.3).

There are, nonetheless, several positive results. It is obvious that the $\mathrm{S}\mathcal{Q}\mathrm{C}_{ev}$ holds for all our logics (except for those based on type I continuous t-norms, because it already failed without truth-constants) since its unrestricted version has been already shown to be true, and the same happens for the $\mathrm{S}\mathcal{R}\mathrm{C}_{ev}$ in WNM t-norm based logics. Regarding canonical completeness properties, the only cases that remain to be checked are those corresponding to the logics $\mathrm{G}\forall(\mathcal{C})$, $\mathrm{NM}\forall(\mathcal{C})$, $\mathrm{L}_\otimes\forall(\mathcal{C})$ and $\mathrm{L}_\star\forall(\mathcal{C})$. In the rest of this section we show that $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$ and $\mathrm{CanFS}\mathcal{Q}\mathrm{C}_{ev}$ always hold for these logics, while we provide only some partial (positive) results in the case of $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ and $\mathrm{CanS}\mathcal{Q}\mathrm{C}_{ev}$.

Theorem 4.11. *The logics* $\mathrm{G}\forall(\mathcal{C})$*,* $\mathrm{NM}\forall(\mathcal{C})$*,* $\mathrm{L}_\otimes\forall(\mathcal{C})$ *and* $\mathrm{L}_\star\forall(\mathcal{C})$ *enjoy the* $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$ *for positively evaluated formulae.*

Proof. Let $*$ denote any of these four t-norms. We have to show that for every formulae $\varphi_1, \ldots, \varphi_k, \psi$ in the language of $\mathrm{L}_*\forall$ and positive constants $\overline{r_1}, \ldots, \overline{r_k}, \overline{s}$:

$$\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \vdash_{\mathrm{L}_*\forall(\mathcal{C})} \overline{s} \to \psi \text{ if, and only if, } \{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \models_{[0,1]_{\mathrm{L}_*(\mathcal{C})}} \overline{s} \to \psi$$

By the deduction theorem and the canonical standard completeness for $\mathrm{L}_*\forall(\mathcal{C})$, a finite deduction of type $\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \vdash_{\mathrm{L}_*\forall(\mathcal{C})} \overline{s} \to \psi$ is equivalent to $\models_{[0,1]_{\mathrm{L}_*(\mathcal{C})}} \&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)^2 \to (\overline{s} \to \psi)$.[21] Thus what we need to prove is the semantical version of the deduction theorem for $\mathrm{L}_*\forall(\mathcal{C})$, i.e. the equivalence between $\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \models_{[0,1]_{\mathrm{L}_*(\mathcal{C})}} \overline{s} \to \psi$ and $\models_{[0,1]_{\mathrm{L}_*(\mathcal{C})}} \&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)^2 \to (\overline{s} \to \psi)$.

[20] Notice that the convention is sound since in the case of Gödel t-norm (the only t-norm which is continuous and WNM at the same time) the completeness properties restricted to evaluated formulae or to positively evaluated formulae are equivalent. The reason is that 0 is the only negative element.

[21] In fact for $\mathrm{G}\forall(\mathcal{C})$ the exponent 2 is redundant since the classical deduction theorem is valid.

In one direction the implication is obvious. For the other one we do it by contraposition. If $\not\models_{[0,1]_{\text{Ł}_*(\mathcal{C})}} \&_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)^2 \to (\overline{s} \to \psi)$, there must exist an Ł$_*\forall(\mathcal{C})$-structure $\langle \mathbf{M}, [0,1]_{\text{Ł}_*(\mathcal{C})}\rangle$ and an evaluation e such that

$$\|\&_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)^2 \to (\overline{s} \to \psi)\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} < 1$$

We have to build an Ł$_*\forall(\mathcal{C})$-structure $\langle \mathbf{M'}, [0,1]_{\text{Ł}_*(\mathcal{C})}\rangle$ and an evaluation of variables e' such that $\|\&_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)^2\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M'},e'} = 1$ and $\|\overline{s} \to \psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M'},e'} < 1$. Observe first that the previous inequality implies that $\|\&_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)^2\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} > \|\overline{s} \to \psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$ and thus $\|\&_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)^2\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} > 0$ which implies that for each $i = 1,\dots,k$, then $\|(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$ is a positive element and thus it is idempotent. Moreover $\|\overline{s} \to \psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} = \neg s \vee \|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} < 1$. We follow the proof by cases:
(i) If $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} = 1$ for every $i \in \{1,\dots,k\}$, then we just take $\mathbf{M'} = \mathbf{M}$ and $e' = e$.
(ii) Suppose there exists a non-empty set of indexes $J \subseteq \{1,\dots,k\}$ such that for all $j \in J$, $\|\overline{r_j} \to \varphi_j\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} = \neg r_j \vee \|\varphi_j\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} = \|\varphi_j\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} < 1$ (because r_j and $\|\overline{r_j} \to \varphi_j\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$ are positive). Thus, for every $j \in J$, $\|\varphi_j\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} > \neg s \vee \|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$. Let $b = \min\{\|\varphi_j\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} \mid j \in J\}$. Assume that $\neg s \vee \|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$ is also positive and take $c = \|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$. Define f as the endomorphism of $[0,1]_{\text{Ł}_*(\mathcal{C})}$ given by $f(x) = 1$ for every $x \geq b$, an ordered bijection between $(\neg b, b)$ and $(0,1)$ such that $f(x) = x$ for $x \in [\neg c, c]$, and $f(x) = 0$ for every $x \leq \neg b$, preserving existing suprema and infima. Now we consider a structure $\mathbf{M'}$ over the same domain as $\mathbf{M}$ with the same interpretation of functional symbols, with the same evaluation of variables $e' = e$, and we will just change the interpretation of the predicate symbols. Indeed, for every n-ary predicate P and arbitrary elements of the domain $m_1,\dots,m_n$, we define $P_{\mathbf{M'}}(m_1,\dots,m_n) = f(P_{\mathbf{M}}(m_1,\dots,m_n))$. Then, since f is a homomorphism that preserves existing suprema and infima, it is obvious that for every Ł$_*\forall$-formula φ we have $\|\varphi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M'},e'} = f(\|\varphi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e})$. An easy computation shows that $\|\&_{i=1,\dots,k}(\overline{r_i} \to \varphi_i)^2\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M'},e'} = 1$ (observe that $\|\overline{r_i} \to \varphi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e} = 1$ for all $i \notin J$ by assumption, and for all $i \in J$ we have $\|\varphi_i\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M'},e'} = 1$), while $\|\overline{s} \to \psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M'},e'} < 1$ (since the value of ψ has not changed). Finally, if $\neg s \vee \|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$ is negative, $\|\psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M},e}$ must be also negative. Then we define the same f without the restriction on $[\neg c, c]$ and it keeps ψ at a negative value and hence under s, so we again have $\|\overline{s} \to \psi\|^{[0,1]_{\text{Ł}_*(\mathcal{C})}}_{\mathbf{M'},e'} < 1$. $\square$

Finally, this theorem can be extended to obtain the CanFS$\mathcal{Q}$C$_{ev}$:

Theorem 4.12. *The logics* G$\forall(\mathcal{C})$*,* NM$\forall(\mathcal{C})$*,* Ł$_\otimes\forall(\mathcal{C})$ *and* Ł$_\star\forall(\mathcal{C})$ *enjoy the* CanFS$\mathcal{Q}$C$_{ev}$*.*

Proof. The proof is analogous to the proof of the previous theorem but we need an additional step based on the technique of [[21], Lemma 5.3.1] extended in Lemma 4.9. After applying the deduction theorem we assume that the resulting formula is not provable, which implies the existence of a structure over some countable chain $\mathcal{A}$ such that the value of the formula is not $\overline{1}^{\mathcal{A}}$. By the result of the mentioned lemma, there exists a structure over a rational chain $\mathcal{B}$ such that the value of the formula is not $\overline{1}^{\mathcal{B}}$. The rest of the proof is then analogous to the previous one. $\square$

Regarding the properties of CanS$\mathcal{R}$C$_{ev}$ and CanS$\mathcal{Q}$C$_{ev}$, as already mentioned above, it remains to be checked the cases of logics G$\forall(\mathcal{C})$, NM$\forall(\mathcal{C})$, $\mathrm{L}_{\otimes}\forall(\mathcal{C})$ and $\mathrm{L}_{\star}\forall(\mathcal{C})$ when the algebra of truth-constants $\mathcal{C}$ has no positive sup-accessible points, i.e. for each $r \in C$ there exists an open interval $(r-\epsilon, r)$ containing no element of C (otherwise CanS$\mathcal{R}$C$_{ev}$ and CanS$\mathcal{Q}$C$_{ev}$ already fail in the propositional case). Two paradigmatic particular examples of algebras of truth-constants satisfying this condition, addressed in the next theorem for G$\forall(\mathcal{C})$, are when $\mathcal{C}$ is finite or it is a strictly decreasing sequence tending to 0.

Theorem 4.13. *Let $\mathcal{C}$ be such that $C \setminus \{0\} = \{t_n \mid n \in \mathbb{N}\}$ where $\langle t_n \rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence with limit* 0*. Then the logic* G$\forall(\mathcal{C})$ *enjoys* CanS$\mathcal{R}$C$_{ev}$ *and* CanS$\mathcal{Q}$C$_{ev}$*.*

Proof. We have to show that for every set of formulae $\{\varphi_i \mid i \in I\} \cup \{\psi\}$ in the language of G$\forall$ and positive constants $\{\overline{r_i} \mid i \in I\} \cup \{\overline{s}\}$:

$\{\overline{r_i} \to \varphi_i \mid i \in I\} \vdash_{\mathrm{G}\forall(\mathcal{C})} \overline{s} \to \psi$ if, and only if, $\{\overline{r_i} \to \varphi_i \mid i \in I\} \models_{[0,1]_{\mathrm{G}(\mathcal{C})}} \overline{s} \to \psi$

In one direction the implication is obvious. For the other one we do it by contraposition. If $\{\overline{r_i} \to \varphi_i \mid i \in I\} \nvdash_{\mathrm{G}\forall(\mathcal{C})} \overline{s} \to \psi$ there must exist a countable G$\forall(\mathcal{C})$-structure $\langle \mathbf{M}, \mathcal{A}\rangle$, and an evaluation e over $\mathcal{A}$ such that $\|\overline{r_i} \to \varphi_i\|^{\mathcal{A}}_{\mathbf{M},e} = \overline{1}^{\mathcal{A}}$ for all $i \in I$ and $\|\overline{s} \to \psi\|^{\mathcal{A}}_{\mathbf{M},e} < \overline{1}^{\mathcal{A}}$. We have to build a G$\forall(\mathcal{C})$-structure $\langle \mathbf{M}', [0,1]_{\mathrm{G}(\mathcal{C})}\rangle$ and an evaluation of variables e' such that $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} = 1$ for all $i \in I$ and $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e'} < 1$. The proof will consist in taking the same domain of individuals $\mathbf{M}' = \mathbf{M}$, the same evaluation $e' = e$, and defining for every n-ary predicate P and arbitrary elements of the domain $m_1, \ldots, m_n$, $P_{\mathbf{M}'}(m_1, \ldots, m_n) = f(P_{\mathbf{M}}(m_1, \ldots, m_n))$, where f is a σ-embedding of $\mathcal{A}$ as G-algebra into $[0,1]_{\mathrm{G}}$ satisfying:

(i) $f(\overline{r_i}^{\mathcal{A}}) \geq r_i$ for all $i \in I$

(ii) $f(\|\psi\|^{\mathcal{A}}_{\mathbf{M},e}) < s$

Notice that such a mapping f solves our problem since being a σ-embedding it holds that, for any G$\forall$-formula φ, $\|\varphi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} = f(\|\varphi\|^{\mathcal{A}}_{\mathbf{M},e})$, and hence (i) yields to $\|\varphi_i\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} \geq r_i$ for all $i \in I$, and (ii) gives us $\|\psi\|^{[0,1]_{\mathrm{G}(\mathcal{C})}}_{\mathbf{M}',e} < s$. Therefore, the rest of the proof is devoted to build the σ-embedding f.

Since $\mathcal{A}$ is a G$(\mathcal{C})$-chain, it defines a filter $F_{\mathcal{A}} = \{r \in C \mid \overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}\}$ of $\mathcal{C}$ such that $\overline{p}^{\mathcal{A}} < \overline{q}^{\mathcal{A}}$ for any $p, q \notin F_{\mathcal{A}}$ and $p < q$. We consider the following cases:

C1: $s \in F_{\mathcal{A}}$ and $\inf_n \overline{t_n}^{\mathcal{A}} = \overline{0}^{\mathcal{A}}$

Let t_m be the greatest element of $C \setminus F_{\mathcal{A}}$. We split the construction of f in two parts. The restriction of f to the interval $[\overline{0}^{\mathcal{A}}, \overline{t_m}^{\mathcal{A}}]$ is taken as any σ-embedding into $[0, t_m]$ such that $f(\overline{t_k}^{\mathcal{A}}) = t_k$ for each $k \geq m$. On the other hand, if $\|\psi\|^{\mathcal{A}}_{\mathbf{M},e} \leq \overline{t_m}^{\mathcal{A}}$, the restriction of f to $[\overline{t_m}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}]$ is taken as any σ-embedding into $[t_m, 1]$. Otherwise, let $\delta \in [0,1]$ be such that $\delta < s$ and $[\delta, s) \cap C = \emptyset$. Then the restriction of f to $[\overline{t_m}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}]$ is taken as any σ-embedding into $[t_m, 1]$ such that $f(\|\psi\|^{\mathcal{A}}_{\mathbf{M},e}) = \delta$.

C2: $s \in F_{\mathcal{A}}$ and there exists $\overline{0}^{\mathcal{A}} < \alpha \in A$ such that $\overline{t_n}^{\mathcal{A}} > \alpha$ for each n.

The construction of the restriction of f to $[\overline{t_m}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}]$ is exactly the same as in C1. Now, the restriction of f to $[\overline{0}^{\mathcal{A}}, \overline{t_m}^{\mathcal{A}}]$ is defined as any σ-embedding into $[0, t_m]$ such that $f(\alpha) = t_{m-1}$. In this case, it holds that $f(\overline{t_k}^{\mathcal{A}}) \geq t_k$ for $k \geq m$.

C3: $s \notin F_{\mathcal{A}}$

In this case, the restriction of f to $[\overline{s}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}]$ can be taken as any σ-embedding into $[s, 1]$ such that $f(\overline{t_i}^{\mathcal{A}}) = t_i$ for all $t_i \notin F_{\mathcal{A}}$ and $t_i \geq s$ (there are finitely-many). The restriction of f to $[\overline{0}^{\mathcal{A}}, \overline{s}^{\mathcal{A}}]$ depends on whether $\inf_n \overline{t_n}^{\mathcal{A}} = \overline{0}^{\mathcal{A}}$ or there exists $\overline{0}^{\mathcal{A}} < \alpha \in A$ such that $\overline{t_n}^{\mathcal{A}} > \alpha$ for each n. Taking t_m as s, then the restriction of f to $[\overline{0}^{\mathcal{A}}, \overline{s}^{\mathcal{A}}]$ in the former case is defined as in C1 and in the latter case as in C2.

Finally, let us notice that being $\mathcal{A}$ a countable G-chain, the above constructions easily generalize when we replace the standard G-algebra $[0,1]_{\mathrm{G}}$ by the G-algebra $[0,1]_{\mathrm{G}}^{\mathbb{Q}}$ over the rational unit interval. □

The proof the theorem above can be easily adapted to the cases considered in the next corollary, and thus we omit the proofs.

Corollary 4.14. *The logic* G$\forall(\mathcal{C})$ *also enjoys the* CanS$\mathcal{R}$C *and* CanS$\mathcal{Q}$C *restricted to positively evaluated formulae when:*

- $\mathcal{C}$ *is finite*
- $C \setminus \{0\} = \{t_n \mid n \in \mathbb{N}\}$*, where* $\langle t_n \rangle_{n \in \mathbb{N}}$ *is a strictly decreasing sequence without limit in* C
- $C \setminus \{0\} = \{t_n \mid n \in \mathbb{N}\} \cup \{\alpha\}$*, where* $\langle t_n \rangle_{n \in \mathbb{N}}$ *is a strictly decreasing sequence with limit* $\alpha \in C$

As expected, one can also straightforwardly extend the above results for G$\forall(\mathcal{C})$ to the other three logics NM$\forall(\mathcal{C})$, Ł$_\otimes\forall(\mathcal{C})$ and Ł$_\star\forall(\mathcal{C})$ when instead of $C \setminus \{0\}$ one considers in the previous corollary the set of positive elements of C.

Corollary 4.15. NM$\forall(\mathcal{C})$*,* Ł$_\otimes\forall(\mathcal{C})$ *and* Ł$_\star\forall(\mathcal{C})$ *enjoy the* CanS$\mathcal{R}$C$_{ev}$ *and* CanS$\mathcal{Q}$C$_{ev}$ *when:*

- $\mathcal{C}$ *is finite*
- $C^+ = \{t_n \mid n \in \mathbb{N}\}$*, where* $\langle t_n \rangle_{n \in \mathbb{N}}$ *is a strictly decreasing sequence without limit in* C^+
- $C^+ = \{t_n \mid n \in \mathbb{N}\} \cup \{\alpha\}$*, where* $\langle t_n \rangle_{n \in \mathbb{N}}$ *is a strictly decreasing sequence with limit* $\alpha \in C$

where C^+ *denotes the set of positive elements of* $\mathcal{C}$*, that is* $C^+ = \{r \in C \mid r > \neg r\}$.

However, it remains open the question whether these positive results also hold for the general case of $\mathcal{C}$ having no positive sup-accessible points, and in particular whether the fulfillment or failure of the CanS$\mathcal{R}$C and CanS$\mathcal{Q}$C properties always come simultaneously. In Table 10 we summary the obtained results about canonical completeness properties. We do not include there the results about non-canonical completeness for (positively) evaluated formulae since, as already discussed in the beginning of this section, they turn out to be the same as for arbitrary formulae.

Logic	Can$\mathcal{R}$C$_{ev}$, CanFS$\mathcal{R}$C$_{ev}$ Can$\mathcal{Q}$C$_{ev}$, CanFS$\mathcal{Q}$C$_{ev}$	CanS$\mathcal{R}$C$_{ev}$ CanS$\mathcal{Q}$C$_{ev}$
$\mathrm{L}_*\forall(\mathcal{C})$, $* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$	No	No
$\mathrm{G}\forall(\mathcal{C})$, $\mathrm{NM}\forall(\mathcal{C})$, $\mathrm{L}_\otimes\forall(\mathcal{C})$, $\mathrm{L}_\star\forall(\mathcal{C})$ C^+ has sup-accessible points	Yes	No
$\mathrm{G}\forall(\mathcal{C})$, $\mathrm{NM}\forall(\mathcal{C})$, $\mathrm{L}_\otimes\forall(\mathcal{C})$, $\mathrm{L}_\star\forall(\mathcal{C})$ C^+ has no sup-accessible points	Yes	?
Other $\mathrm{L}_*\forall(\mathcal{C})$, $* \in$ **WNM-fin**	No	No

Table 10: Canonical standard and rational completeness properties for first-order t-norm based logics with truth-constants restricted to (positively) evaluated formulae.

Acknowledgements: The authors acknowledge partial support of the Spanish project MULOG2 (TIN2007-68005-C04), including FEDER funds of the European Union, and by the Generalitat de Catalunya grant 2005-SGR-00093. The third author also acknowledges partial support from the grant 2006-BP-A-10043 of the Departament d'Educació i Universitats of the Generalitat de Catalunya. We are indebted with professor Petr Hájek for his helpful remarks about conservative expansions, and with the anonymous referees whose reports helped in improving the paper a lot.

References

[1] P. Aglianò, I. M. A. Ferreirim and F. Montagna. Basic Hoops: an Algebraic Study of Continuous t -norms, *Studia Logica* 87 (2007) 73–98.

[2] P. Aglianò and F. Montagna. Varieties of BL-algebras I: general properties, *Journal of Pure and Applied Algebra* 181 (2003) 105–129.

[3] W. J. Blok and D. Pigozzi. Algebraizable logics, *Memoirs of the American Mathematical Society* 396, vol 77, 1989.

[4] C. C. Chang. Algebraic analysis of many-valued logics. *Transactions American Mathematical Society* 88 (1958) 456–490.

[5] R. Cignoli, F. Esteva and L. Godo. On Łukasiewicz logic with truth constants. In *Theoretical Advances and Applications of Fuzzy Logic and Soft Computing* ASC 42, O. Castillo et al. (eds.), Springer, pp. 869- 875, 2007.

[6] R. Cignoli, F. Esteva, L. Godo and A. Torrens. Basic Fuzzy Logic is the logic of continuous t-norms and their residua, *Soft Computing* 4 (2000) 106–112.

[7] P. Cintula. *From Fuzzy Logic to Fuzzy Mathematics*, Ph.D. thesis, University of Prague, 2004.

[8] P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna and C. Noguera. Distinguished algebraic semantics for t-norm based fuzzy logics: methods and algebraic equivalencies, To appear in *Annals of Pure and Applied Logic*, 2008.

[9] A. Di Nola, F. Esteva, L. Godo and F. Montagna. Varieties of BL-algebras. *Soft Computing* 9 (2005) 875–888.

[10] M. Dummett. A propositional calculus with denumerable matrix, *The Journal of Symbolic Logic* 24 (1959) 97–106.

[11] F. Esteva, J. Gispert, L. Godo and F. Montagna. On the standard and Rational Completeness of some axiomatic extensions of Monoidal t-norm Based Logic, *Studia Logica* 71 (2002) 199–226.

[12] F. Esteva, J. Gispert, L. Godo and C. Noguera. Adding truth-constants to logics of a continuous t-norm: axiomatization and completeness results, *Fuzzy Sets and Systems* 158 (2007) 597–618.

[13] F. Esteva and L. Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms, *Fuzzy Sets and Systems*, 124 (2001) 271–288.

[14] F. Esteva, L. Godo and F. Montagna. Equational characterization of the subvarieties of BL generated by t-norm algebras. *Studia Logica* 76, 161-200.

[15] F. Esteva, L. Godo and C. Noguera. On rational weak nilpotent minimum logics, *Journal of Multiple-Valued Logic and Soft Computing* 12, Number 1-2, pp. 9-32, 2006.

[16] F. Esteva, L. Godo and C. Noguera. On expansions of t-norm based logics with truth-constants, To appear in the book *Fuzzy Logics and Related Structures* (S. Gottwald, P. Hájek, U. Höhle and E.P. Klement eds.), 2008.

[17] F. Esteva, L. Godo and C. Noguera. On completeness results for predicate Łukasiewicz, Product, Gödel, and Nilpotent Minimum logics expanded with truth-constants, *Mathware & Soft Computing*, 14 (2007) 233–246.

[18] F. Esteva, L. Godo and C. Noguera. Expanding the (propositional) logic of a t-norm with truth-constants: completeness results for rational semantics, Submitted, 2008.

[19] T. Flaminio. Strong Non-Standard Completeness for Fuzzy Logics, *Soft Computing* 12 (2007) 321–333.

[20] N. Galatos, P. Jipsen, T. Kowalski and H. Ono. *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics, Elsevier, 2007.

[21] P. Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic-Studia Logica Library.* Dordrecht/Boston/London, 1998.

[22] P. Hájek. Basic fuzzy logic and BL-algebras, *Soft Computing* 2 (1998) 124?128.

[23] P. Hájek. What is Mathematical Fuzzy Logic, *Fuzzy Sets and Systems* 157 (2006) 597–603.

[24] P. Hájek and P. Cintula. Triangular Norm Predicate Fuzzy Logics, To appear in the book *Fuzzy Logics and Related Structures* (S. Gottwald, P. Hájek, U. Höhle and E.P. Klement eds.), 2007.

[25] P. Hájek and P. Cintula. On theories and models in fuzzy predicate logics, *The Journal of Symbolic Logic* 71 (2006) 863–880.

[26] P. Hájek, L. Godo and F. Esteva. A complete many-valued logic with product-conjunction, *Archive for Mathematical Logic* 35 (1996) 191–208.

[27] P. Hájek, J. Paris and J. Shepherdson. Rational Pavelka predicate logic is a conservative extension of Łukasiewicz predicate logic, *Journal of Symbolic Logic*, 65 (2000) 669–682.

[28] U. Höhle. Commutative, residuated l-monoids. In Höhle, U. and Klement. E.P. eds., *Non-Classical Logics and Their Applications to Fuzzy Subsets*, Kluwer Acad. Publ., Dordrecht (1995) 55–106.

[29] S. Jenei and F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70:183–192, 2002.

[30] E.P. Klement, R. Mesiar and E. Pap. *Triangular norms*, Kluwer Dordrecht/Boston/London, 1998.

[31] J. Łukasiewicz and A. Tarski. Untersuchungen ber den Aussagenkalkl. *Comptes Rendus de la Societé des Sciences et des Lettres de Varsovie*, cl. iii 23 (1930) 1–21.

[32] C. Noguera, F. Esteva and J. Gispert. On triangular norm based axiomatic extensions of the Weak Nilpotent Minimum logic, *Mathematical Logic Quarterly* 54 (2008) 387–409.

[33] V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic. Part I and Part II. *Kybernetika* 26 (1990) 47–66, and 134–154.

[34] J. Pavelka. On Fuzzy Logic I, II, III. *Zeitschrift fur Math. Logik und Grundlagen der Math.* 25 (1979) 45–52, 119–134, 447–464.

[35] N. Preining. *Complete Recursive Axiomatizability of Gödel Logics*, Vienna University of Technology, Ph.D. thesis, Austria, 2003.

[36] P. Savický, R. Cignoli, F. Esteva, L. Godo and C. Noguera. On product logic with truth-constants, *Journal of Logic and Computation* 16 (2006) 205–225.

[37] L. A. Zadeh. Fuzzy sets, *Information Control* 8 (1965) 338–353.