On the Logical Formalization of Possibilistic Counterparts of States over n-Valued Łukasiewicz Events

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Abstract

Possibility and necessity measures are commonly defined over Boolean algebras. This work consider a generalization of these kinds of measures over MV-algebras as a possibilistic counterpart of the (probabilistic) notion of state on MV-algebras. Two classes of *possibilistic* states over MV-algebras of functions are characterized in terms of (generalized) Sugeno integrals. For reasoning about these representable classes of possibilistic states, we introduce many-valued modal logics based on the Rational Łukasiewicz Logic, that are be shown to be complete with respect to corresponding classes of Kripke models equipped with those states.

Keywords: Possibilistic States, Generalized Sugeno Integral, Łukasiewicz Logic, MV-algebras.

1 Introduction

Probability measures are without a doubt the main tool for modelling and reasoning under uncertainty. In the field of uncertain reasoning, however, many formalisms, upper and lower probabilities [23], Dempster-Shafer plausibility and belief functions [21], possibility and necessity measures [7], have been developed to deal with different notions of non-additive uncertainty.

The most general notion of uncertainty is captured by monotone set functions with two natural boundary conditions. In the literature, these functions have received several names, like *Sugeno measures* [22] or *plausibility measures*[1] [18]. In its simplest form, given a Boolean algebra $\mathcal{U} = (U, \wedge, \vee, \neg, \overline{0}^{\mathcal{U}}, \overline{1}^{\mathcal{U}})$, a Sugeno measure is a mapping $\mu : U \to [0, 1]$ satisfying $\mu(\overline{0}^{\mathcal{U}}) = 0$, $\mu(\overline{1}^{\mathcal{U}}) = 1$, and the monotonicity condition $\mu(x) \leq \mu(y)$ whenever $x \leq^{\mathcal{U}} y$, where $\leq^{\mathcal{U}}$ is the lattice order in $\mathcal{U}$. The class of Sugeno measures encompass the above mentioned classes of measures, i.e. probabilities, upper

[1] The reader should be warned not to confuse this term coined by Halpern with the term "plausibility function" used in the Dempster-Shafer model framework. Plausibility functions, related to the notion of monotone Choquet capacity of order ∞, are indeed just a subclass of the class of plausibility measures.

and lower probabilities, Dempster-Shafer plausibility and belief functions, possibility and necessity measures. In this work, we particularly focus on the latter.

Recall that a *possibility measure* on a (finite) Boolean algebra of events $\mathcal{U} = (U, \wedge, \vee, \neg, \overline{0}^{\mathcal{U}}, \overline{1}^{\mathcal{U}})$ is a Sugeno measure μ^* satisfiying the following $\vee$-decomposition property

$$\mu^*(u \vee v) = \max(\mu^*(u), \mu^*(v)),$$

while a *necessity measure* is a Sugeno measure μ_* satisfying the $\wedge$-decomposition property

$$\mu_*(u \wedge v) = \min(\mu_*(u), \mu_*(v)).$$

Actually, in presence of these decomposition properties, there is no need for the monotonicity condition since it easily follows from each one of them. Possibility and necessity measures are *dual* in the sense that if μ^* is a possibility measure, then the mapping $\mu_*(u) = 1 - \mu^*(\neg u)$ is a necessity measure, and vice versa. If $\mathcal{U}$ is the power set of a finite set X, then any dual pair of measures (μ^*, μ_*) on $\mathcal{U}$ is induced by a *normalized possibility distribution*, namely a mapping $\pi : X \to [0, 1]$ such that, $\max_{x \in X} \pi(x) = 1$, and, for any $A \subseteq X$,

$$\mu^*(A) = \max\{\pi(x) \mid x \in A\} \text{ and } \mu_*(A) = \min\{1 - \pi(x) \mid x \notin A\}.$$

Certainly, it makes sense to consider appropriate extensions of these classes of non-additive measures on algebras of events more general than Boolean algebras, in a similar way the notion of (finitely additive) probability has been generalized in the setting of MV-algebras by means of the notion of *state* [19]. In fact, in this paper, we focus on the investigation and logical formalization of meaningful generalizations of possibility and necessity measures over MV-algebras. Abusing the language, and by analogy with the case of probabilistic states, we will refer to them as *possibilistic states*. In more concrete terms, our aim in this paper is twofold: (*i*) to study possibilistic states over some particular MV-algebras from a measure-theoretic point of view and axiomatically characterize two of their subclasses; (*ii*) to introduce a logical framework to reason about these possibilistic states over finitely-valued Łukasiewicz events (in the sense of equivalent classes of formulas of an n-valued Łukasiewicz logic $Ł_n$), following the approach used in [12] for the case of (probabilistic) states.

We restrict our basic approach to finitely-valued events, since this allows us to get completeness results with respect to the intended semantics, while the case of infinitely-valued events remains an open problem. However, notice that the logics $Ł_n$ can arbitrarily approximate Ł in the sense that, for each theorem φ of Ł, there is a sufficiently large n such that φ is a theorem of $Ł_n$ as well (cf. [1]).

This work is organized as follows. In the next section we introduce the basic background notions concerning finitely and infinitely valued Łukasiewicz logics, along with their algebraic semantics. In Section 3, we define suitable notions of possibilistic states over MV-algebras and provide two axiomatic characterizations that are shown to be equivalent to the existence of a possibility distribution defining the state as a measure from two different forms of the generalized Sugeno integral. In Section 4 and Section 5, we introduce the many-valued modal logic $\mathrm{N}(Ł_n^+, \mathrm{R}Ł)$ to reason about possibilistic states over finitely-valued Łukasiewicz logics. Different semantic models are introduced, and $\mathrm{N}(Ł_n^+, \mathrm{R}Ł)$ is shown to be complete w.r.t. all of them. We also discuss two other kinds of (strong) completeness for $\mathrm{N}(Ł_n^+, \mathrm{R}Ł)$ and $\mathrm{N}(\mathrm{R}Ł, \mathrm{R}Ł)$, the logic obtained by replacing the finitely-valued logic Ł_n^+ by the infinitely-valued Rational Łukasiewicz logic RŁ as the logic for the events. In Section 6, we briefly study the logic $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$, that formalizes a slightly different notion of possibilistic states. Finally, we show that checking the satisfiabiliy of formulas of both $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ and $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is an NP-complete problem. We end with some final remarks.

2 Preliminaries on Łukasiewicz and related logics

The language of Łukasiewicz logic Ł (cf. [6, 15]), consists of a countable set of propositional variables $\{p_1, p_2, \ldots\}$, the binary connective $\to$ and the truth constant $\overline{0}$ (for falsity). Further connectives are defined as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
\neg\varphi & \text{is} & \varphi \to \overline{0}, & \varphi \wedge \psi & \text{is} & \varphi \& (\varphi \to \psi),\\
\varphi \& \psi & \text{is} & \neg(\varphi \to \neg\psi), & \varphi \oplus \psi & \text{is} & \neg(\neg\varphi \& \neg\psi),\\
\varphi \vee \psi & \text{is} & ((\varphi \to \psi) \to \psi), & \varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi) \& (\psi \to \varphi).
\end{array}
$$

The axioms of Łukasiewicz logic are the following:

(Ł1) $\varphi \to (\psi \to \varphi)$, (Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$, (Ł4) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$.

The only inference rule is *modus ponens*, i.e.: from $\varphi \to \psi$ and φ derive ψ.

For each $n \in \mathbb{N}$, the n-valued Łukasiewicz logic Ł_n is the schematic extension of Ł with the axiom schemas:

(Ł5) $(n-1)\varphi \leftrightarrow n\varphi$, (Ł6) $(k\varphi^{k-1})^n \leftrightarrow n\varphi^k$,

for each integer $k = 2, \ldots, n-2$ that does not divide $n-1$, and where $n\varphi$ is an abbreviation for $\varphi \oplus \cdots \oplus \varphi$ (n times) and φ^k is an abbreviation for $\varphi \& \ldots \& \varphi$, ($k$ times).

A *proof* in Ł (Ł_n) is a sequence $\varphi_1, \ldots, \varphi_n$ of formulas such that each φ_i either is an axiom of Ł (Ł_n) or follows from some preceding φ_j, φ_k ($j, k < i$) by modus ponens. As usual, a set of formulas is called a theory. We say that a formula φ can be derived from a theory T, denoted as $T \vdash \varphi$, if there is a proof of φ from a set $T' \subseteq T$. A theory T is said to be consistent if $T \not\vdash \overline{0}$.

The algebraic semantics for Łukasiewicz logic is given by MV-algebras [6], i.e. structures $\mathcal{A} = \langle A, \oplus, \neg, 0\rangle$ satisfying the following equations:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (MV2) $x \oplus y = y \oplus x$,
(MV3) $x \oplus 0 = x$, (MV4) $\neg\neg x = x$,
(MV5) $x \oplus \neg 0 = 0$, (MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

MV algebras can be equivalently presented as commutative bounded integral residuated lattices $\langle A, \otimes, \to, \sqcap, \sqcup, 0, 1\rangle$ satisfying (see [6, 15]):

$$(x \to y) \sqcup (y \to x) = 1; \qquad \text{(Prelinearity)}$$
$$x \sqcap y = x \otimes (x \to y); \qquad \text{(Divisibility)}$$
$$(x \to 0) \to 0 = x. \qquad \text{(Involution)}$$

Indeed, in the signature $\langle \oplus, \neg, 0\rangle$, the monoidal operation $\otimes$ can be defined as $x \otimes y := \neg(\neg x \oplus \neg y)$, while the residuum of $\otimes$ is definable as $x \to y := \neg x \oplus y$. The top element is defined as $1 := \neg 0$, and the order relation is obtained by defining $x \leq y$ iff $x \to y = 1$, while the lattice operations are given by $x \sqcap y := x \otimes (\neg x \oplus y)$ and $x \sqcup y := (x \otimes \neg y) \oplus y$.

For each $n \in \mathbb{N}$, an MV$_n$-algebra is an MV-algebra that satisfies the equations:

(MV7) $(n-1)x = nx$ (MV8) $(kx^{k-1})^n = nx^k$

for each integer $k = 2, \ldots, n-2$ not dividing $n-1$, and where nx is an abbreviation for $x \oplus \cdots \oplus x$ (n times), and x^k is an abbreviation for $x \otimes \cdots \otimes x$, ($k$ times), with $x \otimes y := \neg(\neg x \oplus \neg y)$.

The class of MV-algebras (MV$_n$) forms a variety $\mathbb{MV}$ ($\mathbb{MV}_n$) that also is the equivalent algebraic semantics for Ł (Ł$_n$), in the sense of Blok and Pigozzi [4]. $\mathbb{MV}$ is generated as a quasivariety by the standard MV-algebra $[0,1]_{MV}$, i.e. the MV-algebra over the real unit interval $[0,1]$, where $x \oplus y = \min(x+y, 1)$, and $\neg x = 1 - x$. Each $\mathbb{MV}_n$ is generated by the linearly ordered MV-algebra over the set $S_n = \{0, 1/n, \ldots, (n-1)/n, 1\}$ and whose operations are those of the MV-algebra over $[0,1]$, restricted to S_n.

Interesting examples of MV-algebras are the so-called Łukasiewicz *clans* of functions. Given a non-empty set X, consider the set of functions $[0,1]^X$ endowed with the pointwise extensions of the operations of the standard MV-algebra $[0,1]_{MV}$. Then a (Łukasiewicz) *clan* over X is any subalgebra $\mathcal{C} \subseteq [0,1]^X$, i.e. a set such that

1. if $f, g \in \mathcal{C}$ then $f \oplus g \in \mathcal{C}$,
2. if $f \in \mathcal{C}$ then $\neg f \in \mathcal{C}$,
3. $\overline{0} \in \mathcal{C}$.

Similarly, one can define an Ł$_n$-clan of functions over some set X to be any subalgebra $\mathcal{C} \subseteq (S_n)^X$.

Let *Form* denote the set of Łukasiewicz logic formulas. An evaluation e from *Form* into the standard MV-algebra $[0,1]_{MV}$ is a mapping $e : \mathit{Form} \to [0,1]$ assigning to all propositional variables a value from the real unit interval (with $e(\overline{0}) = 0$) that can be extended to compound formulas as follows:

$$\begin{array}{rclrcl}
e(\neg\varphi) & = & 1 - e(\varphi), & e(\varphi \leftrightarrow \psi) & = & 1 - |e(\varphi) - e(\psi)| \,, \\
e(\varphi \wedge \psi) & = & \min(e(\varphi), e(\psi)), & e(\varphi \vee \psi) & = & \max(e(\varphi), e(\psi)), \\
e(\varphi \oplus \psi) & = & \min(1, e(\varphi) + e(\psi)), & e(\varphi \& \psi) & = & \max(0, e(\varphi) + e(\psi) - 1).
\end{array}$$

An evaluation e is a *model* for a formula φ if $e(\varphi) = 1$. An evaluation e is a model for a theory T, if $e(\psi) = 1$, for every $\psi \in T$. The notions of evaluation and model for Ł$_n$ are defined analogously just replacing $[0,1]$ by S_n as set of truth values.

The fact that $\mathbb{MV}$ is the equivalent algebraic semantics for Łukasiewicz logic and is generated as a quasivariety by the standard MV-algebra implies that the Łukasiewicz logic is finitely strongly standard complete, i.e.: for every finite theory T and every formula φ, $T \vdash \varphi$ iff every model e of T also is a model of φ.

Rational Łukasiewicz logic RŁ is an expansion of Łukasiewicz logic introduced by Gerla in [13], obtained by adding the unary connectives δ_n, for each $n \in \mathbb{N}$, plus the following axioms:

$$\text{(D1)}\ \underbrace{\delta_n\varphi \oplus \cdots \oplus \delta_n\varphi}_{n} \leftrightarrow \varphi, \qquad \text{(D2)}\ \neg\delta_n\varphi \oplus \neg(\underbrace{\delta_n\varphi \oplus \cdots \oplus \delta_n\varphi}_{n-1}).$$

The algebraic semantics for RŁ is given by DMV-algebras (divisible MV-algebras), i.e. structures $\mathcal{A} = \langle A, \oplus, \neg, \{\delta_n\}_{n\in\mathbb{N}}, 0\rangle$ such that $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra and the following equations hold for all $x \in A$ and $n \in \mathbb{N}$:

$$(\delta_n 1) \;\; n(\delta_n x) = x, \qquad (\delta_n 2) \;\; \delta_n x \otimes (n-1)(\delta_n x) = 0.$$

An evaluation e of RŁ formulas into the real unit interval is just a Łukasiewicz logic evaluation extended for the connectives δ_n as follows: $e(\delta_n\varphi) = e(\varphi)/n$.

Notice that in RŁ all rationals in $[0,1]$ are definable as truth constants in the following way:

- $\overline{1/n}$ is definable as $\delta_n\overline{1}$, and
- $\overline{m/n}$ is definable as $m(\delta_n\overline{1})$

since, as easy to check, for any evaluation e, $e(\delta_n\overline{1}) = 1/n$ and $e(m(\delta_n\overline{1})) = \underbrace{(1/n) \oplus \cdots \oplus (1/n)}_{m} = m/n$.

As shown in [13], the variety of DMV-algebras is generated as a quasivariety by the standard DMV-algebra $[0,1]_{DMV}$ (i.e. the expansion of $[0,1]_{MV}$ with the δ_n operations), and hence RŁ is finitely strongly standard complete.

3 A possibilistic counterpart of states and their integral representation

In this section we consider some generalizations of possibility and necessity measures over MV-algebras, and in particular two families of such measures over MV-algebras of functions which admit an integral representation in terms of Sugeno-like integrals.

Although the real unit interval $[0,1]$ is the most usual scale for all kinds of uncertainty measures, any bounded totally ordered set can be actually used (possibly equipped with suitable operations), especially in the case of non-additive measures of a more qualitative nature like possibility and necessity measures.

Definition 3.1
Let $\mathcal{A}$ be an MV algebra, let $\mathcal{L} = (L, \leq, 0^L, 1^L)$ be a bounded totally ordered set, and let $\mu : A \to L$ be an order preserving mapping such that $\mu(0^A) = 0^L$ and $\mu(1^A) = 1^L$. Then:

- μ is called an *L-valued possibility measure* when for all $x, y \in A$

$$\mu(x \vee y) = \max(\mu(x), \mu(y)),$$

- μ is called an *L-valued necessity measure* when for all $x, y \in A$

$$\mu(x \wedge y) = \min(\mu(x), \mu(y)).$$

In what follows we will restrict ourselves to expanded scales $\mathcal{L}$ equipped with the operations $\oplus_L$ and $\neg_L$ making the structure $\mathcal{L} = (L, \oplus_L, \neg_L, 0^L)$ into a (linearly-ordered) MV-algebra. In particular one can choose $\mathcal{L}$ to be the standard MV-algebra $[0,1]_{MV}$ or the standard n-valued MV-algebra $\mathcal{S}_n$.

Definition 3.2
Let μ^* and μ_* be respectively an L-valued possibility and an L-valued necessity over an MV-algebra $\mathcal{A}$. The pair $\langle \mu_*, \mu^* \rangle$ will be called an *L-valued possibilistic state* on $\mathcal{A}$ whenever μ^* and μ_* are a dual pair, i.e. when $\mu^*(x) = \neg_L \mu_*(\neg_A x)$ for all $x \in A$.

Notation: *Since in a possibilistic state $\langle \mu_*, \mu^* \rangle$ one mapping is completely determined by the other, sometimes we may indistinctively refer, by abuse of language, to either μ^* or μ_* as a possibilistic state.*

Let us introduce two notable examples of L-valued possibilistic states on MV-algebras that will play a major role in this work. Let X be a finite set, and let us consider the particular clan over X of the MV-algebra of all functions of X on L, i.e. the algebra $\mathcal{L}^{\mathcal{X}} = (L^X, \oplus_L, \neg_L, \overline{0}, \overline{1})$ whose operations are the pointwise extensions[2] of the operations over L. Moreover, let $\pi : X \to L$ be a normalized[3] possibility distribution. Then, the two pairs of mappings $\mu^*, \mu_* : L^X \to L$ defined below are examples of L-valued possibilistic states:

$$\text{(E1)} \quad \mu^*(f) = \max_{x \in X} \ \pi(x) \otimes_L f(x) \qquad \mu_*(f) = \min_{x \in X} \ \neg_L \pi(x) \oplus_L f(x);$$

$$\text{(E2)} \quad \mu^*(f) = \max_{x \in X} \ \min(\pi(x), f(x)) \quad \mu_*(f) = \min_{x \in X} \ \max(\neg_L \pi(x), f(x)).$$

The above examples are adaptations of existing ones in the literature extending (classical) possibility and necessity measures for fuzzy sets in the framework of possibility theory (see e.g. [9, 8, 14]). Indeed, notice that both (E1) and (E2) are generalizations of the definition of possibility and necessity measures over Boolean algebras. Actually, as we will now see, these two classes of possibilistic states can be represented by a special kind of *fuzzy integrals*, called (generalized) *Sugeno integrals* [22].

Given a L-valued measure $\mu : 2^X \to L$, the *Sugeno integral* of a function $f : X \to L$ with respect to μ is defined as

$$\oint_S f \ d\mu = \max_{i=1,\ldots,n} \ \min(f(x_{\sigma(i)}), \mu(A_{\sigma(i)}))$$

where σ is a permutation of the indices such that $f(x_{\sigma(1)}) \geq f(x_{\sigma(2)}) \geq \ldots \geq f(x_{\sigma(n)})$, and $A_{\sigma(i)} = \{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\}$.

When μ is the (classical) possibility measure on 2^X induced by a (normalized) possibility distribution $\pi : X \to L$, i.e. when $\mu(A) = \max\{\pi(x) \mid x \in A\}$ for every $A \subseteq X$, then the above expression of the Sugeno integral becomes (see e.g. [5])

$$\oint_S f \ d\pi = \max_{x \in X} \ \min(\pi(x), f(x)).$$

When the above minimum operation is replaced by the MV-operation $\otimes_L$ (that interprets the strong conjunction in Łukasiewicz logic), we obtain the so-called generalized

[2] That is, for all $x \in X$ we define $(f \oplus_L g)(x) := f(x) \oplus_L g(x)$, $(f \otimes_L g)(x) := f(x) \otimes_L g(x)$, $(f \vee_L g)(x) := \max(f(x), g(x))$, $(f \wedge_L g)(x) := \min(f(x), g(x))$ and $(\neg_L f)(x) := \neg_L f(x)$; moreover, for each $r \in X$, we will denote by $\overline{r}$ the constant function of value r, i.e. $\overline{r}(x) := r$ for all $x \in X$.

[3] That is, $\max_{x \in X} \pi(x) = 1^L$.

Sugeno integral [22]

$$\oint_{S,\otimes} f\ d\mu = \max_{i=1,\dots,n}\ f(x_{\sigma(i)}) \otimes_L \mu(A_{\sigma(i)}),$$

which, in the case of μ being the possibility measure on 2^X defined by a possibility distribution π becomes

$$\oint_{S,\otimes} f\ d\pi = \max_{x\in X}\ \pi(x) \otimes_L f(x).$$

Therefore, (E2) and (E1) can be seen as two possible definitions of possibilistic states over MV-algebras of functions under the form of a Sugeno and a generalized Sugeno integral, respectively. Notice that (E2) does not actually require the whole MV-algebraic structure of the scale $\mathcal{L}$, since it makes use only of the linear ordering and the involutive negation operation. (E2), instead, specifically relies on the MV-operations.

The next theorem offers an axiomatic characterization of those measures for which there exists a possibility distribution that allows to represent them either in the form of a Sugeno or a generalized Sugeno integral (cf. [2]).

THEOREM 3.3
Let $\mathcal{L}'$ be an MV-subalgebra of the MV-chain $\mathcal{L}$, and let ${\mathcal{L}'}^{\mathcal{X}}$ be the Łukasiewicz clan of all L'-valued functions over some finite set X. Let $\langle \mu_*, \mu^*\rangle$ be an L-valued possibilistic state over ${\mathcal{L}'}^{\mathcal{X}}$. Then, there exists a normalized possibility distribution $\pi : X \to L$ such that:

(i) $\mu^*(f) = \oint_{S,\otimes} f d\pi$ and $\mu_*(f) = \neg_L \oint_{S,\otimes} (\neg_L f) d\pi = \min_{x\in X}(\neg_L \pi(x) \oplus_L f(x))$,

if and only if

$$\mu_*(\overline{r} \oplus_L f) = r \oplus_L \mu_*(f)$$

for all $r \in L'$;

(ii) $\mu^*(f) = \oint_{S} f d\pi$ and $\mu_*(f) = \neg_L \oint_{S} (\neg_L f) d\pi = \min_{x\in X}(\max(\neg_L \pi(x), f(x)))$,

if and only if

$$\mu_*(\overline{r} \vee f) = \max(r, \mu_*(f))$$

for all $r \in L'$,

where $\overline{r}$ stands for the constant function of value r, i.e. $\overline{r}(x) = r$ for all $x \in X$.

PROOF. (i) Suppose that $\langle \mu_*, \mu^*\rangle$ is a possibilistic state over ${\mathcal{L}'}^{\mathcal{X}}$ such that

$$\mu_*(\overline{r} \oplus_L f) = \overline{r} \oplus_L \mu_*(f).$$

It is easy to check that every $f \in {L'}^X$ can be written as

$$f = \bigwedge_{x\in X} \mathbf{x}^c \oplus_L \overline{f(x)},$$

where $\mathbf{x}^c : X \to L'$ is the characteristic function of the complement of the singleton $\{x\}$, i.e. $\mathbf{x}^c(y) = 1^L$ if $y \neq x$ and $\mathbf{x}^c(x) = 0^L$, and $\overline{f(x)}$ stands for the constant function of value $f(x)$.

Now, by applying the axioms of possibilistic states and the assumption that $\mu_*(\overline{r} \oplus f) = \overline{r} \oplus_L \mu_*(f)$, we obtain that

$$\mu_*(f) = \mu_*(\bigwedge_{x \in X} \mathbf{x}^c \oplus_L \overline{f(x)}) = \min_{x \in X} \ \mu_*(\mathbf{x}^c \oplus_L \overline{f(x)}) = \min_{x \in X} \ \mu_*(\mathbf{x}^c) \oplus_L f(x).$$

By putting $\pi(x) = \neg_L \mu_*(\mathbf{x}^c)$, we get

$$\mu_*(f) = \min_{x \in X} \ \neg_L \pi(x) \oplus_L f(x),$$

which, of course, by duality implies that

$$\mu^*(f) = \oint_{X,\otimes} f \ d\pi.$$

The converse is easy.

(ii) The proof is completely analogous to (i), only noticing that

$$\mathbf{x}^c \oplus_L \overline{f(x)} = \mathbf{x}^c \vee \overline{f(x)},$$

since $\mathbf{x}^c$ is a $\{0^L, 1^L\}$-valued function (see [3, 2]). ∎

4 The logic $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ and its semantics

In the rest of the paper we aim at defining and studying properties of completeness and complexity of modal many-valued logics to reason about the necessity of many-valued events (in the sense of the possibilistic states (E1) and (E2) introduced in the previous section), more precisely of formulas of finitely-valued Łukasiewicz logic.

We first consider the logic $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ for dealing with possibilistic states of type (E1). $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is based on the Rational Łukasiewicz logic RŁ, and on the n-valued Lukasiewicz logic expanded with the truth constant $\overline{1/n}$, which will be denoted as Ł_n^+ [4]. Formulas of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ split into two classes: (i) the set $Fm(V)$ of non-modal formulas $\varphi, \psi \ldots$, which are formulas of Ł_n^+ built from set of propositional variables $V = \{p_1, p_2, \ldots\}$ and the truth constant $\overline{1/n}$; (ii) the set $MFm(V)$ of modal formulas $\Phi, \Psi \ldots$, built from atomic modal formulas $N\varphi$, with $\varphi \in Fm(V)$ and N denoting the modality *necessity*, using RŁ. Notice that nested modalities are not allowed.

The axioms of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ are the axioms of Ł_n^+ for non-modal formulas, the axioms of RŁ for modal formulas, plus the following possibilistic state related axioms:

$(N1)$ $\neg N\bot$
$(N2)$ $N(\varphi \wedge \psi) \leftrightarrow (N\varphi \wedge N\psi)$
$(N3)$ $N(\overline{r} \oplus \psi) \leftrightarrow \overline{r} \oplus N\psi$, for each $r \in \{0, 1/n, \ldots, (n-1)/n, 1\}$.

The rules of inference of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ are *modus ponens* (for modal and non-modal formulas); *necessitation*: from from φ derive $N\varphi$; and *monotonicity*: from $\varphi \to \psi$ derive $N(\varphi) \to N(\psi)$.

[4] This logic is axiomatized by adding to the axioms of Ł_n the axioms $n(\overline{1/n})$ and $\neg(\overline{1/n} \otimes (n-1)(\overline{1/n}))$.

The notion of proof in $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$, denoted by $\vdash_N$, is defined as usual from the above axioms and rules.

The semantics of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is given by weak and strong possibilistic Kripke models. A *weak possibilistic Kripke model* (or weak model) for $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is a system $\mathcal{M} = \langle W, e, I\rangle$ where:

- W is a non-empty set whose elements are called *nodes* or *worlds*,
- $e : W \times V \to \{0, 1/n, \ldots, (n-1)/n, 1\}$ is such that, for each $w \in W$, $e(w, \cdot) : V \to \{0, 1/n, \ldots, (n-1)/n, 1\}$ is an evaluation of propositional variables which can be extended to an Ł_n^+-evaluation of (non-modal) formulas of $Fm(V)$ in the usual way.
- For each $\varphi \in Fm(V)$ we define its associated function $\hat{\varphi}_W : W \to \{0, 1/n, \ldots, (n-1)/n, 1\}$, where $\hat{\varphi}_W(w) = e(w, \varphi)$. The set $Fm_W = \{\hat{\varphi}_W \mid \varphi \in Fm(V)\}$ is a clan over W.
- $I : Fm_W \to [0, 1]$ is a possibilistic state over the clan Fm_W, i.e. it satisfies
  (i) $I(\hat{\top}_W) = 1$, $I(\hat{\bot}_W) = 0$,
  (ii) $I(\hat{\varphi}_W \wedge \hat{\psi}_W) = \min(I(\hat{\varphi}_W), I(\hat{\psi}_W))$.
  Moreover, I satisfies the following additional decomposition property:
  (iii) $I(\hat{\overline{r}}_W \oplus \hat{\psi}_W) = r \oplus I(\hat{\psi}_W)$ for each $r \in \{0, 1/n, \ldots, (n-1)/n, 1\}$.

Now, given a formula Φ and a world $w \in W$, the truth value of Φ in $\mathcal{M} = \langle W, e, I\rangle$ at the node w, denoted $\|\Phi\|_{\mathcal{M},w}$, is inductively defined as follows:

- If Φ is a non-modal formula φ, then $\|\varphi\|_{\mathcal{M},w} = e(w, \varphi)$,
- If Φ is an atomic modal formula $N\varphi$, then $\|N\varphi\|_{\mathcal{M},w} = I(\hat{\varphi}_W)$
- If Φ is a non-atomic modal formula, then its truth value is computed by evaluating its atomic modal subformulas, and then by using the truth functions associated to the RŁ-connectives occurring in Φ.

A *strong possibilistic Kripke model* is a system $\mathcal{N} = (W, e, \pi)$ where W and e are defined as in the case of a weak possibilistic Kripke model and π is a possibility distribution on W, i.e. $\pi : W \to [0, 1]$, satisfying $\max_{w \in W} \pi(w) = 1$. Evaluations of formulas of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ in a strong possibilistic Kripke model $\mathcal{N}$ are defined as in the case of weak models except for the case of atomic modal formulas, i.e.:

- if Φ is an atomic modal formula $N(\psi)$, then

$$\|N\psi\|_{\mathcal{N},w} = \inf_{w \in W} (1 - \pi(w)) \oplus e(w, \psi).$$

It is worth mentioning that each strong possibilistic model $\mathcal{M} = \langle W, e, \pi\rangle$ induces a weak possibilistic model $\mathcal{M}' = \langle W, e, I_\pi\rangle$, where $I_\pi : Fm_W \to [0, 1]$ is defined as

$$I_\pi(\hat{\varphi}_W) = \inf\nolimits_{w \in W} (1 - \pi(w)) \oplus e(w, \varphi),$$

which is equivalent to $\mathcal{M}$ in the sense that $\|\Phi\|_{\mathcal{M},w} = \|\Phi\|_{\mathcal{M}',w}$ for any modal formula Φ and any $w \in W$[5]. Due to Theorem 3.3, the converse also holds when W is finite.

[5] Actually, the evaluation of modal formulas in both weak and strong models does not depend on any particular world.

The notions of *model* and 1-*validity* of a formula in a (weak or strong) model or in a class of (weak or strong) models are defined as usual. A (weak or strong) possibilistic Kripke model is said to be *finite* if its corresponding set of worlds W is finite.

It is easy to show that axioms (N1), (N2) and (N3) are 1-valid in each weak or strong possibilistic Kripke model, and the monotonicity and necessity inference rules preseve validity. Notice that that the analogue of the modal logic axiom K for the modality N,

$$N(\varphi \to \psi) \to (N\varphi \to N\psi),$$

is not 1-valid in the class of strong possibilistic Kripke models, as already noticed in [17]. Indeed, let $\mathcal{M} = \langle W, \pi, e\rangle$ be a strong model for $\mathrm{N}(\mathrm{Ł}_3^+, \mathrm{RŁ})$ such that:

- $W = \langle w_1, w_2\rangle$,
- $\pi(w_1) = 0.5,\ \pi(w_2) = 1$,
- $e(w_1, p) = 0.5\ e(w_1, q) = 0,\ e(w_2, p) = 1,\ e(w_2, q) = 1$.

Then it is easy to check that $\|N(p \to q)\|_{\mathcal{M}} = 1$, $\|Np\|_{\mathcal{M}} = 1$, and $\|Nq\|_{\mathcal{M}} = 0.5$, which implies that $\|N(p \to q) \to (Np \to Nq)\|_{\mathcal{M}} = 0.5 < 1$. In the next section we will show that $\mathrm{N}(\mathrm{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is complete w.r.t. both weak and strong models. From that result, the fact that the axiom K fails also w.r.t. weak possibilistic Kripke models will immediately follow.

5 Completeness results for $\mathrm{N}(\mathbf{Ł}_\mathrm{n}^+, \mathrm{R}\mathbf{Ł})$

In this section we are going to show that the logic $\mathrm{N}(\mathrm{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is sound and complete for deductions from finite modal theories with respect to weak and strong models. Also, we will show that completeness for infinite modal theories holds by introducing the notions of Pavelka-style and hyperreal completeness.

5.1 Completeness with respect to weak and strong models

Theorem 5.1
The logic $\mathrm{N}(\mathrm{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is finitely strongly complete with respect to the class of finite weak possibilistic Kripke models.

Proof. Let Γ be a finite modal theory, and Φ be a modal formula. Suppose that $\Gamma \not\vdash_N \Phi$. We show that there is a weak possibilistic model $\mathcal{M}$ of Γ such that $\|\Phi\|_{\mathcal{M}} < 1$. We follow the strategy adopted in [15, 12] that amounts to translating theories over $\mathrm{N}(\mathrm{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ into theories over RŁ. For each modal formula Φ, let $\Phi^\star$ be obtained from Φ by replacing every occurrence of an atomic subformula of the form $N\varphi$ by a new propositional variable p_φ. Then, inductively define the mapping $\star$ from modal formulas into RŁ-formulas as follows:

- $(N(\varphi))^\star = p_\varphi$,
- $(\Phi \to \Psi)^\star = \Phi^\star \to \Psi^\star$,
- $(\neg(\Phi))^\star = \neg(\Phi^\star)$,
- $(\delta_n(\Phi))^\star = \delta_n(\Phi^\star)$

Let, therefore, $\Gamma^\star$ and $\mathcal{F}^\star$ be defined as

$$\Gamma^\star = \{\Psi^\star \mid \Psi \in \Gamma\}$$

and

$$\mathcal{F}^\star = \{\Upsilon^\star \mid \Upsilon \text{ is an instance of } (Ni),\ i = 1,2,3\} \cup \{p_\varphi \mid \text{Ł}_n^+ \vdash \varphi\} \cup \{p_\varphi \to p_\psi \mid \text{Ł}_n^+ \vdash \varphi \to \psi\}$$

respectively.

Using the same technique used in [15] it is not difficult to prove that

$$\Gamma \not\vdash_{\mathrm{N}} \Phi \text{ iff } \Gamma^\star \cup \mathcal{F}^\star \not\vdash_{\mathrm{RŁ}} \Phi^\star. \tag{5.1}$$

Since the set $V^0 \subset V$ of propositional variables appearing in $\Gamma \cup \{\Phi\}$ is finite, without loss of generality we can assume to be working with a finitely generated (over V^0) non-modal language $Fm(V^0)$.

Notice that the Lindenbaum algebra $Fm(V^0)/{\sim_n}$, where $\sim_n$ denotes the relation of provable equivalence in Ł_n^+, is finite (see [6] for more details). This means that there are only finitely many different classes

$$[\varphi]_{\sim_n} = \{\psi \in Fm(V^0) \mid \text{Ł}_n^+ \vdash \varphi \leftrightarrow \psi\}.$$

For each $[\varphi]_{\sim_n}$, let $\varphi^\sharp$ be the representative of the class. We define a further translation:

- For each modal formula Φ, let $\Phi^\sharp$ be the formula resulting from the substitution of each propositional variable p_φ occurring in $\Phi^\star$ by $p_{\varphi^\sharp}$,
- If $\Phi = \Theta \to \Lambda$ then $\Phi^\sharp = \Theta^\sharp \to \Lambda^\sharp$.
- If $\Phi = \flat\Theta$ then $\Phi^\sharp = \flat\Theta^\sharp$ with $\flat \in \{\neg, \delta_n\}$.

Consequently, we define $\Gamma^\sharp$ and $\mathcal{F}^\sharp$ as:

$$\Gamma^\sharp = \{\Psi^\sharp \mid \Psi^\star \in \Gamma^\star\}$$

and

$$\mathcal{F}^\sharp = \{\Upsilon^\sharp \mid \Upsilon \text{ is an instance of } (\mathrm{N}i),\ i = 1,2,3\} \cup \{p_{\varphi^\sharp} \mid \text{Ł}_n \vdash \varphi\} \cup \{p_{\varphi^\sharp} \to p_{\psi^\sharp} \mid \text{Ł}_n^+ \vdash \varphi \to \psi\}.$$

As shown in [12]:

Lemma 5.2
$\Gamma^* \cup \mathcal{F}^* \vdash_{\mathrm{RŁ}} \Phi^*$ iff $\Gamma^\sharp \cup \mathcal{F}^\sharp \vdash_{\mathrm{RŁ}} \Phi^\sharp$.

Therefore, we obtain:

$$\Gamma \not\vdash_{\mathrm{N}} \Phi \quad \text{iff} \quad \Gamma^\sharp \cup \mathcal{F}^\sharp \not\vdash_{\mathrm{RŁ}} \Phi^\sharp. \tag{5.2}$$

Since $\Gamma^\sharp \cup \mathcal{F}^\sharp$ is a finite theory and since RŁ is finitely strongly complete, we know that there exists an RŁ-evaluation v that is a model of $\Gamma^\sharp \cup \mathcal{F}^\sharp$ such that $v(\Phi^\sharp) < 1$.

Let now $\mathcal{M}$ be the system $\mathcal{M} = \langle \Omega_n, e, I\rangle$, where Ω_n is the class of all the Ł_n^+-evaluations of the propositional variables in V^0; $e : \Omega_n \times V^0 \to \{0, 1/n, \ldots, n-1/n, 1\}$ is defined as $e(w, q) = w(q)$ if $q \in V^0$ and $e(w, q) = 0$ otherwise; $I : Fm_{\Omega_n} \to [0,1]$ is defined as $I(\hat{\varphi}_{\Omega_n}) = v(p_{\varphi^\sharp})$. For the sake of simplicity, we will write $\hat{\varphi}$ instead of $\hat{\varphi}_{\Omega_n}$. Notice that Ω_n is finite. In fact, V^0 is finite, and so there are only finitely many functions from V^0 into $\{0, 1/n, \ldots, n-1/n, 1\}$. In order to prove that $\mathcal{M}$ is a weak possibilistic Kripke model for $\mathrm{N}(\text{Ł}_{\mathrm{n}}^+, \mathrm{RŁ})$ we just have to show that I is indeed a possibilistic state:

- If $Ł_n^+ \vdash \varphi \leftrightarrow \psi$, then $Ł_n^+ \vdash \varphi^\sharp \leftrightarrow \psi^\sharp$, and so RŁ $\vdash p_{\varphi^\sharp} \leftrightarrow p_{\psi^\sharp}$. Obviously $I(\hat{\varphi}) = I(\hat{\psi})$.
- Similarly, if $Ł_n^+ \vdash \varphi$, then $I(\hat{\varphi}) = 1$; and if $Ł_n^+ \vdash \varphi \to \psi$, then $I(\hat{\varphi}) \leq I(\hat{\psi})$.
- Since $Ł_n^+ \vdash \neg\bot$, then RŁ $\vdash \neg p_{\bot^\sharp}$, which means $I(\hat{\bot}) = 0$.
- As an instance of axiom N2 we have $p_{\varphi\wedge\psi} \leftrightarrow p_\varphi \wedge p_\psi$. Then, it is easy to see that $I(\hat{\varphi} \wedge \hat{\psi}) = \min(I(\hat{\varphi}), I(\hat{\psi}))$.
- Finally, from axiom N3, we get that $p_{\overline{r}\oplus\psi} \leftrightarrow \overline{r} \oplus p_\psi$, and consequently $I(\hat{\overline{r}} \oplus \hat{\varphi}) = \hat{r} \oplus I(\hat{\varphi})$.

Then we obtain that $\mathcal{M}$ is a finite weak possibilistic Kripke model for $\mathrm{N}(Ł_\mathrm{n}^+, \mathrm{RŁ})$. Moreover it is fairly easy to observe that $\mathcal{M}$ clearly is a model for Γ, but $\|\Phi\|_\mathcal{M} < 1$. This ends the proof of the theorem. ∎

We now proceed to prove completeness w.r.t. strong Kripke models.

Theorem 5.3
The logic $\mathrm{N}(Ł_\mathrm{n}^+, \mathrm{RŁ})$ is finitely strongly complete with respect to the class of strong possibilistic finite Kripke models.

Proof. Let $\mathcal{M}$ be the finite weak possibilistic model (Ω_n, e, I) defined in the proof of the previous theorem. Theorem 5.1 shows that I is indeed a possibilistic state over the clan of $Ł_n^+$-evaluations over the formulas $Fm(V^0)$. Since $I(\hat{\overline{r}} \oplus \hat{\varphi}) = \hat{r} \oplus I(\hat{\varphi})$, and since Ω_n is finite, Theorem 3.3 ensures the existence of a normalized possibility distribution $\pi : \Omega_n \to [0, 1]$ such that

$$I(\hat{\varphi}) = \bigwedge_{w\in\Omega_n} \neg\pi(w) \oplus \hat{\varphi}(w).$$

Hence, $\mathcal{N} = \langle\Omega_n, e, \pi\rangle$ is the desired strong (finite) model. ∎

5.2 Pavelka-style and hyperreal completeness

From Theorem 5.1 it can be seen that completeness fails for deductions from infinite theories. In fact, if Γ is an infinite modal theory such that $\Gamma \not\vdash \Phi$, then the propositional theory $\Gamma^\sharp \cup N^\sharp$ is infinite as well, hence, since RŁ is not strongly standard complete, the previous strategy does not allow us to define a weak (strong) possibilistic model where Φ fails to be true. Also notice that, if we consider the logic N(RŁ, RŁ), obtained replacing the finitely-valued logic $Ł_n^+$ by the infinitely-valued logic RŁ as the logic for the events[6], then the translation $\sharp$ defined in the proof of Theorem 5.1, would lead to an infinite theory $\Gamma^\sharp \cup N^\sharp$, even if the modal theory Γ is finite[7].

This observation points out that, in order to recover a notion of strong completeness for both $\mathrm{N}(Ł_\mathrm{n}^+, \mathrm{RŁ})$, and N(RŁ, RŁ), we have to look for (possibly different) notions of strong completeness for RŁ.

The first notion we are going to use is due to Hájek [15]. Whenever a logic $\mathcal{L}$ allows the definition of rational truth constants in its language (as is the case of RŁ), it

[6] Where axiom (N3) is considered for every rational $r \in [01] \cap \mathbb{Q}$.

[7] This depends on the fact that the variety of DMV-algebras is not *locally finite* (cf. [6, 13]), then, if V^0 is a finite set of variables, the Lindenbaum DMV-algebra $Fm(V^0)/\sim$, is not finite.

makes sense to consider completeness *a la* Pavelka. Let Γ be a (finite or countable) theory of $\mathcal{L}$, and let Φ be a formula. Then,

- the *provability degree* of Φ over Γ is defined as as

$$|\Phi|_\Gamma = \sup\{r \in [0,1] \cap \mathbb{Q} \mid \Gamma \vdash_{\mathcal{L}} \overline{r} \to \Phi\},$$

- the *truth degree* of Φ over Γ is defined as as

$$\|\Phi\|_\Gamma = \inf\{v(\Phi) \mid v \text{ is a model for } \Gamma\}.$$

If, for every theory Γ and every formula Φ, $|\Phi|_\Gamma = \|\Phi\|_\Gamma$, then $\mathcal{L}$ is said to be *Pavelka-style complete*. Clearly, Pavelka completeness is a *strong completeness* in the sense that no restriction is imposed on the cardinality of Γ.

Gerla proved in [13] that the logic RŁ enjoys Pavelka-style completeness. From this fact, and defining the truth degree of a modal formula Φ over a modal theory Γ by means of weak possibilistic models, it is fairly easy to prove that:

Theorem 5.4
The logics $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ and $\mathrm{N}(\mathrm{RŁ}, \mathrm{RŁ})$ are Pavelka-style complete w.r.t. weak possibilistic models.

Proof. We display the proof for $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ only, since the case of $\mathrm{N}(\mathrm{RŁ}, \mathrm{RŁ})$ is analogous (cf. [10, Theorem 5.5.2]).

Let $\Gamma \cup \{\Phi\}$ be an arbitrary modal theory of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$. Let $\Gamma^\sharp$, $N^\sharp$ and $\Phi^\sharp$ be obtained as in the proof of Theorem 5.1. Since RŁ enjoys Pavelka-style completeness,

$$|\Phi^\sharp|_{\Gamma^\sharp \cup N^\sharp} = \|\Phi^\sharp\|_{\Gamma^\sharp \cup N^\sharp}.$$

Moreover it is easy to see (cf. (5.2)) that $|\Phi|_\Gamma = |\Phi^\sharp|_{\Gamma^\sharp \cup N^\sharp}$, hence, to prove the claim, it is sufficient to show that $\|\Phi^\sharp\|_{\Gamma^\sharp \cup N^\sharp} = \|\Phi\|_\Gamma$. This follows from the fact that every $[0,1]$-model for $\Gamma^\sharp \cup N^\sharp$ can be translated into a weak possibilistic model for Γ and vice versa. ∎

We introduce now a second notion of strong completeness for $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ and $\mathrm{N}(\mathrm{RŁ}, \mathrm{RŁ})$, namely a strong completeness with respect to weak possibilistic models. A *hyperreal weak possibilistic model* is a structure (W, e, I^*) where:

- W and e are as in the above considered possibilistic Kripke models with the only exception that, for the case of a model for $\mathrm{N}(\mathrm{RŁ}, \mathrm{RŁ})$, $e : W \times V \to [0,1]$ is such that, for each $w \in W$, $e(w, \cdot) : V \to [0,1]$ is an evaluation of propositional variables which extends to an RŁ-evaluation of (non-modal) formulas of $Fm(V)$ in the usual way.
- $I^* : Fm_W \to [0,1]^*_{DMV}$ is a possibilistic state taking value in the DMV-algebra $[0,1]^*_{DMV}$, the latter being a non trivial ultrapower of the standard DMV-algebra $[0,1]_{DMV}$.

The main result which will allows us to show completeness of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ and $\mathrm{N}(\mathrm{RŁ}, \mathrm{RŁ})$ with respect to the classes of hyperreal-valued weak possibilistic Kripke models introduced above is the following:

Lemma 5.5 ([11])
The logic RŁ is sound and strongly complete with respect to the class of DMV-algebras defined as ultrapowers of $[0,1]_{DMV}$. In other words, if $\Gamma \cup \{\phi\}$ is an arbitrary theory of RŁ such that $\Gamma \not\vdash_{\mathrm{RL}} \varphi$, then there exists a non-trivial ultrapower $[0,1]^*_{DMV}$ of $[0,1]_{DMV}$ and a evaluation v into $[0,1]^*_{DMV}$, such that $v(\gamma) = 1$ for all $\gamma \in \Gamma$, and $v(\varphi) < 1$.

Theorem 5.6
The logics $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ and $\mathrm{N}(\text{RŁ}, \text{RŁ})$ are strongly complete w.r.t. the related classes of hyperreal weak possibilistic finite Kripke models.

Proof. The strong completeness of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ with respect to hyperreal weak possibilistic models is easily provable using Lemma 5.5 and the proof of Theorem 5.1. Therefore, if $T \cup \{\Phi\}$ is a countable theory of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ such that $T \not\vdash_N \Phi$, let $\mathcal{M} = (W, e, I^*)$ be a hyperreal weak possibilistic model such that $\|T\|_{\mathcal{M}} = 1$, and $\|\Phi\|_{\mathcal{M}} < 1$.

The case of $\mathrm{N}(\text{RŁ}, \text{RŁ})$ is analogous and left to the reader. ∎

6 The logic $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$

In the two previous sections we have been concerned with the logic $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ which captures reasoning about possibilistic states of type (E1). In this section we turn our attention to a logic to reason about possibilistic states of the type (E2), directly related to Sugeno integrals. Therefore, we introduce the logic $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$, where Q stands for *qualitative*, whose axioms are those of Ł_n^+ for non-modal formulas, the axioms of RŁ for modal formulas, plus the following possibilistic state related axioms:

$$
\begin{array}{rll}
(K) & N(\varphi \to \psi) \to (N\varphi \to N\psi) & \\
(N1) & \neg N\bot & \\
(N2) & N(\varphi \wedge \psi) \leftrightarrow (N\varphi \wedge N\psi) & \\
(QN3) & N(\overline{r} \vee \psi) \leftrightarrow \overline{r} \vee N\psi, & \text{for each } r \in \{0, 1/n, \ldots, (n-1)/n, 1\}
\end{array}
$$

The rules of inference are *modus ponens* (for modal and non-modal formulas) and *necessitation*, i.e., from from φ derive $N\varphi$.

It is worth pointing out that the most relevant differences between $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ and $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ are the axiom $(QN3)$ (that differs from $(N3)$) and the axiom (K) (from which the monotonicity rule is now derivable)[8]. Nevertheless, the analogy with $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ is evident. For this reason, we will merely outline the Kripke-based semantics for $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ and formulate completeness results without proofs, that can be easily retrieved by following the same argument carried out for $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$.

The semantics of $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ is given by corresponding weak and strong possibilistic Kripke models. A *weak q-possibilistic Kripke model* (or weak model) for $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ is a system $\mathcal{M} = \langle W, e, I\rangle$ where:

- W, e are as in the possibilistic models of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$
- $I : Fm_W \to [0,1]$ is a possibilistic state over the clan $Fm_W = \{\hat{\varphi}_W \mid \varphi \in Fm(V)\}$ which further satisfies:

[8]Notice that the axiom K it is sound under the $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ semantics (see [17, Lemma 5.1]), while, as shown above, it is not sound under the $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \text{RŁ})$ semantics.

– $I(\hat{\bar{r}}_W \vee \hat{\psi}_W) = \max(r, I(\hat{\psi}_W))$.

A *strong q-possibilistic Kripke model* is a system $\mathcal{N} = (W, e, \pi)$ where W and e are defined as in the case of a weak possibilistic Kripke model and π is a possibility distribution on W, i.e. $\pi : W \to \{0, 1/n, .., 1\}$ satisfying $\max_{w\in W} \pi(w) = 1$. Evaluations of formulas of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ in a strong possibilistic Kripke model $\mathcal{N}$ are defined as in the case of weak models except for the case of atomic modal formulas, i.e.:

- if Φ is an atomic modal formula $N(\psi)$, then

$$\|N\psi\|_{\mathcal{N}} = \inf_{w\in W} \max(1 - \pi(w), e(w, \psi)).$$

Following the strategy adopted for $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$, we can prove:

Theorem 6.1
The logic $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is sound and finitely strongly complete with respect to the class of weak and strong q-possibilistic (finite) Kripke models. Moreover, $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ is Pavelka-style compete and sound and strongly complete w.r.t. the class of hyperreal weak q-possibilistic models.

7 Complexity issues

In this section we will provide results concerning the complexity of the set of theorems and of satisfiable formulas of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ and $\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$. First of all, we need to recall some definitions and preliminary results.

Let A be a rational matrix having m rows and n columns, b be a rational column vector, c be a rational row vector, and d be a rational number. The tuple (A, b, c, d) is said to be a *particular LP-problem* whose size is the number of bits necessary to represent all those rationals as fractions of dyadic numbers. The *general* LP-problem reads: given any particular LP-problem (A, b, c, d), does the system $Ax \le b$, $cx > d$ have a solution?, i.e. is there a vector $x = (x_1, \dots, x_n)$ such that $\sum_{k=1}^{n} a_{lk}x_k \le b_k$ for each l, and $\sum_{k=1}^{n} c_k x_k > d$? As proved in [20], the general LP-problem is in NP.

A *particular MIP-problem* (MIP-problem for short) is a tuple (A, b, c, d, k) where (A, b, c, d) is a particular LP-problem and k represents the additional condition saying that $x_k, \dots, x_n$ must be Boolean (zeros or ones).

Lemma 7.1 ([15])
Any MIP problem is NP-complete.

Definition 7.2
Let Φ be a modal formula in the language of $\mathrm{N}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ ($\mathrm{QN}(\text{Ł}_\mathrm{n}^+, \mathrm{RŁ})$ respectively). Then Φ is said to be:

- *satisfiable* if there is a weak (strong) possibilistic model $\mathcal{M}$ such that $\|\Phi\|_{\mathcal{M}} = 1$;
- (<1)-*satisfiable* if there is a weak (strong) possibilistic model $\mathcal{M}$ such that $\|\Phi\|_{\mathcal{M}} < 1$;

Of course the previous definition applies to any logic whose semantics is based on (a subset) of the unit interval $[0, 1]$.

In [13], Gerla showed that the problem of checking if a formula φ of RŁ is either satisfiable or (<1)-satisfiable is in NP, since it can be reduced to the solvability of a MIP-problem.

LEMMA 7.3
Let Φ be a formula of $\mathrm{N}(\mathrm{Ł_n}, \mathrm{RŁ})$ ($\mathrm{QN}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ respectively). Then:

(1) Φ is satisfiable in a strong (weak) possibilistic model iff Φ is satisfied by a finite weak (strong) possibilistic model.
(2) Φ is (<1)-satisfiable in a weak (strong) possibilistic model iff Φ is (<1)-satisfied in a finite weak (strong) possibilistic model.

PROOF. (1). One direction is trivially true. In order to prove the other one, let Φ be a formula of $\mathrm{N}(\mathrm{Ł_n}, \mathrm{RŁ})$, and let $\mathcal{M}' = (W', e', \pi')$ be a strong model which satisfies Φ. Let V^0 be the finite set of propositional variables occurring in Φ. Now, let W be the finite set of all the functions form V^0 into $\{0, 1/n, \ldots, (n-1)/n, 1\}$. Define $e : W \times V \to \{0, 1/n, \ldots, (n-1)/n, 1\}$ as $e(w, p) = w(p)$, if $p \in V^0$, and 0 otherwise. Let $[w]_{V_0} = \{w' \in W' \mid \forall p \in V^0, e(w, p) = e'(w', p)\}$. Finally, let $\pi : W \to [0, 1]$ be defined as follows: for all $w \in W$,

$$\pi(w) = \max\{\pi'(w') \mid w' \in [w]_{V_0}\}.$$

Call $\mathcal{M} = (W, e, \pi)$ the finite model so defined. Then, the claim is that for each modal formula Ψ whose underlying set of propositional variables is in V_0, $\|\Psi\|_{\mathcal{M}'} = \|\Psi\|_{\mathcal{M}}$. Actually, we have to show the claim for atomic modal formulas, the case of RŁ connectives being easy.

Let hence $\Psi = N\gamma$ where γ is built from variables in V_0. Then, noting that $e(w, \gamma) = e'(w', \gamma)$ for all $w' \in [w]_{V_0}$, the claim follows observing that

$$\|N\gamma\|_{\mathcal{M}'} = \inf_{w \in W} \neg\pi(w) \oplus e(w, \gamma) = \inf_{w \in W} \left(\inf_{w' \in [w]_{V_0}} \neg\pi'(w') \oplus e(w', \gamma) \right) = \|N\gamma\|_{\mathcal{M}}$$

holds just using the definition of π. Thus, in particular, $\|\Phi\|_{\mathcal{M}'} = \|\Phi\|_{\mathcal{M}}$, and hence $\mathcal{M} \models \phi$, and the item (1) is settled for strong models.

As for weak models, (1) easily follows considering the (finite) weak model definable from (W, e, π).

(2). If Φ is (<1)-satisfiable, then it is easy to see that there exists a $t \in \mathbb{N}$ such that $\Phi \to \overline{1 - 1/t}$ is satisfiable, and, from (1) there exists a finite model which satisfies $\Phi \to \overline{1 - 1/t}$.

The proofs for $\mathrm{QN}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ are analogous. ∎

The following is a well-known result of linear programming (see for instance [20]) that will be needed in the rest of the section.

LEMMA 7.4
If a system of s linear equalities and/or inequalities has a non-negative solution, then it has a non-negative solution with at most $s + 1$ positive entries.

Now we are ready to prove that the sets of satisfiable and (<1)-satisfiable formulas of $\mathrm{N}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ and $\mathrm{QN}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ are NP-complete.

THEOREM 7.5
The sets of satisfiable formulas, and (<1)-satisfiable formulas of $\mathrm{N}(Ł_n^+, RŁ)$ and $\mathrm{QN}(Ł_n^+, RŁ)$ are NP-complete.

PROOF. We will follow the lines of [16, Theorem 2.1].

We begin with $\mathrm{N}(Ł_n^+, RŁ)$. Let $\Phi(N\psi_1, \ldots, N\psi_k)$ be a formula in $\mathrm{N}(Ł_n^+, RŁ)$. A formula of RŁ is obtained whenever any $N\psi_i$ in $\Phi(N\psi_1, \ldots, N\psi_k)$ is replaced by a new propositional variable p_i. Therefore, the set of satisfiable RŁ formulas can be reduced to the set of satisfiable formulas of $\mathrm{N}(Ł_n^+, RŁ)$. This shows that our problem is NP-hard.

So as to show NP-completeness, consider the following NP-algorithm. As above, let $\Phi(p_1, \ldots, p_k)$ be the RŁ-formula obtained by replacing every Φ-subformula of the form $N\psi_i$ by p_i. Call $MIP(1)$ the MIP-problem associated to the satisfiability of $\Phi(p_1, \ldots, p_k)$ in RŁ.

Among all the variables occurring in $MIP(1)$, let $z_1, \ldots, z_k$ be those associated to the propositional variables $p_1, \ldots, p_k$.

From Lemma 7.3 (and keeping the same notation), we must get values $\pi(w)$, for $w \in \{0, 1/n, \ldots, (n-1)/n, 1\}^{V^0}$, and by Theorem 3.3 we have to require that $\pi(w)$ satisfies:

$$\max_w \ \pi(w) = 1,$$

and for all $i = 1, \ldots k$

$$z_i = \min_w \ \neg\pi(w) \oplus w(\psi_i).$$

The above conditions define a system of $k+1$ linear equations in $\{0, 1/n, \ldots, (n-1)/n, 1\}^{V^0}$ variables, which, by Lemma 7.4, admits a positive solution iff they have a solution with at most $k+2$ positive entries. This allows us to keep the size of the problem polynomial. In fact now, guess $k+1$ mutually different vectors $w_1, \ldots, w_{k+1}$ in $\{0, 1/n, \ldots, (n-1)/n, 1\}^{V^0}$, then compute the values $\alpha_{i,j} = w_j(\psi_i)$[9] (for every $i = 1, \ldots k$, and every $j = 1, \ldots, k+1$), and finally define $k+2$ equations:

$$\max_{j=1}^{k+1} \ \pi(w_j) = 1,$$

and for all $i = 1, \ldots k$,

$$z_i = \min_{j=1}^{k+1} \ \neg\pi(w_j) \oplus \alpha_{i,j}.$$

Call $MIP(2)$ the MIP-problem defined by adding to $MIP(1)$ the above conditions. Then Φ is satisfiable iff $MIP(2)$ admits a solution.

From the above, we can conclude that the set of satisfiable formlas of $\mathrm{N}(Ł_n^+, RŁ)$ is NP-hard, is in NP, and so it is NP-complete.

As for (<1)-satisfiable formulas, the claim follows replacing, in the above algorithm, $MIP(1)$ by the NP-algorithm checking if an RŁ-formula is (<1)-satisfiable (see [13]).

As for $\mathrm{QN}(Ł_n^+, RŁ)$, notice that the proof is very similar, with the difference that the condition to be satisfied by the variables z_i is for all $i = 1, \ldots k$

$$z_i = \min_w \ \max(\neg\pi(w), w(\psi_i)).$$

[9] These values can be computed in polynomial time in the length of Φ (see [16])

This ends the proof of the theorem. ∎

The following corollary can be immediately obtained observing that a formula Φ is a theorem of $\mathrm{N}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ ($\mathrm{QN}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ respectively) iff Φ is not (<1)-satisfiable.

Corollary 7.6
The sets of theorems of $\mathrm{N}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ and $\mathrm{QN}(\mathrm{Ł}_n^+, \mathrm{RŁ})$ are both co-NP-complete.

8 Final remarks

In this work we have focused both on a measure-theoretic and on a logical study of possible notions of possibilistic states over MV-algebras that generalize the notion of possibility and necessity measures commonly defined over Boolean algebras of events.

The approach adopted in this paper (and in [12]) can be followed in order to study suitable notions of measures over algebras of events which are generalizations of Boolean algebras.

In our future work we plan to investigate the logic of possibilistic states over events associated to the infinitely-valued Łukasiewicz logic. In fact, the problem of establishing completeness for this logic, with respect to the intended standard semantics, remains open.

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