On the Equational Characterization of Continuous t-Norms

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Abstract A (continuous) t-norm is called equationally definable when the corresponding standard BL-algebra $[\mathbf{0,1}]_*$ defined by $*$ and its residuum is the only (up to isomorphism) standard BL-algebra that generates the same variety $Var([\mathbf{0,1}]_*)$. In this chapter we check that a continuous t-norm $*$ is equationally definable if and only if the t-norm is a finite ordinal sum of copies of the three basic continuous t-norms, i.e. Łukasiewicz, Gödel and Product t-norms.

1 Introduction

A core constituent of *fuzzy logic in narrow sense* [15], from where the discipline of Mathematical fuzzy logic has been intensively developed in the last two decades [5, 10, 11, 14], is the family of residuated many-valued logical calculi with truth values on the real unit interval [0, 1], and with min, max, a (left-continuous) t-norm $*$ and its residuum $\rightarrow_*$ as basic truth functions, interpreting respectively the lattice meet and joint connectives, a strong conjunction and its adjoint implication. These logics are also known as *t-norm based fuzzy logics*.

In this framework, Hájek introduced in [11, 12] the so-called *Basic Fuzzy logic*, BL for short, to capture the 1-tautologies common to all many-valued calculi in [0, 1] defined by a *continuous* t-norm and its residuum, as proved in [4]. Thus, BL is in fact a common sublogic of three well-known fuzzy logics: Łukasiewicz's infinitely-valued logic, Gödel's infinitely-valued logic and Product logic, corresponding to the three basic t-norms, i.e. Łukasiewicz, minimum and product t-norms.

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The variety of BL-algebras constitutes the algebraic semantics of Hájek's BL, which is generated by the so-called *standard* BL-algebras $[\mathbf{0},\mathbf{1}]_*$, that is, the BL-algebras defined on the real unit interval $[0, 1]$, and that in turn are induced by continuous t-norms $*$ and their residuum $\rightarrow_*$. Some subvarieties of **BL** generated by a single standard BL-chain $[\mathbf{0},\mathbf{1}]_*$ are well-known, in particular the subvarieties of MV algebras, Gödel algebras and Product algebras, the algebraic counterparts of Łukasiewicz, Gödel and Product logics respectively. These varieties are respectively generated by the standard algebras defined by Łukasiewicz, minimum and product t-norms, and are fully described and equationally characterized in the literature. A step further was done in [8], where all varieties $Var([\mathbf{0},\mathbf{1}]_*)$ of BL-algebras generated by a single standard BL-chain $[\mathbf{0},\mathbf{1}]_*$ was proved to be finitely axiomatizable.

Then the question arises of whether such an axiomatization of $Var([\mathbf{0},\mathbf{1}]_*)$ (i.e. a set of equations) univocally characterizes $*$ itself, in the sense of whether $[\mathbf{0},\mathbf{1}]_*$ is the only (up to isomorphism) standard BL-algebra that generates the same variety $Var([\mathbf{0},\mathbf{1}]_*)$. When this is so, we say that $*$ is *equationally definable*.

As a rather direct consequence of results in [8], in this short note, and after introducing some needed preliminaries, we check in Sect. 3 that a continuous t-norm is equationally definable if and only if the t-norm is a finite ordinal sum of the three basic continuous t-norms, while in Sect. 4 we show how to effectively find a set of equations of $Var([\mathbf{0},\mathbf{1}]_*)$ for an arbitrary equationally definable continuous t-norm $*$.

2 Preliminaries

We start with some elementary and well-known definitions and results about t-norms, just for the sake of the paper being self-contained. A t-norm is a binary operation on $[0, 1]$ that is commutative, associative, non-decreasing (monotone) in both variables and that have 0 as absorbent and 1 as unity. A t-norm is continuous if it is continuous as real function of two variables. The three basic continuous t-norms are minimum (min), product (the usual product of reals, $\odot$) and Łukasiewicz (denoted $*_L$ and defined by $x *_L y = \max(0, x + y - 1)$). The greatest and smallest continuous t-norms are the minimum and the Łukasiewicz t-norms respectively, i.e., for all continuous t-norm $*$ and for all $x, y \in [0, 1]$, we have $x *_L y \le x * y \le \min(x, y)$.

The following are some basic results on continuous t-norms, see e.g. [13] for further details and results:

- Any continuous t-norm is an ordinal sum of (possibly infinitely-many) copies[1] of the minimum, product and Łukasiewicz t-norms.
- A t-norm $*$ is continuous if and only if it satisfies the divisibility condition: for all $x, y \in [0, 1]$ with $x > y$ there exists $z \in [0, 1]$ such that $y = x * z$.

[1] If we allow for at most a countable number of degenerated components with a single idempotent element.

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- Each left-continuous t-norm $*$ uniquely defines a binary operation $\to_*$, called the residuum of $*$, that satisfies the following condition: for all $x, y, z \in [0, 1]$, $x * y \leq z$ if and only if $x \leq y \to_* z$ (*residuation or adjunction condition*).
- The residuum $\to_*$ of a left-continuous t-norm $*$ is actually defined as $x \to_* y = \max\{z \in [0, 1] : x * z \leq y\}$ (*residuated implication*).
- A left-continuous t-norm $*$ is continuous if and only if the following equation is satisfied: for all $x, y \in [0, 1]$, $x * (x \to_* y) = \min(x, y)$ (*Divisibility equation*).

On the oher hand, it is also well known that the algebraic counterpart of Hájek's BL logic [11] is given by the variety of *BL-algebras*, i.e. algebraic structures $\mathbf{A} = (A, \wedge, \vee, *, \to, 0, 1)$ satisfying:

- $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice,
- $(A, *, 1)$ is a commutative monoid with unit 1,
- $*$ and $\to$ form and adjoint pair, i.e. they satisfy the residuation condition: for all $x, y, z \in A$, $x * y \leq z$ if and only if $x \leq y \to z$,
- Prelinearity: for all $x, y \in A$, $(x \to y) \vee (y \to x) = 1$,
- Divisibility: for all $x, y \in A$, $x * (x \to y) = x \wedge y$.

In other words, BL-algebras are a subclass of residuated lattices, namely, the class of bounded, commutative, integral residuated lattices further satisfying pre-linearity and divisibility.

A *standard BL-chain* is a BL-algebra defined over the real unit interval $[0, 1]$. It is easy to prove that:

- A continuous t-norm and its residuum defines a standard BL-chain,
- Each standard BL-chains is defined by a continuous t-norm and its residuum.

The last items shows that there is a bijection between continuous t-norms and standard BL-chains. From now on, we will denote by $[\mathbf{0}, \mathbf{1}]_*$ the BL-algebra $([0, 1], \min, \max, *, \to_*, 0, 1)$ defined by a continuous t-norm $*$ and its residuum.

The ordinal sum representation for continuous t-norms extends to an ordinal sum representation for standard BL-chains in the obvious way, the only new thing to consider is the definition of the residuum over the whole ordinal sum in terms of the residuum over each component. Using a similar representation for BL-chains, in [4] it was proved that the logic BL is complete with respect to the class of standard BL-chains, or in other words, that the whole variety of BL-algebras is generated by the class of standard BL-chains.

A related class of algebraic structures is that of *hoops*. In what follows we introduce some basic definitions and results about hoops and the decomposition theorem for BL-chains as ordinal sums of hoops that we will use in the next section, see [2, 3, 9] for more details.

Definition 1 A *hoop* is an algebraic structure $\mathbf{A} = (A, *, \to, 1)$ such that:

- $*$ is a binary commutative operation with unit 1, i.e. $x * y = y * x$ and $1 * x = x$ for all $x, y \in A$

- $\to$ is a binary operation satisfying:
  - for all $x \in A$, $x \to x = 1$,
  - for all $x, y, z \in A$, $(x * y) \to z = x \to (y \to z)$,
  - for all $x, y \in A$, $x * (x \to y) = y * (y \to x)$.

The associated order relation is defined by: $x \leq y$ if $x \to y = 1$.

A *basic* hoop is a hoop satisfying the following condition:

- $((x \to y) \to z) * (y \to x) \to z) \to z = 1$

A *Wajsberg* hoop is a hoop satisfying the following condition:

- for all $x, y \in A$, $(x \to y) \to y = (y \to x) \to x$.

A *cancellative* hoop is a hoop such that:

- for all $x, y, z \in A$, $x * y \leq x * z$ implies that $y \leq z$.

From this definition, one can check the following facts and properties:

(i) $\leq$ as defined above is indeed an ordering and 1 is maximal
(ii) $*$ is associative
(iii) $*$ is monotonically increasing w.r.t. $\leq$: $x \leq y$ implies $x * z \leq y * z$
(iv) $(*, \to)$ is an adjoint pair: $x \to y \leq z$ iff $x * y \leq z$
(v) $x * (x \to y) \leq y$
(vi) $1 \to x = x$

Furthermore, regarding the classes of basic, Wajsberg and cancellative hoops, the following relationship among them hold: every Wajsberg hoop is basic and each cancellative hoop is Wajsberg (hence basic as well). Note that hoops have an greatest element, but they may lack a least element. A hoop $\mathbf{A} = (A, *, \to, 1)$ is called *bounded* if $(A, \leq)$ has a least element. Then it turns out that cancellative hoops coincide with *unbounded* Wajsberg hoops, while bounded Wajsberg hoops coincide with MV-algebras.
Prominent examples of Wajsberg hoops are the following:

- **2**, defined on a set of two elements $\{a, 1\}$, that is in fact a two-element Boolean algebra.
- Ł $= ([0, 1], *_Ł, \to_Ł, 1)$, the (bounded) Wajsberg hoop defined over $[0, 1]$ by the Łukasiewicz t-norm and its residuum.
- C $= ((0, 1], \odot, \to_\odot, 1)$, the (unbounded) cancellative hoop defined over $(0, 1]$ by the product t-norm and its residuum.

A similar construction to the ordinal sums for t-norms and BL-chains can be also defined for hoops.

Definition 2 Let $(I, \leq)$ be a totally ordered set, and for all $i \in I$ let $\mathbf{A}_i = (A_i, *_i, \to_i, 1)$ be a hoop such that $A_i \cap A_j = \{1\}$ for every $j \neq i$. Then the ordinal sum of this family is the structure $\bigoplus_{i \in I} \mathbf{A_i} = (\bigcup_{i \in I} A_i, *, \to, 1)$, where the operations are defined as follows:

$$x * y := \begin{cases} x *_i y & \text{if } x, y \in A_i, \\ x & \text{if } x \in A_i \setminus \{1\}, y \in A_j, \text{ and } i < j, \\ y & \text{if } y \in A_i \setminus \{1\}, x \in A_j, \text{ and } i < j. \end{cases}$$

$$x \to y := \begin{cases} x \to_i y & \text{if } x, y \in A_i, \\ y & \text{if } x \in A_i, y \in A_j, \text{ and } i > j, \\ 1 & \text{otherwise.} \end{cases}$$

Notice that in an ordinal sum of hoops, the greatest element is common to all the hoops and to the ordinal sum as well. For instance, the product standard chain $[0, 1]_\Pi = ([0, 1], \odot, \to_\odot, 0, 1)$, viewed as a bounded hoop, can be decomposed as the ordinal sum of **2** and C, i.e. $[0, 1]_\Pi = \mathbf{2} \oplus \mathrm{C}$. Actually, in [1] the authors prove that any BL-chain, viewed as a bounded basic hoop, can be decomposed as an ordinal sum of linearly ordered Wajsberg hoops. Restricted to standard BL-chains, this result amounts to say that any standard BL-chain, as a hoop, can be decomposed as an ordinal sum of (suitably arranged) copies of the Wajsberg hoops **2**, C and Ł. In this way, besides viewing the standard product algebra as the ordinal sum of **2** plus C, we can understand the standard Gödel chain as being isomorphic to the ordinal sum of continuum many of copies of **2** (one for each element of a Gödel component), while the standard Łukasiewicz chain $[0, 1]_Ł = ([0, 1], *_Ł, \to_Ł, 0, 1)$ coincides with Ł as hoop.

As already mentioned, regarding the ordinal sums of hoops just defined, one can notice that the main difference with respect to the ordinal sum of BL-chains is that the top elements of the components are identified with the top element of the ordinal sum. Therefore, for instance, when considering the decomposition of a BL-chain as an ordinal sum of Wajsberg hoops (**2**, C or Ł in the case of standard BL-chains), the top of any component is the top of the ordinal sum, and given two consecutive components, the bottom (if it exists) of the second component is not in the first component. Notice also that the decomposition of any standard BL-chain as ordinal sum of hoops has always a first component that is either **2** (if it is an SBL-chain[2]) or Ł otherwise.

Finally recall that a set of equations determine a *variety* (or equational class) of algebraic structures. By inspecting their definition, it is clear that the classes of hoops, basic hoops and Wajsberg hoops are indeed varieties. The class of cancellative hoops turns out to be a variety as well, since the condition used in Definition 2 can be shown to be equivalent to the validity of the equation $x = y \to (y * x)$.

Thus it is interesting to know how the varieties generated by the main three prominent Wajsberg hoops, **2**, C and Ł, are related to each other. To do so we consider the following three terms:

- $e_Ł(x) = (x \to x^2) \vee ((x \to x^3) \to x^2)$
- $e_C(x) = (x \to x^2)$
- $e_{\mathbf{2}}(x) = (x \to x^3) \to x^2$

[2] That is, a standard BL-chain defined by an strict continuous t-norm.

where x^n stands for $x * \overset{n}{\ldots} * x$. An easy computation shows that the equation $e_Ł(x) = 1$ is valid in **2** and C and not in Ł, $e_C(x) = 1$ is a valid equation in **2** and neither in C nor in Ł, and finally, the equation $e_{\mathbf{2}}(x) = 1$ is valid in C and neither in **2** nor in Ł.

Therefore, it is clear that C and Ł do not belong to variety of hoops $Var(\mathbf{2})$ generated by **2**, while **2** and Ł do not belong to the variety $Var(C)$ generated by C. On the other hand, it is easy to check that both **2** and C belong to the variety of hoops $Var(Ł)$ generated by Ł, since **2** is a subhoop of Ł and C is a subhoop of the well-known Chang algebra, which is an MV-algebra, and thus belongs to $Var(Ł)$.

Summarising, we have

$$\mathbf{2}, C \in Var(Ł), \qquad C, Ł \notin Var(\mathbf{2}), \qquad \mathbf{2}, Ł \notin Var(C),$$

and thus, the following strict inclusions among varieties hold:

$$Var(\mathbf{2}) \subset Var(Ł), \qquad Var(C) \subset Var(Ł).$$

3 Characterization of Standard BL-Chains that Are Equationally Definable

Let us denote by $[\mathbf{0}, \mathbf{1}]_*$ either the standard BL-chain, or its corresponding hoop when no confusion exists, defined over $[0, 1]$ by a continuous t-norm $*$ and its residuum $\rightarrow_*$. The goal of this section is to characterize those continuous t-norms $*$ that admit an *equational characterization* in the sense that the variety $Var([\mathbf{0}, \mathbf{1}]_*)$ is uniquely generated by $[\mathbf{0}, \mathbf{1}]_*$, that is, for any other standard BL-chain $[\mathbf{0}, \mathbf{1}]_\circ$ with $\circ$ being a t-norm non isomorphic to $*$, $Var([\mathbf{0}, \mathbf{1}]_*) \neq Var([\mathbf{0}, \mathbf{1}]_\circ)$. In such a case, we can say that the set of equations defining $Var([\mathbf{0}, \mathbf{1}]_*)$ characterize $*$.

Actually, generalizing the well-known Mostert and Shields representation theorem of continuous t-norms, Hájek showed in [12] that every standard BL-chain $[\mathbf{0}, \mathbf{1}]_*$ can be isomorphically decomposed as an ordinal sum (over a bounded ordered index set) of Gödel, Łukasiewicz and Product BL-chain components. However, as hoops, each Gödel BL-chain is isomorphic to an ordinal sum of (possibly infinite) copies of **2**, while Łukasiewicz and Product components on a closed real interval are isomorphic to Ł and $\Pi = \mathbf{2} \oplus C$ respectively. Then any standard BL-chain, as a hoop, will be isomorphic to a (possibly infinite) ordinal sum of Wajsberg hoops Ł, C and **2**.

The following definition and proposition are particular cases of more general definitions and results given in [8], and therefore here we only state them without proofs.

Definition 3 (i) We will denote by Fin the set of ordinal sums (as hoops) of finitely-many copies of Ł, 2 and C, and whose first component is either Ł or 2.

(ii) Let $\mathbf{A}$ be a standard BL-chain whose decomposition as ordinal sum of hoops is $\mathbf{A} = \mathbf{A_0} \oplus (\bigoplus_{i \in I} \mathbf{A_i})$. Then $Fin(\mathbf{A})$ is the set of all finite ordinal sums $\bigoplus_{i=0,\dots,n} \mathbf{B_i}$ of Wajsberg hoops satisfying the following conditions:

- Each $\mathbf{B_i}$ is either $\mathbf{2}$, C or Ł,
- $\mathbf{B_0}$ is either $\mathbf{2}$ or Ł,
- There are components $\mathbf{A_0} < \mathbf{A_1} < \cdots < \mathbf{A_n}$ of $\mathbf{A}$ such that for every $i = 0, \dots, n$: (i) if $\mathbf{B_i} =$ Ł then $\mathbf{A_i}$ is isomorphic to Ł; (ii) if $\mathbf{B_i} =$ C, then $\mathbf{A_i}$ is isomorphic either to C or to Ł; and (iii) if $\mathbf{B_i} = \mathbf{2}$, then $\mathbf{A_i}$ is isomorphic either to $\mathbf{2}$ or to Ł.

Example 1 Consider the standard BL-chain $\mathbf{A} = G \oplus \text{Ł} \oplus \Pi$. Then, for instance, $\mathbf{2} \oplus \text{Ł}$ and $\mathbf{2} \oplus \mathbf{2} \oplus \text{Ł} \oplus C$ are in $Fin(\mathbf{A})$, while neither $\text{Ł} \oplus \mathbf{B}$ for any $\mathbf{B} \in Fin$, nor $\mathbf{2} \oplus \text{Ł} \oplus \text{Ł}$ are in $Fin(\mathbf{A})$.

As shown next, the set of $Fin([\mathbf{0}, \mathbf{1}]_*)$ of BL-chains univocally determines the variety $V([\mathbf{0}, \mathbf{1}]_*)$ induced by the t-norm $*$.

Proposition 1 (c.f. Theorem 3.9 of [8]) *Let* $[\mathbf{0}, \mathbf{1}]_*, [0, 1]_\circ$ *be two standard BL-chains. Then* $Var([\mathbf{0}, \mathbf{1}]_*) \subseteq Var([\mathbf{0}, \mathbf{1}]_\circ)$ *if, and only if,* $Fin([\mathbf{0}, \mathbf{1}]_*) \subseteq Fin([\mathbf{0}, \mathbf{1}]_\circ)$. *Hence,* $Var([\mathbf{0}, \mathbf{1}]_*) = Var([\mathbf{0}, \mathbf{1}]_\circ)$ *if, and only if,* $Fin([\mathbf{0}, \mathbf{1}]_*) = Fin([\mathbf{0}, \mathbf{1}]_\circ)$.

Notation convention: In the following, given two continuous t-norms $*$ and $\circ$, we will write $* \equiv \circ$ to denote that they isomorphic in the usual sense of t-norms, that is, when there exists an increasing bijection $f : [0, 1] \to [0, 1]$ such that, for any $x, y \in [0, 1]$, $x \circ y = f^{-1}(f(x) * f(y))$.

The following lemma is straightforward to check.

Lemma 1 *If* $*$ *and* $\circ$ *are two continuous t-norms such that both* $[0, 1]_*$ *and* $[0, 1]_\circ$ *have a finite ordinal sum decomposition in terms of BL-components, then* $* \equiv \circ$ *if, and only if, they have the same decomposition,*

From the above proposition and lemma, the characterization of the equationally definable standard BL-chains follows.

Proposition 2 *A continuous t-norm* $*$ *admits an equational characterization if, and only if, the corresponding standard BL-chain* $[\mathbf{0}, \mathbf{1}]_*$ *can be decomposed as an ordinal sum with finitely-many copies of components* Ł, G *and* Π.

Proof First we prove that for a continuous t-norm $*$ whose decomposition as ordinal has a finite number of components, $Var([\mathbf{0}, \mathbf{1}]_\circ) = Var([\mathbf{0}, \mathbf{1}]_*)$ if and only if $\circ \equiv *$ (the components of their decomposition as ordinal sums are the same). By the previous proposition, this is equivalent to prove that if $\circ$ is a continuous t-norm such that $\circ \not\equiv *$, then $Fin(\circ) \neq Fin(*)$. We prove this claim by cases, adapting a more general proof in [8]:

- If the decomposition of $[\mathbf{0},\mathbf{1}]_\circ$ has more components than the decomposition of $[\mathbf{0},\mathbf{1}]_*$ then it is evident that there exist BL-chains in $Fin(\circ)$ that are not in $Fin(*)$. For example let $\circ$ be a continuous t-norm obtained as Ł $\oplus\, G$, and let $*$ be a continuous t-norm obtained as Ł $\oplus\, \Pi \oplus G$. Then it is clear that $2 \oplus C \in Fin([\mathbf{0},\mathbf{1}]_*)$ but $2 \oplus C \notin Fin([\mathbf{0},\mathbf{1}]_\circ)$.
- An analogous reasoning proves the statement when the decomposition of $[\mathbf{0},\mathbf{1}]_\circ$ has more components than the decomposition of $[\mathbf{0},\mathbf{1}]_*$.
- If the number of components of the decomposition $[\mathbf{0},\mathbf{1}]_*$ and $[\mathbf{0},\mathbf{1}]_\circ$ is the same, then they need to differ in some component and thus we can find BL-chains that are in $Fin(*)$ and not in $Fin(\circ)$ and viceversa. For example, let $\circ$ be the continuous t-norm obtained as Ł $\oplus$ Ł $\oplus\, G$ and let $*$ be the continuous t-norm obtained as Ł $\oplus\, \Pi \oplus G$. Then we have that $2 \oplus C \in Fin([\mathbf{0},\mathbf{1}]_*)$ but $2 \oplus C \notin Fin([\mathbf{0},\mathbf{1}]_\circ)$, while Ł $\oplus$ Ł $\in Fin([\mathbf{0},\mathbf{1}]_\circ)$ and Ł $\oplus$ Ł $\notin Fin([\mathbf{0},\mathbf{1}]_*)$.

In the case the decomposition of $[\mathbf{0},\mathbf{1}]_*$ has infinitely many components, it is easy to prove that there exist infinitely-many continuous t-norms $\circ$ such that $* \not\equiv \circ$ but $Fin([\mathbf{0},\mathbf{1}]_*) = Fin([\mathbf{0},\mathbf{1}]_\circ)$. We do not formally prove the statement but we give some examples:

- If the decomposition of $[\mathbf{0},\mathbf{1}]_*$ consists of an infinite number of Łukasiewicz components Ł, then any other standard BL-chain $[\mathbf{0},\mathbf{1}]_\circ$ whose decomposition begins with an Ł component and contains infinitely many Łukasiewicz components together with (finitely or infinitely many) components Π or G, defines the same variety, namely, the full variety of BL-algebras, see [1].
- If the decomposition of $[\mathbf{0},\mathbf{1}]_*$ begins with a $\mathbf{2}$ component and contains an infinite number of Łukasiewicz components, then any other standard BL-chain $[\mathbf{0},\mathbf{1}]_\circ$ whose decomposition begins with a $\mathbf{2}$ component and contains infinitely many Łukasiewicz components together with (finitely or infinitely many) components Π or G, defines the same variety, namely, the full variety of SBL-algebras, see [1].

4 How to Find a Set of Equations of an Equationally Definable t-Norm

After identifying in the last section which t-norms are equationally definable, in this section we show how to find an effective set of equations for each of them, again relying in results from [8]. It has to be remarked that the equations actually characterise the variety generated by the standard algebra $[\mathbf{0},\mathbf{1}]_*$ for a given equationally definable t-norm $*$, and hence the equations will involve not only the operation corresponding to the t-norm but the operation corresponding to its residuum as well.

First we introduce an equation that will have a key role in axiomatizing the varieties $V([\mathbf{0},\mathbf{1}]_*)$.

Definition 4 Let $\mathbf{A}$ be a BL-chain whose decomposition as ordinal sum of Wajsberg hoops has finitely many components, i.e., $\mathbf{A} = \bigoplus_{\mathbf{i=0,1,\ldots,n}} \mathbf{A_i}$. Then we will denote by e_A the following equation on $n+1$ variables,

$$\left[\left(\bigwedge_{i=0,\dots,n-1}((x_{i+1}\to x_i)\to x_i)\ *\ (\neg\neg x_0\to x_0)\right)\to\left(\bigvee_{i=0,\dots,n} x_i\right)\right]\vee\bigvee_{i=0,\dots,n} e_i^A(x_i)=1 \qquad (e_A)$$

where $e_i^A(x)=e_{Ł}(x)$ if $\mathbf{A_i}=Ł$, $e_i^A(x)=e_C(x)$ if $\mathbf{A_i}=C$, and $e_i^A(x)=e_{\mathbf{2}}(x)$ if $\mathbf{A_i}=\mathbf{2}$.

Notation convention: for the sake of a simpler notation, from now on we will use $Fin(*)$ and $Var(*)$ to respectively denote $Fin([\mathbf{0},\mathbf{1}]_*)$ and $Var([\mathbf{0},\mathbf{1}]_*)$.

Lemma 2 *Let $*$ be a continuous t-norm whose corresponding standard BL-chain has a decomposition as ordinal sum with finitely many components Ł, Π and G, and let $\mathbf{A}\in Fin$. Then e_A is valid in all BL-chains $\mathbf{B}\in Fin(*)$ if and only if $\mathbf{A}\notin Fin(*)$.*

And from this result, we can prove the following equational characterization as a particular case of a more general result in [8, Theorem 5.2].

Proposition 3 *Let $*$ be a continuous t-norm whose corresponding standard BL-chain $[\mathbf{0},\mathbf{1}]_*$ has a decomposition as ordinal sum with finitely many components Ł, Π and G. Then,*

$Var(*)$ is axiomatized by the set of equations $AX(*)=\{e_B:\mathbf{B}\in Fin(*^{\perp})\}$,

where $Fin(*^{\perp})=Fin\backslash Fin(*)$.

Note that $AX(*)$ may contain an infinite number of equations. However we can do it better. Actually, one can show that one needs only a finite subset of $AX(*)$ to axiomatize $Var(*)$. Indeed, it is only necessary to keep from $Fin(*^{\perp})$ only those BL-chains that are *minimal* in the following sense. Define an ordering relation in the set Fin as follows: for all $\mathbf{A},\mathbf{B}\in Fin$, define $A\preceq B$ if $\mathbf{A}\in Var(\mathbf{B})$. And denote by $Min(*^{\perp})$ the minimal elements of $Fin(*^{\perp})$ with respect to the order $\preceq$. It is then clear that it is enough to consider the set of equations corresponding to the BL-chains of $Min(*^{\perp})$, and moreover, it can be shown that $Min(*^{\perp})$ is always finite, and hence that $Var(*)$ can be axiomatized by a finite set of equations.

Proposition 4 *Let $*$ be a continuous t-norm whose decomposition as ordinal sum of t-norms has finitely many components. Then:*

(i) The set $Min(*^{\perp})$ is finite.
(ii) $Var(*)$ is axiomatized by the finite set of equations

$$AX_{min}(*)=\{e_B:\mathbf{B}\in Min(*^{\perp})\}.$$

Following [8], given an arbitrary continuous t-norm $*$ and its decomposition as ordinal sum of Ł, G and Π components, an algorithmic procedure to find the set $Min(*^{\perp})$ can be given. The idea to find the minimal elements of Fin which are not

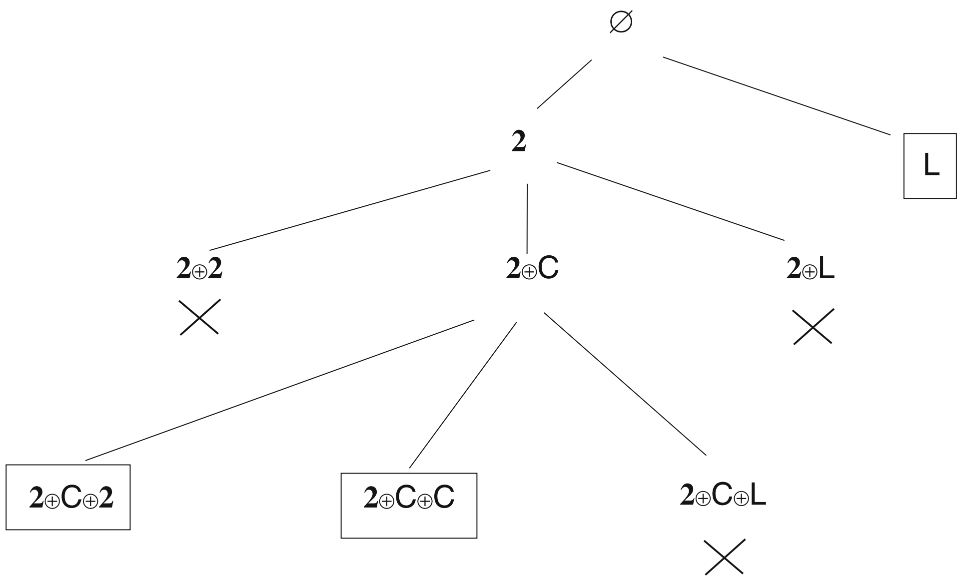

Fig. 1 Analysis for $* = G \oplus Ł$

in $Fin(*)$ is to iteratively checking ordinal sums from Fin of increasing length (1, 2, 3, etc.). At a given step i, a given current ordinal sum $\mathbf{B}$ of length i is checked whether there is another non-discarded ordinal sum $\mathbf{B}'$ of length $\leq i$ such that $\mathbf{B}' \preceq \mathbf{B}$. If so, the current ordinal sum is discarded for further analysis at step $i + 1$. Otherwise $\mathbf{B} \in Min(*^{\perp})$ only if $\mathbf{B}$ is checked to not belong to $Fin(*)$. At next step $i + 1$, only those non-discarded ordinal sums at step i are expanded with a new component, and the procedure starts over. This iterative procedure ends in a finite number of steps. We exemplify this procedure with two examples.

Example 2 Consider a continuous t-norm $*$ isomorphic to $G \oplus Ł$. The above iterative procedure, depicted in Fig. 1 as a spanning tree, yields:

$$Min(*^{\perp}) = \{Ł, \mathbf{2} \oplus C \oplus \mathbf{2}, \mathbf{2} \oplus C \oplus C\}.$$

Example 3 Consider a continuous t-norm $*$ isomorphic to $G \oplus Ł \oplus \Pi \oplus Ł$. The above iterative procedure, depicted in Fig. 2, yields:

$$Min(*^{\perp}) = \{Ł, \mathbf{2} \oplus C \oplus Ł \oplus \mathbf{2}, \mathbf{2} \oplus C \oplus Ł \oplus C\}.$$

Therefore using the result of the previous proposition, we automatically have a finite set of equations $AX_{min}(*)$ univocally characterising $*$, since the only continuous t-norm algebra (up to isomorphism) belonging to $Var(*)$ is $[\mathbf{0}, \mathbf{1}]_*$ itself.

Dedication

This short note is dedicated to Peter Klement in the occasion of his retirement. We are deeply indebted to Peter, not only for his outstanding and numerous scientific

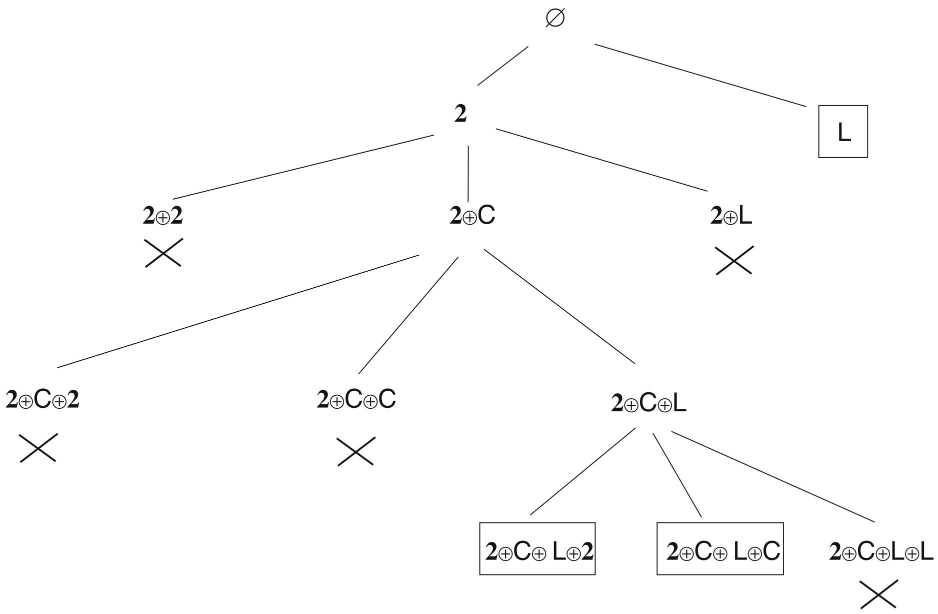

Fig. 2 Analysis for $* = G \oplus Ł \oplus \Pi \oplus Ł$

contributions to the field of fuzzy logic, but also for his incredible task of fostering the exchange of ideas and the collaboration among researchers in our community, mainly (but not only) through his Linz Seminars on Fuzzy Set Theory since 1979. Congratulations Peter!

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