Lexicographic Refinements in the Context of Possibilistic Decision Theory

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Abstract

In Possibilistic Decision Theory (PDT), decisions are ranked by a pessimistic and an optimistic qualitative criteria. The preference relations induced by these criteria have been axiomatized by corresponding sets of rationality postulates, both à la Neumann-Morgenstern and à la Savage. In this paper we first address a particular issue regarding the axiomatic systems of PDT à la von Neumann and Morgenstern. Namely, we show how to adapt the axiomatic systems for the pessimistic and optimistic criteria when finiteness assumptions in the original model are dropped. Second, we show that a recent axiomatic approach by Giang and Shenoy using binary utilities can be captured by preference relations defined as lexicographic refinements of the above two criteria. We also provide an axiomatic characterization of these lexicographic refinements.

Keywords: decision theory, possibility theory

1 Introduction: the basic framework of possibilistic decision theory

In [3], an axiomatic qualitative counterpart of Von Neumann and Morgenstern's Expected Utility Theory was proposed by Dubois and Prade where uncertainty is modeled by possibility distributions on the set of states or situations instead of probability distributions.

In this framework, there is a (finite) set S of possible states and the uncertainty about what is the actual state of the world t is represented by a normalized possibility distribution $\pi_0 : S \to V$ with values on a finite linearly ordered (l.o.) uncertainty scale $(V, \leq, 0_V, 1_V)$. Decisions are modeled as mappings $d : S \to X$ from situations to a finite set of possible consequences (or prizes) X, where $d(s)$ denotes the consequence obtained by decision d when the state s occurs. Each decision d induces a (normalized) possibility distribution on consequences $\pi_d : X \to V$ defined as

$$\pi_d(x) = \max\{\pi_0(s) \mid s \in S, d(s) = x\},$$

A normalized possibility distribution on X is also called a possibilistic lottery. We shall also use the expression $[\pi(x_1)/x_1, \pi(x_2)/x_2 ... \pi(x_n)/x_n]$ to denote a lottery π with the convention that impossible consequences (consequences x with $\pi(x) = 0$) are omitted from the list. The set of lotteries will be denoted by $\Pi(X)$. Notice that $\Pi(X)$ is closed under the operation of standard possibilistic mixture defined as follows. Given n possibility distributions $\pi_1, ..., \pi_n$ and n values from V, $\lambda_1, ..., \lambda_n$ such that $\max(\lambda_1, ..., \lambda_n) = 1_V$, then $[\lambda_1/\pi_1, ..., \lambda_n/\pi_n]$ is the (normalized) possibility distribution defined as

$$[\lambda_1/\pi_1, ..., \lambda_n/\pi_n](x) = \max_{i=1,...,n} \min(\lambda_i, \pi_i(x)) \quad (1)$$

This possibilistic mixture construct allows to express not only simple lotteries but also compound lotteries. Notice that each consequence $x \in X$ can be viewed also as a lottery π_x where $\pi_x(x) = 1_V$ and $\pi_x(y) = 0_V$ for $y \neq x$. When no confusion exists, we will use x to also denote π_x. With this convention, we can consider X as included in $\Pi(X)$. Similarly, we shall also denote by A both a subset $A \subseteq X$ and the possibility distribution on X such that $\pi(z) = 1_V$ if $z \in A$ and 0_V otherwise.

In the framework of possibilistic decision theory à la Von Neumann and Morgenstern, from the decision maker point of view, a decision d is equivalently expressed by the induced lottery π_d, hence, ranking decisions amounts to rank lotteries. Therefore, the main concern will be on the definition of preference relations in the set of possibility distributions on consequences (i.e. possibilistic lotteries) and the axiomatic systems of rationality postulates which characterize them.

Given a utility function $u : X \to U$ representing the decision maker's preferences on consequences, where $(U, \leq_U, 0_U, 1_U)$ is a finite linearly ordered utility scale (a consequence x is preferred to x' whenever $u(x) >_U u(x')$), the basic possibilistic model introduced by Dubois and Prade [3] propose to define two preference relations among lotteries according to an optimistic or a pessimistic criterion represented by Sugeno-like integrals which generalize the well-known Wald's maximin and maximax criteria. Namely,

$$d \preceq^- d' \text{ iff } QU^-(\pi_d \mid u) \leq_Q U^-(\pi'_d \mid u),$$
$$d \preceq^+ d' \text{ iff } QU^+(\pi_d \mid u) \leq_Q U^+(\pi'_d \mid u),$$

where

$$QU^-(\pi_d \mid u) = \min_{x \in X} \max(n(\pi_d(x)), u(x)),$$
$$QU^+(\pi_d \mid u) = \max_{x \in X} \min(h(\pi_d(x)), u(x))$$

with $n = n_U \circ h$, n_U being the reversing involution on U and $h : V \to U$ is an *onto* order-preserving mapping linking the uncertainty and utility scales.

Since $\mathcal{U}^-(d)$ evaluates to what extent all possible consequences of d are good, $\mathcal{U}^-$ models a pessimistic criterion, while $\mathcal{U}^+(d)$ represents an optimistic behavior by evaluating to what extent at least one possible consequence is good. Notice that both criteria are qualitative in the sense that they only involve the minimum, the maximum and an order-reversing operators.

In [3, 1], the authors study two axiomatic systems in the style of von Neumann and Morgenstern (VNM). Namely, the first set of rationality postulates $\mathcal{S}_P$ for a preference ordering $\sqsubseteq$ on lotteries is the following one:

A1 (*structure*): $\sqsubseteq$ is a total pre-order (i.e. $\sqsubseteq$ is reflexive, transitive, total.).

A2$^-$ (*uncertainty aversion*): $\pi \leq \pi' \Rightarrow \pi' \sqsubseteq \pi$.

A3 (*independence*):
$\pi_1 \sim \pi_2 \Rightarrow [\alpha/\pi_1, \beta/\pi] \sim [\alpha/\pi_2, \beta/\pi]$.

A4$^-$ (*continuity*):
$\forall \pi \in \Pi(X)\ \exists \lambda \in V$ s.t. $\pi \sim [1/\overline{x}, \lambda/\underline{x}]$.

A4$^-$ (*continuity*):
$\forall \pi \in \Pi(X)\ \exists \lambda \in V$ s.t. $\pi \sim [1/\overline{x}, \lambda/\underline{x}]$,

where $\pi_1 \sim \pi_2$ means $\pi_1 \sqsubseteq \pi_2$ and $\pi_2 \sqsubseteq \pi_1$, and $\overline{x}$ and $\underline{x}$ denote respectively a best and a worst consequence according to $\sqsubseteq$. The second system $\mathcal{S}_O$ consists of **A1**, **A3** and

A2$^+$ (*uncertainty attraction*): $\pi \leq \pi' \Rightarrow \pi \sqsubseteq \pi'$.

A4$^+$ (*continuity*):
$\forall \pi \in \Pi(X)\ \exists \lambda \in V$ s.t. $\pi \sim [\lambda/\overline{x}, 1/\underline{x}]$.

It is shown that a preference relation satisfying the first axiom system $\mathcal{S}_P$ can be represented by a pessimistic utility function QU^-, and preference relations obeying the second system $\mathcal{S}_O$ can be represented by a optimistic utility function QU^+. In [4], these two utility functionals are justified by axiom systems in the style of Savage. The difference is that in the VNM approach, a possibility function on states is assumed to be given, whereas in the latter approach such a function is deduced from a preference relation on the set of actions.

These axiom systems were extended in [1] to cope with generalized possibilistic mixtures operations $[\lambda_1/\pi_1, ..., \lambda_n/\pi_n]_\star$ induced by a t-norm-like operation $\star$ on V:

$$[\lambda_1/\pi_1, ..., \lambda_n/\pi_n]_\star(x) = \max_{i=1,...,n} \lambda_i \star \pi_i(x) \qquad (2)$$

Then, if one replaces the original mixtures (1) by these ones in the above axiomatic systems, call them $\mathcal{S}_P^\star$ and $\mathcal{S}_O^\star$, it can be shown that the preference relations obeying $\mathcal{S}_P^\star$ and $\mathcal{S}_O^\star$ can still be represented respectively by the following pessimistic and optimistic utilities:

$$QU_\star^-(\pi_d \mid u) = \min_{x \in X} n(\pi_d(x) \star \lambda_x), \qquad (3)$$
$$QU_\star^+(\pi_d \mid u) = \max_{x \in X} h(\pi_d(x) \star \mu_x), \qquad (4)$$

with $n(\lambda_x) = u(x) = h(\mu_x)$, and n is as above.

In this paper we first show that a recent axiomatic approach by Giang and Shenoy [6] using special bi-valued utilities can be captured by preference relations defined as a lexicographic refinement of the above two pessimistic and optimistic criteria. Then we show how to adapt the axiomatic systems for the pessimistic and optimistic criteria when we abandon the assumption that the uncertainty and utility scales U and V are finite and we take the real unit interval [0, 1] as a common scale. Finally, using this new framework, we also provide a new axiomatic characterization of those lexicographic refinements, improving a first approach described in [2]. Notice that refinements of possibilistic criteria by means of lexicographic orderings (leximax, leximin) have been also used in [5], not on the criteria themselves but to compare the representative vectors $((\pi_d(x_1), u(x_1)), ..., (\pi(x_n), u(x_n)))$ induced by each decision d.

2 Giang-Shenoy's utility systems

Remaining in the same possibilistic framework, in [6] Giang and Shenoy propose the next system $\mathcal{S}_B$ of four axioms for a preference relation $\preceq$ on lotteries with a min-based mixture operation (1).

B1 (*Total pre-order*): $\preceq$ is reflexive, transitive and complete.

B2 (*Qualitative monotonicity*): for any $\lambda, \mu \in V$ with $\max(\lambda, \mu) = 1$, $[\lambda/\overline{x}, \mu/\underline{x}] \preceq [\lambda/\overline{x}, \mu/\underline{x}]$ if $\begin{cases} (\lambda \le \lambda' \text{ and } \mu = \mu' = 1) \text{ or} \\ (\lambda < 1 \text{ and } \lambda' = 1) \text{ or} \\ (\lambda = \lambda' = 1 \text{ and } \mu \ge \mu') \end{cases}$

B3 (*Substitutability*): $\pi_1 \sim \pi_2 \Rightarrow [\alpha/\pi_1, \mu/\pi] \sim [\alpha/\pi_2, \mu/\pi]$.

B4 (*continuity*): $\forall x \in X, \exists \lambda, \mu \in V$ s.t. $x \sim [\lambda/\overline{x}, \mu/\underline{x}]$.

The authors show that this axiomatic system $\mathcal{S}_B$ is weaker than $\mathcal{S}_P$ and $\mathcal{S}_O$ and that preference relations satisfying axioms B1 through B4 are representable by utility functions PU similar to QU^+ but taking values on a two-dimensional scale, but still linearly ordered. Indeed, given a finite linearly ordered utility scale $(W, \le)$, one can define a corresponding *binary*[1] utility scale $(U_W, \underline{\ll})$, where $U_W = \{(a, b) \mid a, b \in W, \max(a, b) = 1\}$ and the strict part of $\underline{\gg}$ is defined as:

[1] This is the term used by the authors.

$$(a, b) \ll (a', b') \text{ iff } (a < a') \text{ or } (b > b').$$

This gives the following linear ordering:

$$(0, 1) \ll ... \ll (x, 1) \ll ... \ll (1, 1)$$
$$\ll ... \ll (1, x) \ll ... \ll (1, 0),$$

for any $0 < x < 1$ of W. Then, given a pair of order preserving mappings $k_1, k_2 : V \to W$ such that $k_i(0) = 0, k_i(1) = 1$, and a binary utility assessment of consequences $u : X \to U_W$, the utility function $PU : \Pi(X) \to U_W$ is defined as

$$PU(\pi \mid k, u) = \overline{\max}_{x \in X} \overline{\min}(k(\pi(x)), u(x))$$

where $k(v) = (k_1(v), k_2(v))$ for any $v \in V$ and $\overline{\min}$ and $\overline{\max}$ denote the point-wise extension of min and max to $W \times W$. This kind of utility function PU induces a total pre-ordering among possibility distributions

$$\pi \preceq \pi' \text{ if } PU(\pi \mid k, u) \underline{\ll} PU(\pi' \mid k, u)$$

that satisfies the system $\mathcal{S}_B$, and conversely.

It can be shown that such pre-ordering can be seen in fact as a lexicographic ordering in terms of suitable evaluations of the pessimistic and optimistic utilities QU^- and QU^+. This is based on the following trivial observation.

Lemma 1 *Let $a, b, a', b' \in W$ such that $\max(a, b) = 1$, and let $n : W \to W$ the order-reversing involution on W. Then:*

$$(a, b) \underline{\ll} (a', b') \text{ iff } (a, n(b)) \le_{lex} (a', n(b')) \text{ iff }$$
$$(n(b), a) \le_{lex} (n(b'), a')$$

where $\le_{lex}$ is the lexicographic ordering on $W \times W$ induced by the ordering $\le$ on W.

Now, given $u : X \to U_W$, if we consider its projections $u_1, u_2 : X \to U$, i.e. $u(x) = (u_1(x), u_2(x))$ with the condition $\max(u_1(x), u_2(x)) = 1$ for any $x \in X$, then we can express $PU(\pi \mid u, k) = (PU^1(\pi \mid k_1, u_1), PU^2(\pi \mid k_2, u_2))$, where

$$PU^i(\pi \mid k_i, u_i) = \max_{x \in X} \min(k_i(\pi(x)), u_i(x))$$

for $i = 1, 2$. Noticing that $n(PU^2(\pi \mid k_2, u_2)) = QU^-(\pi \mid k_2, u_2^*)$, where $u_2^*(x) = n(u_2(x))$, the next representation it is just a matter of routine checking.

Theorem 1 *The preference ordering induced by $PU(\cdot \mid k, u)$ is the lexicographic refinement of the ordering induced by $QU^+(\cdot \mid k_1, u_1)$ by the ordering induced by $QU^-(\cdot \mid k_2, u_2^*)$ (or viceversa). That is, for any lottery π:*

$$\begin{gathered} PU(\pi \mid k, u) \lll PU(\pi' \mid k, u) \\ \textit{iff} \\ (QU^+(\pi \mid k_1, u_1), QU^-(\pi \mid k_2, u_2^*)) \leq_{lex} \\ (QU^+(\pi' \mid k_1, u_1), QU^-(\pi' \mid k_2, u_2^*)) \\ \textit{iff} \\ (QU^-(\pi \mid k_2, u_2^*), QU^+(\pi \mid k_1, u_1)) \leq_{lex} \\ (QU^-(\pi' \mid k_2, u_2^*), QU^+(\pi' \mid k_1, u_1)) \end{gathered}$$

However, notice that in this representation u_1 and u_2^* are not independent utility assignments. Indeed, since $\max(u_1(x), u_2(x)) = 1$ for all x, then $\min(n(u_1(x)), u_2^*(x)) = 0$, i.e. $u_1(x) < 1$ implies $u_2^*(x) = 0$.

3 Pessimistic and optimistic utilities on [0, 1]

It is not difficult to adapt the pessimistic and optimistic utility axiomatic systems $\mathcal{S}_P^\star$ and $\mathcal{S}_O^\star$ to preference relations defined over $\Pi_{[0,1]}(X)$, the set possibilistic lotteries with $V = [0, 1]$, and mixture operations (2) defined by an arbitrary t-norm operation $\otimes$. In fact, it is enough to introduce a uniqueness condition of the paramaters λ and μ in axioms A4$^-$ and $A4^+$, and moreover the whole framework becomes much notationally simpler. Indeed let us consider the axiomatic systems $\mathcal{S}_{P!}^{\otimes} = \{\mathbf{A1}, \mathbf{A2}^-, \mathbf{A3}, \mathbf{A4!}^-\}$ and $\mathcal{S}_{O!}^{\otimes} = \{\mathbf{A1}, \mathbf{A2}^+, \mathbf{A3}, \mathbf{A4!}^+\}$, where

A4!$^-$: for all $\pi \in \Pi_{[0,1]}(X)$ there exists *a unique* λ such that $\pi \sim [1/\overline{x}, \lambda/\underline{x}]_\otimes$

A4!$^+$: for all $\pi \in \Pi_{[0,1]}(X)$ there exists *a unique* μ such that $\pi \sim [\mu/\overline{x}, 1/\underline{x}]_\otimes$

In the following $\oplus$ will denote the corresponding dual t-conorm of $\otimes$ (i.e. $\oplus(x, y) = 1 - \otimes(1 - x, 1 - y)$).

Theorem 2 *A binary relation $\preceq$ on $\Pi_{[0,1]}(X)$ satisfies the axioms $\mathcal{S}_{P!}^{\otimes}$ iff there exists $u : X \to [0, 1]$ such that, for any $\pi_1, \pi_2 \in \Pi_{[0,1]}(X)$, $\pi_1 \preceq \pi_2$ iff $QU_\otimes^-(\pi_1 \mid u) \leq QU_\otimes^-(\pi_2 \mid u)$, where $QU_\otimes^-(\pi \mid u) = \min_{x \in X} \oplus(1 - \pi(x), u(x))$,*

Proof: 1. For each $x \in X$ there exists a unique λ_x such that $x \sim [1/\overline{x}, \lambda_x/\underline{x}]_\otimes$. Then define $u : X \to [0, 1]$ by

$$u(x) = 1 - \lambda_x.$$

It is clear that $u(\overline{x}) = 1$. Notice that by axiom A4!$^-$ $\underline{x} \sim [1/\overline{x}, \mu/\underline{x}]_\otimes$ for some μ, then by axiom A3, $\underline{x} = [1/\underline{x}, 1/\underline{x}]_\otimes \sim [1/[1/\overline{x}, \mu/\underline{x}]_\otimes, 1/\underline{x}]_\otimes = [1/\overline{x}, 1/\underline{x}]_\otimes$, hence $u(\underline{x}) = 0$.

Notice that, by axiom A4!$^-$, one can check that $[1/\overline{x}, \mu/\underline{x}]_\otimes \sim [1/\overline{x}, \lambda/\underline{x}]_\otimes$ iff $\lambda = \mu$.

We define $QU(\pi) = 1 - \lambda_\pi$, where $\pi \sim [1/\overline{x}, \lambda_\pi/\underline{x}]_\otimes$. So defined, due to axiom A4!$^-$, QU is well defined and represents $\preceq$. We want to prove that $QU = QU_\otimes^-(\cdot \mid u)$:

- It is easy to check that QU and $QU_\otimes^-(\cdot \mid u)$ coincide over the lotteries $[1/\overline{x}, \lambda/\underline{x}]_\otimes$. Moreover, by definition of u, $QU(u) = u(x)$ for all $x \in X$.
- $QU([1/x, \lambda/y]_\otimes) = \min(u(x), \oplus(1-\lambda, u(y)))$. Indeed, A4!$^-$ guarantees there exist α and β such that $x \sim [1/\overline{x}, \alpha/\underline{x}]_\otimes$ and $y \sim [1/\overline{x}, \beta/\underline{x}]_\otimes$. Using A3, we have

  $$\begin{aligned} [1/x, \lambda/y] &\sim [1/[1/\overline{x}, \alpha/\underline{x}], \lambda/[1/\overline{x}, \beta/\underline{x}]] \\ &= [1/\overline{x}, \max(\alpha, \otimes(\lambda, \beta))/\underline{x}]. \end{aligned}$$

  Hence $QU([1/x, \lambda/y]) = 1 - \max(\alpha, \otimes(\lambda, \beta)) = \min(1 - \alpha, \oplus(1 - \lambda, 1 - \beta)) = \min(u(x), \oplus(1 - \lambda, u(y)))$.
- $QU([1/\pi_1, 1/\pi_2]_\otimes) = \min(QU(\pi_1), Q(\pi_2))$. Indeed, there exist α and β such that $[1/\pi_1, 1/\pi_2]_\otimes \sim [1/[1/\overline{x}, \alpha/\underline{x}]_\otimes, 1/[1/\overline{x}, \alpha/\underline{x}]_\otimes]_\otimes = [1/\overline{x}, \max(\alpha, \beta)/\underline{x}]_\otimes$, therefore $QU([1/\pi_1, 1/\pi_2]_\otimes) = 1 - \max(\alpha, \beta) = \min(1 - \alpha, 1 - \beta) = \min(QU(\pi_1), Q(\pi_2))$.
- $QU(\pi) = \min_{x \in X} \oplus(1 - \pi(x), u(x))$. Since π is normalized, let x_j such that $\pi(x_j) = 1$. Without loss of generality assume $j = 1$. Defining $\pi_i = [1/x_1, \pi(x_i)/x_i]_\otimes$ for $i > 1$, then $\pi = \max_{i>1} \pi_i$, hence $QU(\pi) = \min_i(QU(\pi_i) = \min_i \min(u(x_1), \oplus(1 - \pi(x_i), u(x_i))) = \min_{x \in X} \oplus(1 - \pi(x), u(x))$.

This ends the proof. $\square$

In an analogous form, one can also prove that a binary relation $\preceq$ on $\Pi_{[0,1]}(X)$ satisfies the axioms $\mathcal{S}^{\otimes}_{O!}$ iff there exists $u : X \to [0,1]$ such that the utiltiy $QU^{+}_{\otimes}(\cdot \mid u)$, defined as $QU^{+}_{\otimes}(\pi \mid u) = \max_{x\in X} \otimes(\pi(x), u(x))$, represents $\preceq$.

4 Lexicographic Refinements: new postulates

Let $u_1, u_2 : X \to [0,1]$ be a pair of utility assignments such that with $u_1^{-1}(1) \cap u_2^{-1}(1) \neq \emptyset \neq u_1^{-1}(0) \cap u_2^{-1}(0)$ and let $\overline{x}$ and $\underline{x}$ such that $u_1(\overline{x}) = u_2(\overline{x}) = 1$ and $u_1(\underline{x}) = u_2(\underline{x}) = 0$. Furthermore, for a given t-norm $\otimes$, consider the two preference orderings on $\Pi_{[0,1]}(X)$:

$\pi \preceq^{-}_{u_i} \pi'$ if $QU^{-}_{\otimes}(\pi \mid u_i) \leq QU^{-}_{\otimes}(\pi' \mid u_i)$,
$\pi \preceq^{+}_{u_i} \pi'$ if $QU^{+}_{\otimes}(\pi' \mid u_i) \leq QU^{+}_{\otimes}(\pi' \mid u_i)$.

Notation: in the rest of this paper, and for the sake of a simpler notation, we will denote the $\otimes$-mixture operation on $\Pi_{[0,1]}(X)$ simply by [...] and not $[...]_{\otimes}$.

Let $F^{u_1,u_2}_{\otimes} : \Pi_{[0,1]}(X) \to U \times U$ be the binary utility functional defined by

$$F^{u_1,u_2}_{\otimes}(\pi) = (QU^{-}_{\otimes}(\pi \mid u_1), QU^{+}_{\otimes}(\pi \mid u_2)).$$

We can define then on $\Pi_{[0,1]}(X)$ the total preordering $\preceq_{lex}$ induced by $F^{u}_{\otimes}$ and by the lexicographic ordering $\leq_{lex}$ on $U \times U$. Namely,

$$\pi \preceq^{lex}_{u_1,u_2} \pi' \text{ iff}_{DEF}\ F^{u_1,u_2}_{\otimes}(\pi) \leq_{lex} F^{u_1,u_2}_{\otimes}(\pi').$$

In other words, $\pi \preceq^{lex}_{u_1,u_2} \pi'$ if either $\pi \prec^{-}_{u_1} \pi'$ or $[\pi \sim^{-}_{u_1} \pi'$ and $\pi \preceq^{+}_{u_2} \pi'\ ]$. The following properties of $\preceq^{lex}_{u}$ are very interesting.

Proposition 1 *The following properties hold:*
(i) $\pi \preceq^{-}_{u_1} \pi'$ *iff* $[1/\pi, 1/\overline{x}] \preceq^{lex}_{u_1,u_2} [1/\pi', 1/\overline{x}]$.
(ii) $\pi \preceq^{+}_{u_2} \pi'$ *iff* $[1/\pi, 1/\underline{x}] \preceq^{lex}_{u_1,u_2} [1/\pi', 1/\underline{x}]$.
(iii) For all $x, x' \in X$, $x \preceq^{lex}_{u_1,u_2} x'$ *iff* $(u_1(x), u_2(x)) \leq_{lex} (u_1(x'), u_2(x'))$.

In view of these properties, let us consider the system postulates $\mathcal{S}^{\otimes}_{PO} = \{\mathbf{A1}, \mathbf{A3}, \mathbf{L2}, \mathbf{L4!}, \mathbf{L5}_{PO}\}$ for a preference relation $\preceq$ on $\Pi(X)_{[0,1]}$ where

L2: if $\pi \leq \pi'$ then $[1/\pi', 1/\overline{x}] \preceq [1/\pi, 1/\overline{x}]$ and $[1/\pi, 1/\underline{x}] \preceq [1/\pi', 1/\underline{x}]$.

L4!: for all $\pi \in \Pi(X)_{[0,1]}$, there exist *unique* $\lambda, \mu \in [0,1]$ such that $[1/\pi, 1/\overline{x}] \sim [1/\overline{x}, \lambda/\underline{x}]$ and $[1/\pi, 1/\underline{x}] \sim [\mu/\overline{x}, 1/\underline{x}]$,

L5$_{PO}$: $\pi \preceq \pi'$ iff either $[1/\pi, 1/\overline{x}] \prec [1/\pi', 1/\overline{x}]$ or $([1/\pi, 1/\overline{x}] \sim [1/\pi', 1/\overline{x}]$ and $[1/\pi, 1/\underline{x}] \preceq [1/\pi', 1/\underline{x}])$.

where as usual $\underline{x}$ and $\overline{x}$ denote respectively a minimal and maximal elements of $\preceq$ over X.

Lemma 2 *If* A1, L2, L4! *and* $\mathrm{L5_{OP}}$ *hold, then* $\underline{x} \preceq \pi \preceq \overline{x}$ *for all* $\pi \in \Pi(X)_{[0,1]}$

Let $\overline{x}$ and $\underline{x}$ be respectively a maximal and minimal element of X w.r.t. $\preceq$. We define two new relations $\preceq^{-}$ and $\preceq^{+}$ on $\Pi(X)_{[0,1]}$ by putting $\pi \preceq^{-} \pi'$ iff $[1/\pi, 1/\overline{x}] \preceq [1/\pi', 1/\overline{x}]$ and $\pi \preceq^{+} \pi'$ iff $[1/\pi, 1/\underline{x}] \preceq [1/\pi', 1/\underline{x}]$.

Lemma 3 *Let* $\preceq$ *satisfy* A1, L2, A3 *and* L4!. *Let* $\overline{x}$ *and* $\underline{x}$ *be a maximal and minimal element of* X *w.r.t.* $\preceq$. *Then:*
(i) $\underline{x} \preceq^{-} \pi \preceq^{-} \overline{x}$, *and* $\underline{x} \preceq^{+} \pi \preceq^{+} \overline{x}$, *for all* π.
(ii) $\preceq^{-}$ *satisfies the axioms* $\mathcal{S}^{\otimes}_{P!}$.
(iii) $\preceq^{+}$ *satisfies the axioms* $\mathcal{S}^{\otimes}_{O!}$.

Theorem 3 *A preference ordering* $\preceq$ *on* $\Pi(X)_{[0,1]}$ *satisfies the system of postulates* $\mathcal{S}^{\otimes}_{PO}$ *if, and only if, there exist two mapping* $u_1, u_2 : X \to [0,1]$ *with* $u_1^{-1}(1) \cap u_2^{-1}(1) \neq \emptyset \neq u_1^{-1}(0) \cap u_2^{-1}(0)$ *such that, for all* $\pi, \pi' \in \Pi(X)_{[0,1]}$,

$$\pi \preceq \pi' \textit{ iff } F^{u_1,u_2}_{\otimes}(\pi) \leq_{lex} F^{u_1,u_2}_{\otimes}(\pi').$$

Proof: One direction is easy. As for the other direction, assume $\preceq$ satisfies (A1) through (L5$_{PO}$). By Lemma 3, its associated relations $\preceq^{-}$ and $\preceq^{+}$ satisfy the axioms $\mathcal{S}^{\otimes}_{P!}$ and $\mathcal{S}^{\otimes}_{O!}$ respectively. Therefore, by Theorems 1 and 2, we have:

- there exists $u_1 : X \to [0,1]$ such that $\preceq^{-} = \preceq^{-}_{u_1}$
- there exists $u_2 : X \to [0,1]$ such that $\preceq^{+} = \preceq^{+}_{u_2}$

By Lemma 3, $u_1(\overline{x}) = u_2(\overline{x}) = 1$ and $u_1(\underline{x}) = u_2(\underline{x}) = 0$. For every $x \in X$, by axiom A4!$^{-}$, there exists λ_x such that $x \sim^{-} [1/\overline{x}, \lambda_x/\underline{x}]$, hence $QU^{-}_{\otimes}(x \mid u_1) = QU^{-}_{\otimes}([1/\overline{x}, \lambda_x/\underline{x}] \mid u_1)$, hence $u_1(x) = 1 - \lambda_x$. On the other hand, by axiom A4!$^{+}$, there exists μ_x such that $x \sim^{+} [1/\underline{x}, \mu_x/\overline{x}]$, hence $u_2(x) = \mu_x$.

Finally, by Axiom $L5_{PO}$, $\preceq$ is the lexicographic ordering defined by $\preceq_u^-$ and $\preceq_u^+$, in other words, we have for all π and π', $\pi \preceq \pi'$ iff $F_\times^{u_1,u_2}(\pi) \leq_{lex} F_\times^{u_1,u_2}(\pi')$. $\square$

It is easy to check that if we replace axiom $\mathbf{L5}_{PO}$ by axiom $\mathbf{L5}_{OP}$, where

$\mathbf{L5}_{OP}$: $\pi \preceq \pi'$ iff either $[1/\pi, 1/\underline{x}] \prec [1/\pi', 1/\underline{x}]$ or $([1/\pi, 1/\underline{x}] \sim [1/\pi', 1/\underline{x}]$ and $[1/\pi, 1/\overline{x}] \preceq [1/\pi', 1/\overline{x}])$.

then the axiom system $\mathcal{S}_{OP}^{\otimes} = \{\mathbf{A1}, \mathbf{A3}, \mathbf{L2}, \mathbf{L4!}, \mathbf{L5}_{OP}\}$ captures the preference relations over lotteries defined as the lexicographic refinement of an ordering induced by an optimistic criterion by an ordering induced by a pessimistic criterion.

In these representations, the utility assignments u_1 and u_2 are unrelated, except by the condition $u_1^{-1}(1) \cap u_2^{-1}(1) \neq \emptyset \neq u_1^{-1}(0) \cap u_2^{-1}(0)$ which says that they share a maximal and a minimal element of X. Finally we will show how to add suitable postulates in order to guarantee that $u_1 = u_2$ or that $min(u_1, 1 - u_2) = 0$. In fact, consider the following two postulates for all $x \in X$:

L6: there exists λ such that $[1/x, 1/\overline{x}] \sim [1/\overline{x}, (1-\lambda)/\underline{x}]$ and $[1/x, 1/\underline{x}] \sim [\lambda/\overline{x}, 1/\underline{x}]$

L7: if $[1/x, 1/\overline{x}] \sim [1/\overline{x}, \lambda/\underline{x}]$ with $\lambda < 1$, then $[1/x, 1/\underline{x}] \sim [1/\overline{x}, 1/\underline{x}]$

Theorem 4 *A preference ordering $\preceq$ on $\Pi(X)_{[0,1]}$ satisfies the system of postulates $\mathcal{S}_{PO}^{\otimes}$ plus* **L6** *if, and only if, there exists a single mapping $u : X \to [0,1]$ with $u^{-1}(1) \neq \emptyset \neq u^{-1}(0)$ such that, for all $\pi, \pi' \in \Pi(X)_{[0,1]}$,*

$$\pi \preceq \pi' \text{ iff } F_\otimes^{u,u}(\pi) \leq_{lex} F_\otimes^{u,u}(\pi').$$

Theorem 5 *A preference ordering $\preceq$ on $\Pi(X)_{[0,1]}$ satisfies the system of postulates $\mathcal{S}_{PO}^{\otimes}$ plus* **L7** *if, and only if, there exist two mapping $u_1, u_2 : X \to [0,1]$ with $u_1^{-1}(1) \cap u_2^{-1}(1) \neq \emptyset \neq u_1^{-1}(0) \cap u_2^{-1}(0)$ and with $u_2(x) = 1$ if $u_1(x) > 0$, such that, for all $\pi, \pi' \in \Pi(X)_{[0,1]}$,*

$$\pi \preceq \pi' \text{ iff } F_\otimes^{u_1,u_2}(\pi) \leq_{lex} F_\otimes^{u_1,u_2}(\pi').$$

As a corollary of this last theorem, when $\otimes = \min$, the system $\mathcal{S}_{PO}^{\otimes}$ plus **L7** would be equivalent to Giang and Shenoy's system $\mathcal{S}_B$.

Finally, let us notice that in a very recent paper [7], Giang and Shenoy still propose another axiomatic system for decision making where uncertainty is modeled by likelihood functions. Their system of postulates is very similar to the system $\mathcal{S}_B$ described in Section 2, but using [0, 1] as uncertainty and utility scales and lotteries with a $\otimes$-mixture operation with $\otimes$ being the product. By analogous reasons of those in Section 2, the system in [7] would be then be very close to our system $\mathcal{S}_{PO}^{\otimes}$ plus **L7**, for $\otimes$ = product.

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References

[1] D. Dubois, L. Godo, H. Prade, and A. Zapico. On the possibilistic-based decision model: From decision under uncertainty to case-based decision. *Intl. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 7(6):631–670, 1999.

[2] D. Dubois, L. Godo, H. Prade, and A. Zapico. Advances in Qualitative Decision Theory: Refined Rankings. In proc. of IBERAMIA-2000, LNAI 1952, Springer, pp. 427-436, 2000.

[3] D. Dubois and H. Prade. Possibility theory as a basis for qualitative decision theory. In *14th Int. Joint Conf. on Artificial Intelligence (IJCAI'95)*, pages 1924–1930, Montreal, 1995.

[4] D. Dubois, H. Prade and R. Sabbadin. Decision-theoretic foundation of qualitative possibility theory. *European Journal of Operational Research* 128 (2001), 478495.

[5] H. Fargier and R. Sabbadin. Qualitative decision under uncertainty: back to expected utility. *Artificial Intelligence* Volume 164 (1-2), 2005, pp. 245-280.

[6] P. Giang and P. Shenoy. Two Axiomatic Approaches to Decision Making Using Possibility Theory *European Journal of Operational Research*, Vol. 162, No. 2, 2005, pp. 450–467.

[7] P. Giang and P. Shenoy. Decision Making on the Sole Basis of Statistical Likelihood. *Artificial Intelligence*, Volume 165 (2), 2005, pp. 137-163