

# Implication-based qualitative integrals and a logical formalization

Didier Dubois<sup>1</sup>[0000-0002-6505-2536], Lluís Godo<sup>2</sup>[0000-0002-6929-3126], and Agnès Rico<sup>3</sup>[0009-0006-0589-6557]

<sup>1</sup> IRIT - CNRS & Université de Toulouse, 118 Route de Narbonne, 31062 Toulouse cedex 09, France

`didier.dubois@irit.fr`

<sup>2</sup> IIIA-CSIC, Campus de la UAB, 08193 Bellaterra, Spain

`godo@iiia.csic.es`

<sup>3</sup> ERIC, Université de Lyon, 43 bld du 11 novembre, 69100 Villeurbanne, France

`agnes.rico@univ-lyon1.fr`

**Abstract.** We consider a general logic which attaches uncertainty degrees to declarative statements using monotonic set functions with boundary conditions. In a previous paper, Dellunde et al. considered possibility theory as a natural uncertainty framework to extend Gödel logic with a modality  $N$  such that the truth-value of a formula  $N\varphi$  can be interpreted as the degree of necessity of  $\varphi$ . A link was established between Gödel implication and such modal Gödel logic associated to a possibilistic measure. The aim of this note is to extend these results to more general monotonic set functions using Gödel implication-based integrals and reciprocal of Gödel implication-based integrals.

**Keywords:** Qualitative integral · Gödel implication · Two layer many-valued modal logic.

## 1 Introduction

Possibilistic logic in its full modal-like syntax uses Boolean logic statements  $\varphi$  embedded inside a modality connective  $N_\lambda$ , say  $N_\lambda(\varphi)$  expressing the statement that the degree of certainty (or necessity) of  $\varphi$  is no less than  $\lambda$  [6]. If instead of Boolean logic, a many-valued logic is used, it requires to evaluate the degree of certainty of a fuzzy event  $F$ , using a many-valued implication  $\rightarrow$  in the form  $N(F) = \inf_{x \in X} \pi(x) \rightarrow \mu_F(x)$  where  $\pi$  is the possibility distribution describing the epistemic state of an agent [2].

In [2] several (two-layer) modal fuzzy logics trying to capture reasoning with different notions of necessity are presented. Three of these logics are of particular interest for this work:

- 1) A logic where  $N(F)$  is defined using Gödel implication,
- 2) A logic where  $N(F)$  is defined using Kleene-Dienes implication,
- 3) A logic where  $N(F)$  is defined using the reciprocal of Gödel implication.

In [2] these logics were called  $N_{HG}(C)$ ,  $NG_{KD}(C)$  and  $NG_{RG}(C)$  respectively. In fact, the three kinds of necessities above appear to be particular cases of qualitative integrals [3, 4]. Qualitative integrals are monotonic aggregation functionals used to evaluate objects or alternatives depending on several criteria or points of view, and on a finite linearly ordered scale equipped only with an involutive negation. They have been also formalised as generalised quantifiers in [7]. Their definitions are based on monotonic set functions named capacities. Particular cases of these capacities are possibility measures (maxitive capacities) and necessity measures (minitive capacities). Qualitative integrals are defined using conjunction (called in this case Sugeno integrals) or an implication (sometimes called co-integrals). Particular cases of the latter are those defined with Kleene-Dienes conjunction or implication, Gödel implication or reciprocal Gödel implication.

In this paper our aim is to present a logical formalization generalising [2] capable of coping with reasoning about qualitative integrals. More in detail, we will present two systems of modal logics on top of a (finite-valued) Gödel fuzzy logic, one axiomatising both the Sugeno and the Gödel implication-based integrals, and another axiomatising the integral based on the reciprocal of Gödel implication. To this end, we will first provide two new characterizations of these qualitative integrals as functionals. The logical systems that we present here generalize both the fuzzy logics  $NHG(C)$  and  $NGRG(C)$  in [2] for maxitive and minitive capacities, and the modal-like logic BC in [1] to reason about Boolean capacities encapsulating formulas in propositional logic.

The paper comes as follows: Section 2 is devoted to background and notations: basic possibilities and necessities; Sugeno integrals and its variants based on Gödel implication and its reciprocal. Section 3 presents new results concerning the reciprocal of Gödel implication-based integrals. Section 4 deals with the extension of basic possibilities and basic necessities to Sugeno integrals and its implication-based variants. Section 5 presents an associated two-layer logic formalism. Section 6 contains some conclusions and prospects for future work.

## 2 Background and Notations

We consider objects evaluated according to some criteria  $\mathcal{X} = \{x_1, \dots, x_n\}$  that are valued on a finite scale  $L = \{0 = \lambda_0 < \lambda_1 < \dots < \lambda_m = 1\}$ . We assume that  $L$  is equipped with a unary order reversing involutive operation denoted by  $1 - \cdot$  and named negation. On  $L$  the minimum operator is denoted by  $\min$  and the maximum by  $\max$ , although we will also use the notations  $\wedge$  and  $\vee$  respectively without danger of confusion.

An object  $f$  may be seen as a function  $f : \mathcal{X} \rightarrow L$ , or equivalently as a vector  $f = (f_1, \dots, f_n) \in L^n$ , where  $f_i = f(x_i)$  is the local evaluation of  $f$  with respect to  $x_i$ . For any  $a \in L$ , we will denote by  $\underline{a}$  the constant function on  $X$  of value  $a$ . The set of all functions is denoted  $L^{\mathcal{X}}$ .

Along the paper, we will use  $\rightarrow_G$ ,  $\rightarrow_{KD}$  and  $\rightarrow_{rG}$  to refer respectively to Gödel, Kleene-Dienes and the reciprocal of Gödel implication on  $L$ , i.e. for all

$u, v \in L$ ,  $u \rightarrow_G v = 1$  if  $u \leq v$  and  $u \rightarrow_G v = v$  otherwise,  $u \rightarrow_{KD} v = \max(1 - u, v)$ , and  $u \rightarrow_{rG} v = (1 - v) \rightarrow_G (1 - u)$ .

## 2.1 Basic possibilities and basic necessities

In [2] some generalizations of possibility and necessity measures for fuzzy events were introduced together with their axiomatic characterizations. We recall them here customized to our setting.

**Definition 1.** A mapping  $\Lambda : L^{\mathcal{X}} \rightarrow L$  such that  $\Lambda(\underline{a}) = a$  for all  $a \in L$  is called a basic necessity if for any  $f, g \in L^{\mathcal{X}}$  it satisfies

$$- \Lambda(f \wedge g) = \min(\Lambda(f), \Lambda(g))$$

and is called a basic possibility if for any  $f, g \in L^{\mathcal{X}}$  it satisfies

$$- \Lambda(f \vee g) = \max(\Lambda(f), \Lambda(g)).$$

Then, the following characterizations are proved to hold for some particular families of basic necessities and possibilities definable from a possibility distribution on the set criteria  $\mathcal{X}$ . This is in fact equivalent to say that these basic possibilities and necessities on fuzzy subsets of  $\mathcal{X}$  (objects in our setting) are definable from a possibility or a necessity measure on (classical) subsets of  $\mathcal{X}$ .

**Theorem 1.** Let  $N : L^{\mathcal{X}} \rightarrow L$  be a basic necessity and  $\Pi : L^{\mathcal{X}} \rightarrow L$  a basic possibility. The following conditions hold:

$$- \Pi(f) = \bigvee_{x \in \mathcal{X}} \min(\pi(x), f(x)) \text{ for some possibility distribution } \pi : \mathcal{X} \rightarrow L \text{ iff } \Pi \text{ satisfies the following condition for any function } f \text{ and any } a \in L:$$

$$(1) \Pi(\underline{a} \wedge f) = \min(a, \Pi(f)) .$$

$$- N(f) = \bigwedge_{x \in \mathcal{X}} \pi(x) \rightarrow_{KD} f(x) \text{ for some possibility distribution } \pi : \mathcal{X} \rightarrow L \text{ iff } N \text{ satisfies the following condition for any function } f \text{ and any } a \in L:$$

$$(2) N(\underline{a} \rightarrow_{KD} f) = a \rightarrow_{KD} N(f).$$

$$- N(f) = \bigwedge_{x \in \mathcal{X}} \pi(x) \rightarrow_{rG} f(x) \text{ for some possibility distribution } \pi : \mathcal{X} \rightarrow L \text{ iff } N \text{ satisfies the following condition for any function } f \text{ and any } a \in L:$$

$$(3) N(\underline{a} \rightarrow_{rG} f) = a \rightarrow_G N(f).$$

$$- N(f) = \bigwedge_{x \in \mathcal{X}} \pi(x) \rightarrow_G f(x) \text{ and } \Pi(f) = \bigvee_{x \in \mathcal{X}} \min(\pi(x), f(x)) \text{ for some possibility distribution } \pi : \mathcal{X} \rightarrow L \text{ iff } \Pi \text{ satisfies (3) and the following condition is satisfied for any function } f \text{ and any } a \in L :$$

$$(4) N(f \rightarrow_G \underline{a}) = \Pi(f) \rightarrow_G a.$$

It is worth noticing that, in contrast to what happens with  $\rightarrow_G$  and  $\rightarrow_{rG}$ , for the necessity based on  $\rightarrow_{KD}$  we do not have the expected property that if  $\pi = f$ , then  $N(f) = 1$ . On the other hand, for  $\rightarrow_{rG}$  we have that  $N(f) \geq a$  iff  $\pi \leq a \rightarrow_{KD} f = \max(1 - a, f)$ , in conformity with possibilistic logic, where  $\max(1 - a, f)$  is the largest possibility distribution such that  $N(f) \geq a$ . We do not have this property for the necessities based on  $\rightarrow_G$  or  $\rightarrow_{KD}$ .

## 2.2 Capacities

Capacities, also known as fuzzy measures or plausibility measures, are generalizations of possibility distributions that are classically used in qualitative decision theory where objects are evaluated according to several points of view or criteria [5]. They are monotonic set functions representing interaction and weights of subsets of criteria. They are used to define integrals in qualitative decision theory. Namely, a capacity is a set function  $\mu : 2^{\mathcal{X}} \rightarrow L$  such that  $B \subseteq C$  implies  $\mu(B) \leq \mu(C)$ ,  $\mu(\emptyset) = 0$  and  $\mu(\mathcal{X}) = 1$ . The conjugate capacity  $\mu^c$  of a capacity  $\mu$  is defined as  $\mu^c(A) = 1 - \mu(\bar{A})$ , for all  $A \subseteq \mathcal{X}$ .

Particular cases, important in decision theory, are when we consider possibility and necessity measures. A possibility measure is a maxitive capacity  $\Pi$ :

$$\forall A, B \subseteq \mathcal{X}, \quad \Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$$

In the following,  $\forall x_i \in \mathcal{X}$ ,  $\Pi(\{x_i\})$  is denoted by  $\pi_i$ . We then have that  $\forall A \subseteq \mathcal{X}$ ,  $\Pi(A) = \max_{i=1}^n \pi_i$ . On the other hand, a necessity measure is a minitive capacity  $N$ , i.e.,

$$\forall A, B \subseteq \mathcal{X}, \quad N(A \cap B) = \min(N(A), N(B)).$$

Capacities are used to define Sugeno integrals and its variants we deal with in this paper: Gödel implication-based integrals and reciprocal of Gödel implication-based integrals.

## 2.3 Sugeno integrals

We recall the basic definitions for the Sugeno integral and for its corresponding residuated variant based on Gödel implication.

**Definition 2 ([3]).** *The Sugeno integral of  $\mathbf{f}$  with respect to  $\mu$  is:*

$$S_\mu(\mathbf{f}) = \max_{A \subseteq \mathcal{X}} (\mu(A) \otimes_{KD} \min_{i \in A} f_i) = \min_{A \subseteq \mathcal{X}} (\mu^c(A) \rightarrow_{KD} \max_{i \in A} f_i)$$

where  $\otimes_{KD}$  is Kleene-Dienes conjunction i.e.  $\alpha \otimes_{KD} \beta = \min(\alpha, \beta)$ ,  $\forall \alpha, \beta \in L$ .

Notice that, when  $\mu$  is a possibility measure (a maxitive capacity), the Sugeno integral simplifies to  $S_\mu(\mathbf{f}) = \max_{x \in \mathcal{X}} \min(\pi(x), f(x))$ , where  $\pi(x) = \mu(\{x\})$ , and hence it is nothing but a basic possibility as introduced in Def. 1.

The values of the capacity  $\mu$  can be recovered from the Sugeno integral itself, as  $S_\mu(1_A) = \mu(A)$ , where  $1_A$  denotes the characteristic function of a subset  $A \subseteq \mathcal{X}$ .

The two following characterization results are proved in [3, 4]. They use the notion of comonotonic functions.

**Definition 3.** *Two functions  $f, g : \mathcal{X} \rightarrow L$  are comonotonic if and only if  $f_i < f_j$  implies  $g_i \leq g_j$  and  $g_i < g_j$  implies  $f_i \leq f_j$ .*

It is worth noticing that any among min-comonotonicity or max-comonotonicity of a functional  $I$  implies non-decreasingness of  $I$ . We provide a proof for the sake of being self-contained.

**Lemma 1.** *Let  $I : L^{\mathcal{X}} \rightarrow L$  be a mapping satisfying one of the two following conditions:*

- max-comonotonicity:  $I(f \vee g) = \max(I(f), I(g))$  for all  $f, g$  comonotonic.
- min-comonotonicity:  $I(f \wedge g) = \min(I(f), I(g))$  for all  $f, g$  comonotonic.

Then,  $I$  is non-decreasing, that is, if  $f \leq g$  then  $I(f) \leq I(g)$ .

*Proof.* Suppose  $f \leq g$  and let  $\mathcal{X} = \{x_1, \dots, x_n\}$  be ordered such that  $f_1 \leq \dots \leq f_n$ . One can build a function  $h$  such that  $f \leq h \leq g$  and such that  $f, h$  are comonotonic, and  $h, g$  are comonotonic as well. Namely, define  $h$  as follows, proceeding iteratively from  $i = n$  to  $i = 1$ :

- for  $i = n$ : define  $h(x_n) = f_n$ ,
- for  $i = k$ , with  $k < n$ : if  $g(x_k) > g(x_{k+1})$ , then  $h(x_k) = f_{k+1}$ ,  
otherwise  $h(x_k) = f_k$ .

It can be easily checked that  $f \leq h \leq g$  and that the comonotonicities between  $f$  and  $h$ , and between  $h$  and  $g$  hold.

Now suppose  $I$  is min-comonotonic. Then we have:  $I(\min(f, h)) = I(f) = \min(I(f), I(h)) \leq I(h)$  and  $I(\min(h, g)) = I(h) = \min(I(h), I(g)) \leq I(g)$ , so we have  $I(f) \leq I(g)$ . A similar proof can be devised in case of assuming  $f$  to be max-comonotonic.  $\square$

Sugeno integral can be characterized using either max-comonotonicity or min-comonotonicity. We recall the following results presented in [11].

**Proposition 1.** *Let  $I : L^{\mathcal{X}} \rightarrow L$  be a mapping. There is a capacity  $\mu : 2^{\mathcal{X}} \rightarrow L$  such that  $I = S_{\mu}$  if and only if*

- $I(1_{\mathcal{X}}) = 1$
- $I(f \vee g) = \max(I(f), I(g))$  for all  $f, g$  comonotonic.
- $I(\underline{a} \wedge f) = \min(a, I(f))$ , for all  $a \in L$ .

**Proposition 2.** *Let  $I : L^{\mathcal{X}} \rightarrow L$  be a mapping. There is a capacity  $\mu : 2^{\mathcal{X}} \rightarrow L$  such that  $I = S_{\mu}$  if and only if*

- $I(0_{\mathcal{X}}) = 0$
- $I(f \wedge g) = \min(I(f), I(g))$  for all  $f, g$  comonotonic.
- $I(\underline{a} \vee f) = \max(a, I(f))$  for all  $a \in L$ .

We conclude this part by noticing that the following duality relation holds:

$$\forall f \in L^{\mathcal{X}}, S_{\mu}(f) = 1 - S_{\mu^c}(1 - f).$$

## 2.4 Gödel Implication-based Integral

The Gödel implication-based integral (also called cointegral in [4]) is defined like the Sugeno integral, replacing the Kleene-Dienes conjunction by Gödel implication, in the min-max expression of Sugeno integral.

**Definition 4 ([3]).** *The Gödel implication-based integral of  $\mathbf{f}$  with respect to  $\mu$  is:*

$$I_\mu^G(\mathbf{f}) = \min_{A \subseteq \mathcal{X}} (\mu^c(A) \rightarrow_G \max_{x_i \in A} f_i).$$

Unlike the case of the Sugeno integral, the values of the capacity  $\mu$  cannot be directly recovered from the integral  $I_\mu^G$ , but one can check that,

$$\forall a \in L, \forall A \subseteq \mathcal{X} \quad I_\mu^G(1_A \rightarrow_G a) = \mu^c(A) \rightarrow_G a, \text{ for any subset } A \subseteq \mathcal{X}.$$

We next recall a characterisation theorem for a functional to be a Gödel integral, see [3].

**Theorem 2.** *Let  $I : L^\mathcal{X} \rightarrow L$  be a mapping. There is a capacity  $\mu$  such that  $I(\mathbf{f}) = I_\mu^G(\mathbf{f})$  for every  $\mathbf{f} \in L^\mathcal{X}$  if and only if*

- $I(\mathbf{f} \wedge \mathbf{g}) = \min(I(\mathbf{f}), I(\mathbf{g}))$  for any comonotonic functions  $\mathbf{f}, \mathbf{g} \in L^\mathcal{X}$
- $\exists$  a capacity  $\rho$  and a binary operation  $\star$  such that  
 $I(\underline{a} \vee 1_A) = \rho^c(\bar{A}) \star a, \forall a \in L$
- $I(\underline{1}) = 1$  and  $I(\underline{0}) = 0$ .

*In this case  $\rho = \mu$  and  $\star = \rightarrow_G$*

When the capacity is a necessity measure  $N$ , the Gödel implication-based integral simplifies to this expression:  $I_N^G(\mathbf{f}) = \bigwedge_{i=1}^n \pi(x_i) \rightarrow_G f(x_i)$ , where  $\pi(x_i) = 1 - N(\overline{\{x_i\}})$ . In such a case,  $I_N^G(\mathbf{f})$  is again nothing but a basic necessity as introduced in Def. 1.

## 3 Integral with the reciprocal of Gödel implication

The integral based on the reciprocal of Gödel implication and on qualitative capacity  $\mu$ , denoted  $I_\mu^{rG}$ , is:

$$I_\mu^{rG}(f) = \bigwedge_{A \subseteq X} \mu^c(A) \rightarrow_{rG} (\bigvee_{x \in A} f(x)) = \bigwedge_{A \subseteq X} (1 - \bigvee_{x \in A} f(x)) \rightarrow_G (1 - \mu^c(A)).$$

Even if already introduced in [3], this is the first time it is considered, so we need to prove some theoretical results. It is easy to check that when  $\mu$  is a necessity measure  $N$  the above expression can be simplified to this expression:

$$I_N^{rG}(f) = \bigwedge_{x \in X} \pi(x) \rightarrow_{rG} f(x).$$

where  $\pi(x) = N(\overline{\{x\}})$ . In such a case,  $I_N^{rG}(\mathbf{f})$  is again nothing but a basic necessity as introduced in Def. 1. Next we present a characterization theorem for the reciprocal Gödel integral. Let us start with some technical lemmas. Before that, note that, the following equality holds:

$$\forall t, u, v \in L, \quad u \rightarrow_{rG} (t \rightarrow_{KD} v) = t \rightarrow_G (u \rightarrow_{rG} v).$$

Indeed,  $u \rightarrow_{rG} (t \rightarrow_{KD} v) = u \rightarrow_{rG} ((1-t) \vee v) = (1 - ((1-t) \vee v)) \rightarrow_G 1 - u = (t \wedge (1-v)) \rightarrow_G 1 - u = t \rightarrow_G ((1-v) \rightarrow_G (1-u)) = t \rightarrow_G (u \rightarrow_{rG} v)$ .

**Lemma 2.**  $I_\mu^{rG}(1_A) = \mu(A)$ .

*Proof.*  $I_\mu^{rG}(1_A) = \bigwedge_{B \subseteq X} (1 - (\bigvee_{x \in B} 1_A(x))) \rightarrow_G (1 - \mu^c(B)) = \bigwedge_{B \subseteq X, B \subseteq \bar{A}} (1 - (\bigvee_{x \in B} 1_A(x))) \rightarrow_G (1 - \mu^c(B)) = \bigwedge_{B \subseteq X, B \subseteq \bar{A}} 1 \rightarrow_G (1 - \mu^c(B)) = \bigwedge_{B \subseteq X, B \subseteq \bar{A}} 1 - \mu^c(B) = 1 - \mu^c(\bar{A}) = \mu(A)$ .  $\square$

**Lemma 3.**  $(KDG) I_\mu^{rG}(a \rightarrow_{KD} f) = a \rightarrow_G I_\mu^{rG}(f), \forall a \in L$

*Proof.*  $I_\mu^{rG}(a \rightarrow_{KD} f) = \bigwedge_{A \subseteq X} \mu^c(A) \rightarrow_{rG} \bigvee_{x \in A} (a \rightarrow_{KD} f(x)) = \bigwedge_{A \subseteq X} \mu^c(A) \rightarrow_{rG} (a \rightarrow_{KD} \bigvee_{x \in A} f(x)) = \bigwedge_{A \subseteq X} a \rightarrow_G [\mu^c(A) \rightarrow_{rG} \bigvee_{x \in A} f(x)] = a \rightarrow_G [\bigwedge_{A \subseteq X} \mu^c(A) \rightarrow_{rG} \bigvee_{x \in A} f(x)] = a \rightarrow_G I_\mu^{rG}(f)$ .  $\square$

These interesting properties are instrumental to show the following characterisation for this integral.

**Theorem 3.** *Let  $I : L^X \rightarrow L$ . Then,  $I$  satisfies:*

1.  $I(f \wedge g) = \min(I(f), I(g))$  if  $f$  and  $g$  are comonotonic
2.  $I(\underline{1}) = 1$
3.  $I(a \wedge f) = \min(a, I(f)), \forall a \in L$
4.  $I(a \rightarrow_{KD} f) = a \rightarrow_G I(f), \forall a \in L$

*if, and only if, there exists a capacity  $\mu$  such that  $I = I_\mu^{rG}$ .*

*Proof.* That  $I_\mu^{rG}$  satisfies the above properties is clear. Conversely, without loss of generality, we may suppose that the criteria are ordered according to  $f$ . So we assume to have  $f_1 \leq \dots \leq f_n$  and we denote by  $A_i$  the sets  $\{x_i, x_{i+1}, \dots, x_n\}$ . Note that  $\bar{A}_{i+1} = \{x_1, \dots, x_i\}$ . It is then easy to check that  $f$  can be expressed as  $f = \bigwedge_{i=1}^n 1_{\bar{A}_{i+1}} \rightarrow_G f_i$ . Furthermore, for all  $k \in \{1, \dots, n-1\}$ , it is easy to check that the functions

$$\bigwedge_{i=1}^k 1_{\bar{A}_i} \rightarrow_G f_i \quad \text{and} \quad 1_{\bar{A}_{k+1}} \rightarrow_G f_{k+1}$$

are comonotonic. Therefore, by the min-comonotonicity property of  $I$  we have, for all  $k$ ,  $I(\bigwedge_{i=1}^{k+1} 1_{\bar{A}_{i+1}} \rightarrow_G f_i) = \min(I(\bigwedge_{i=1}^k 1_{\bar{A}_{i+1}} \rightarrow_G f_i), I(1_{\bar{A}_{k+2}} \rightarrow_G f_{k+1}))$ . Hence, we also have  $I(f) = \bigwedge_{i=1}^n I(1_{\bar{A}_{i+1}} \rightarrow_G f_i)$ .

Since the characteristic functions  $1_{\overline{A_{i+1}}}$  are Boolean, we have  $1_{\overline{A_{i+1}}} \rightarrow_G f_i = 1_{\overline{A_{i+1}}} \rightarrow_{KD} f(x_i) = (1 - f(x_i)) \rightarrow_{KD} (1 - 1_{\overline{A_{i+1}}})$ . So we have

$$I(f) = \bigwedge_{i=1}^n I((1 - \overline{f(x_i)}) \rightarrow_{KD} (1 - 1_{\overline{A_{i+1}}})) = \bigwedge_{i=1}^n [1 - f(x_i) \rightarrow_G I(1 - 1_{\overline{A_{i+1}}})]$$

$$= \bigwedge_{i=1}^n [1 - I(1_{A_{i+1}}) \rightarrow_{rG} f(x_i)].$$

Then, define a mapping  $\mu : 2^X \rightarrow L$  by putting  $\mu(A) = I(1_A)$  for all  $A \subseteq X$ . So defined,  $\mu$  is indeed a capacity: it is non-decreasing by Lemma 1,  $\mu(X) = I(1_X) = I(\underline{1}) = 1$ , and  $\mu(\emptyset) = I(1_\emptyset) = I(\underline{0}) = 0$ . We finally have:

$$I(f) = \bigwedge_{i=1}^n [1 - \mu(A_{i+1}) \rightarrow_{rG} f(x_i)] = \bigwedge_{i=1}^n [\mu^c(\overline{A_{i+1}}) \rightarrow_{rG} f(x_i)] = I_\mu^G(f). \quad \square$$

## 4 From basic necessities and possibilities to Sugeno and Gödel implication integrals

In this section, we attempt to generalise the basic necessity  $N$  with a Gödel integral and the basic possibility  $\Pi$  with a Sugeno integral.

In decision theory, when we want to move from an integral with respect to a capacity (or fuzzy measure) to an integral with respect to a possibility measure (resp. necessity measure), we ask to keep the maxitivity (resp. minitivity) with respect to all functions. Otherwise, it suffices to ask for this property only for comonotonic functions.

We present next a characterization result (without proof) for Gödel implication-based integrals which is very similar to the one recalled in Theorem 2. Namely, it relies on the fact that  $1_A \rightarrow_G \underline{a}$  and  $1_{\overline{A}} \vee \underline{a}$  are the same function since the characteristic functions  $1_A$  are Boolean. We also note that the capacity is explicitly determined by one of the conditions.

**Theorem 4.** *Let  $I : L^X \rightarrow L$  be a mapping. The following conditions hold:*

- (1) *If  $I(f) = I_\mu^G(f)$  then:*
- $I(\min(f, g)) = \min(I(f), I(g))$  for any comonotonic functions  $f, g \in L^X$
  - $I(1_A \rightarrow_G \underline{a}) = \mu^c(A) \rightarrow_G \underline{a}$
  - $I(\underline{1}) = 1$  and  $I(\underline{0}) = 0$ .

- (2) *Conversely, if  $I$  is such that:*
- $I(\min(f, g)) = \min(I(f), I(g))$  for any comonotonic functions  $f, g \in L^X$
  - $I(1_A \rightarrow_G \underline{a}) = \sigma^c(A) \rightarrow_G \underline{a}$ , where  $\sigma^c(A) = \min\{a \mid I(1_A \rightarrow_G \underline{a}) = 1\}$
  - $I(\underline{1}) = 1$  and  $I(\underline{0}) = 0$
- then  $\sigma$  is a capacity and  $I(f) = I_\sigma^G(f)$ .*

Next, we show a couple of homogeneity-like properties of the integral based on Gödel implication when the functions are combined with constant functions via Gödel implication.

**Proposition 3.** (a)  $I_\mu^G(f \rightarrow_G \underline{a}) = S_{\mu^c}(f) \rightarrow_G \underline{a} = (1 - S_\mu(1 - f)) \rightarrow_G \underline{a}$ .  
 (b)  $\underline{a} \rightarrow_G I_\mu^G(f) = I_\mu^G(\underline{a} \rightarrow_G f)$ .

*Proof.* (a)  $I_\mu^G(f \rightarrow_G a) = \min_{A \subseteq \mathcal{C}} \mu^c(A) \rightarrow_G \max_{i \in A} (f_i \rightarrow_G a) = \min_{A \subseteq \mathcal{C}} \mu^c(A) \rightarrow_G ((\min_{i \in A} f_i) \rightarrow_G a) = \min_{A \subseteq \mathcal{C}} ((\mu^c(A) \wedge \min_{i \in A} f_i) \rightarrow_G a) = (\max_{A \subseteq \mathcal{C}} (\mu^c(A) \wedge \min_{i \in A} f_i)) \rightarrow_G a = S_{\mu^c}(f) \rightarrow_G a.$

(b)  $I_\mu(a \rightarrow f) = \bigwedge_{A \subseteq X} \mu^c(A) \rightarrow_G \bigvee_{x \in A} (a \rightarrow_G f(x)) = \bigwedge_{A \subseteq X} \bigvee_{x \in A} \mu^c(A) \rightarrow_G (a \rightarrow_G f(x)) = \bigwedge_{A \subseteq X} \bigvee_{x \in A} a \rightarrow_G (\mu^c(A) \rightarrow_G f(x)) = \bigwedge_{A \subseteq X} a \rightarrow_G (\mu^c(A) \rightarrow_G \bigvee_{x \in A} f(x)) = a \rightarrow_G \bigwedge_{A \subseteq X} (\mu^c(A) \rightarrow_G \bigvee_{x \in A} f(x)) = a \rightarrow_G I_\mu^G(f). \quad \square$

Using this relationship we can show the following combined representation of both Sugeno and Gödel implication integrals based on the same capacity, generalising (4) of Theorem 1 for basic possibilities and necessities.

**Theorem 5.** *Let  $I_G, I_S : L^{\mathcal{X}} \rightarrow L$  be two mappings. There is a capacity  $\mu : 2^{\mathcal{X}} \rightarrow L$  such that  $I_G(f) = I_\mu^G(f)$  and  $I_S(f) = S_{\mu^c}(f)$  for every  $f \in L^{\mathcal{X}}$  if and only if*

- (c1)  $I_G(\min(h, g)) = \min(I_G(h), I_G(g))$ , for all comonotonic  $h, g \in L^{\mathcal{X}}$
- (c2)  $I_S(\max(h, g)) = \max(I_S(h), I_S(g))$ , for all comonotonic  $h, g \in L^{\mathcal{X}}$
- (c3)  $I_G(f \rightarrow_G \underline{a}) = I_S(f) \rightarrow_G a$ ,  $\forall a \in L$
- (c4)  $I_S(\underline{a} \wedge f) = \min(a, I_S(f))$ ,  $\forall a \in L$
- (c5)  $I_G(\underline{1}) = I_S(\underline{1}) = 1$  and  $I_G(\underline{0}) = I_S(\underline{0}) = 0$ .

*Proof.* The left-to-right implication is clear as the  $I_\mu^G$  and  $S_\mu$  integrals satisfy all the properties. Conversely, assume  $I_G$  and  $I_S$  satisfy all these properties. Define  $\mu$  such that  $\mu^c(A) = I_S(1_A)$  for every  $A \subseteq \mathcal{X}$ . Then  $\mu$  is a capacity and  $I_G$  satisfies all conditions of Theorem 2. Hence, we have  $I_G = I_\mu^G$ . On the other hand,  $I_S$  also satisfies all the conditions of Proposition 1, hence there is a capacity  $\mu'$  such that  $I_S = S_{\mu'}$ . Then, by the third property above, we have, for all  $A$  and  $a \in L$ ,  $I_\mu^G(1_A \rightarrow_G \underline{a}) = S_{\mu'}(1_A) \rightarrow_G a$ , that is,  $\mu^c(A) \rightarrow_G a = \mu'(A) \rightarrow_G a$ . If we take  $a = \mu^c(A)$ , this equality yields  $1 = \mu'(A) \rightarrow_G \mu^c(A)$ , i.e.  $\mu'(A) \leq \mu^c(A)$ . Conversely, taking  $a = \mu'(A)$ , we obtain  $\mu^c(A) \rightarrow_G \mu'(A) = 1$ , i.e.  $\mu^c(A) \leq \mu'(A)$ . Therefore, it follows that  $\mu^c = \mu'$ .  $\square$

## 5 $\text{IRG}_n^\sim$ : a two-layer logic for reasoning about Gödel implication-based and Sugeno Integrals

In this section, we will define a modal two-layer logic on top of  $\text{RG}_n^\sim$ , the expansion of  $n$ -valued Gödel logic with truth-constants  $\bar{r}$ , one for each  $r \in C_n = \{0, 1/(n-1), \dots, (n-2)/(n-1), 1\}$ , and with an involutive negation  $\sim$ , see e.g. [12, 8]. Following previous approaches [9, 2], the idea is to introduce two modal operators  $I$  and  $S$  such that, for every formula  $\varphi$  of the logic  $\text{RG}_n^\sim$ ,  $I\varphi$  and  $S\varphi$  are fuzzy propositions whose truth-values are to be interpreted as the values of the Gödel integral and Sugeno integral of  $\varphi$  respectively. The logics we will define next are built over a two-layer language: non-modal formulas to describe and reason about objects or fuzzy events, and modal formulas (without nesting of the operators  $I$  and  $S$ ) to represent and reason about the certainty/worthiness of the non-modal formulas. Since Gödel and Sugeno models are basically qualitative models (the comparative ordering is what matters rather than absolute

degrees), we choose finite-valued Gödel logic (with truth-constants and with involutive negation) as underlying fuzzy logic to reason about the modal  $I\varphi$  and  $S\varphi$  formulas.

Due to the semantics of the logic  $\text{RG}_n^\sim$ , using the implication connective  $\rightarrow$  from Gödel logic, one can express that the Gödel integral of  $\varphi$  is at least  $r$  or at most  $s$  (with  $r, s \in C_n$ ) by the formulas  $\bar{r} \rightarrow I\varphi$  and  $I\varphi \rightarrow \bar{s}$ , while a formula  $I\varphi \rightarrow I\psi$  expresses the qualitative comparative statement “ $\psi$  is at least as certain/worthy as  $\varphi$ ”.

### 5.1 Syntax of $\text{IRG}_n^\sim$

Let us fix a *finite* set of propositional variables  $V$  and the set  $C_n = \{0, 1/(n-1), \dots, (n-2)/(n-1), 1\}$  used as a scale.

The language of  $\text{IRG}_n^\sim$  is built by expanding the language of  $\text{RG}_n^\sim$  with two modal operators under some restrictions. More specifically, formulas are built from the set  $V$  of propositional variables and truth-constants  $\bar{r}$  for  $r \in C_n$ , by using Gödel logic connectives  $\wedge, \vee, \rightarrow$ , an involutive negation  $\sim$  and two modal operators  $I$  and  $S$ . The restriction is that modalities cannot be nested, so *atomic modal* formulas are of the form  $I\varphi$  and  $S\varphi$  where  $\varphi$  is propositional. Then atomic modal formulas can be freely combined with other formulas (propositional or modal) with the above (propositional) connectives. A formula will be called *purely modal* when it is a propositional combination of atomic modal formulas.

In the language, we also introduce three definable connectives,  $\neg\varphi := \varphi \rightarrow \bar{0}$  (Gödel negation),  $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$  (Gödel equivalence), and  $\Delta\varphi := \neg\sim\varphi$  (Baaz-Monteiro operator), and two definable (conjugate) modal operators,  $I^c\varphi := \sim I(\sim\varphi)$  and  $S^c\varphi := \sim S(\sim\varphi)$ .

Note that so defined, the language has two layers of formulas, one of purely propositional formulas, denoted  $\text{Fm}(V, C_n)$ , and another of non-nested modal formulas.

### 5.2 Semantics of $\text{IRG}_n^\sim$

A *capacity model* is a pair  $\mathcal{M} = (\Omega, \mu)$  where:

- $\Omega = \{\omega : V \rightarrow C_n\}$  is the set of evaluations of propositional variables into  $C_n$ . Each  $\omega \in \Omega$  is called a *world* from  $\mathcal{M}$ .
- Each  $\omega : V \rightarrow C_n$  is extended to a truth-evaluation of non-modal formulas of  $\text{Fm}(V, C_n)$  in the usual way using the  $\text{RG}_n^\sim$  logic truth functions and such that  $\omega(\bar{r}) = r$ , for each  $r \in C_n$ . In particular, as expected, note that  $\omega(\sim\varphi) = 1 - \omega(\varphi)$  and that  $\omega(\Delta\varphi) = 1$  if  $\omega(\varphi) = 1$  and  $\omega(\Delta\varphi) = 0$  otherwise,
- For each  $\varphi \in \text{Fm}(V, C_n)$  we define its associated function  $f_\varphi : \Omega \rightarrow C_n$ , where  $f_\varphi(\omega) = \omega(\varphi)$ .
- $\mu : 2^\Omega \rightarrow C_n$  is a capacity

Now, given an arbitrary formula  $\Phi$ , the truth value of  $\Phi$  in a world  $\omega$  of  $\mathcal{M} = (\Omega, \mu)$ , denoted by  $\|\Phi\|_{\mathcal{M}, \omega}$ , is inductively defined as follows:

- if  $\Phi$  is a propositional formula from  $\text{Fm}(\mathbb{V}, C_n)$ , then  $\|\Phi\|_{\mathcal{M}, \omega} = \omega(\Phi)$
- If  $\Phi$  is an atomic modal formula  $I\varphi$ , then  $\|I\varphi\|_{\mathcal{M}, \omega} = I_\mu^G(f_\varphi)$
- If  $\Phi$  is an atomic modal formula  $S\varphi$ , then  $\|S\varphi\|_{\mathcal{M}, \omega} = S_\mu(f_\varphi)$
- If  $\Phi$  is a non-atomic modal formula, then its truth value  $\|\Phi\|_{\mathcal{M}, \omega}$  is computed by evaluating its subformulas, and then by using the truth functions associated to the  $\text{RG}_n^\sim$ -connectives occurring in  $\Phi$ .

Note that the evaluation of purely modal formulas do not depend of the particular world but on the whole model, i.e. if  $\Phi$  is purely modal, then  $\|\Phi\|_{\mathcal{M}, \omega} = \|\Phi\|_{\mathcal{M}, \omega'}$  for every  $\omega, \omega' \in \Omega$ . In that case, we will simply write  $\|\Phi\|_{\mathcal{M}}$  without danger of confusion.

A world  $\omega$  of  $\mathcal{M}$  satisfies a formula  $\Phi$ , written  $\omega \models_{\mathcal{M}} \Phi$ , when  $\|\Phi\|_{\mathcal{M}, \omega} = 1$ . A formula  $\Phi$  logically follows from a set of modal formulas  $\Gamma$ , written  $\Gamma \models \Phi$  if, for any capacity model  $\mathcal{M} = (\Omega, \mu)$  and  $\omega \in \Omega$ ,  $\|\Phi\|_{\mathcal{M}, \omega} = 1$  whenever  $\|\Psi\|_{\mathcal{M}, \omega} = 1$  for all  $\Psi \in \Gamma$ .

If  $\Gamma \cup \{\Phi\}$  is a set of purely modal logics, then  $\Gamma \models \Phi$  iff for capacity model  $\mathcal{M} = (\Omega, \mu)$ ,  $\|\Phi\|_{\mathcal{M}} = 1$  whenever  $\|\Psi\|_{\mathcal{M}} = 1$  for every  $\Psi \in \Gamma$ .

We will denote by  $\mathcal{CAP}(C_n)$  the class of capacity models with values on  $C_n$ .

### 5.3 An axiomatization for $\text{IRG}_n^\sim$

Let  $\Omega = \{\omega : \mathbb{V} \rightarrow C_n\}$  be the set of valuations of propositional variables in  $C_n$ , which is finite. For each  $\omega \in \Omega$  consider the formula  $\chi_\omega = \bigwedge_{p \in \mathbb{V}} \Delta(p \leftrightarrow \omega(p))$ . This formula captures at the syntactical level the assignment of truth-values to propositional variables by the valuation  $\omega$ .

Further, for each non-modal formula  $\varphi$  built from a set of propositional variables  $q_1, \dots, q_k$ , denoted as  $\varphi = t(q_1, \dots, q_k)$ , and each  $\omega \in \Omega$ , let us consider the formula  $\hat{\varphi}_\omega$  obtained by replacing each propositional variable  $q_i$  by the truth-constant  $\omega(q_i)$ , that is,  $\hat{\varphi}_\omega = t(\omega(q_1), \dots, \omega(q_k))$ . Thus, the formula  $\hat{\varphi}_\omega$  is nothing but a syntactical combination of truth-constants representing the truth-value  $\varphi$  gets under the valuation  $\omega$ .

These formulas will be used in the below axiomatization to faithfully account for the monotonicity of the operator  $I$  and the commonotonicity conditions required for the  $\wedge$  and  $\vee$  decomposition properties of the operators  $I$  and  $S$  respectively.

Axioms of  $\text{IRG}_n^\sim$  are the following:

- ( $\text{RG}_n^\sim$ ) axioms of  $\text{RG}_n^\sim$ , for non-modal and modal formulas
- (I1)  $I(\bar{r} \rightarrow \varphi) \leftrightarrow \bar{r} \rightarrow I(\varphi)$
- (I2)  $I(\top), \neg I(\perp)$
- (I3)  $S(\varphi \wedge \bar{r}) \leftrightarrow S(\varphi) \wedge \bar{r}$
- (I4)  $S(\top), \neg S(\perp)$
- (I5)  $I(\varphi \rightarrow \bar{r}) \leftrightarrow S(\varphi) \rightarrow \bar{r}$
- (RMin)  $[\bigwedge_{\omega, \omega' \in \Omega} \Delta(\hat{\varphi}_\omega \rightarrow \hat{\varphi}_{\omega'}) \leftrightarrow \Delta(\hat{\psi}_\omega \rightarrow \hat{\psi}_{\omega'})] \rightarrow [I(\varphi \wedge \psi) \leftrightarrow I\varphi \wedge I\psi]$
- (RMax)  $[\bigwedge_{\omega, \sigma \in \Omega} \Delta(\hat{\varphi}_\omega \rightarrow \hat{\varphi}_{\omega'}) \leftrightarrow \Delta(\hat{\psi}_\omega \rightarrow \hat{\psi}_{\omega'})] \rightarrow [S(\varphi \vee \psi) \leftrightarrow S\varphi \vee S\psi]$

The rules of inference of  $\text{IRG}_n^\sim$  are:

- (MP) *modus ponens* (for modal and non-modal formulas)
- (RmonI) *monotonicity* for  $I$ : if  $\varphi \rightarrow \psi$  is a theorem of  $\text{RG}_n^\sim$ ,  $I\varphi \rightarrow I\psi$  is a theorem of  $\text{IRG}_n^\sim$
- (RmonS) *monotonicity* for  $S$ : if  $\varphi \rightarrow \psi$  is a theorem of  $\text{RG}_n^\sim$ ,  $S\varphi \rightarrow S\psi$  is a theorem of  $\text{IRG}_n^\sim$

The semantical condition represented by the antecedent in the axiom (RMon) captures the condition that  $f_\varphi \leq f_\psi$ , while the (same) antecedents in the axioms (RMin) and (RMax) amount to require that the functions  $f_\varphi$  and  $f_\psi$  are comonotonic.

The above axioms and rules define in the usual way a notion of proof, denoted  $\vdash_{\text{IRG}_n^\sim}$ . Following the same approach of [2] with the necessary modifications, one can prove the following completeness result for  $\text{IRG}_n^\sim$ .

**Theorem 6.**  $\text{IRG}_n^\sim$  is sound and complete for purely modal theories with respect to the class  $\mathcal{CAP}(C_n)$  of capacity-Kripke models.

*Proof.* Let  $\Gamma$  and  $\Phi$  be a modal theory and a modal formula respectively, and assume  $\Gamma \not\vdash_{\text{IRG}_n^\sim} \Phi$ . We will prove that  $\Gamma \not\models \Phi$ , i.e. that there exists a capacity model of  $M_0$  and a world  $v \in \Omega$  satisfying  $\Gamma$  but not  $\Phi$ . We follow the strategy adopted in [2] that amounts to translating theories over  $\text{IRG}_n^\sim$  into theories over  $\text{RG}_n^\sim$ . For each modal formula  $\Psi$ , let us denote by  $\Psi^*$  the corresponding  $\text{RG}_n^\sim$ -formula obtained from  $\Psi$  by considering any atomic modal subformula of the form  $I\varphi$  or  $S\varphi$  as new propositional variables. Define, therefore,  $\Gamma^* = \{\Psi^* \mid \Psi \in \Gamma\}$  and

$$\begin{aligned} \text{Ax}^* = & \{\mathcal{Y}^* \mid \mathcal{Y} \text{ is an instance of axiom (Ii), } i = 1, 5\} \cup \\ & \{(I\varphi)^* \rightarrow (I\psi)^*, (S\varphi)^* \rightarrow (S\psi)^* \mid \vdash_{\text{RG}_n^\sim} \varphi \rightarrow \psi\} \cup \\ & \{(I\varphi)^*, (S\varphi)^* \mid \vdash_{\text{RG}_n^\sim} \varphi\} \cup \\ & \{(I\varphi \wedge \psi)^* \leftrightarrow (I\varphi)^* \wedge (I\psi)^*, (S\varphi \vee \psi)^* \leftrightarrow (S\varphi)^* \vee (S\psi)^* \mid \\ & \quad \vdash_{\text{RG}_n^\sim} \bigwedge_{\omega, \sigma \in \Omega} (\hat{\varphi}_\omega \rightarrow \hat{\varphi}_\sigma) \leftrightarrow (\hat{\psi}_\omega \rightarrow \hat{\psi}_\sigma)\} \end{aligned}$$

Using the same technique first used in [12] in a probabilistic context, one obtains

$$\Gamma \not\vdash_{\text{IRG}_n^\sim} \Phi \text{ iff } \Gamma^* \cup \text{Ax}^* \not\vdash_{\text{RG}_n^\sim} \Phi^*. \quad (1)$$

Now, since  $\text{RG}_n^\sim$  is strongly complete [12, Theorem 4.2.21], we know there exists a  $\text{RG}_n^\sim$ -evaluation  $v$  that is a model of  $\Gamma^* \cup \text{Ax}^*$  such that  $v(\Phi^*) < 1$ .

Define, for any non-modal formula  $\varphi$ ,  $\mathbb{I}(f_\varphi) = v((I\varphi)^*)$  and  $\mathbb{S}(f_\varphi) = v((S\varphi)^*)$ . Since  $\text{Ax}^*$  contains all the translations of all the theorems of  $\text{IRG}_n^\sim$  (f.i. the formulas  $(I(\psi \rightarrow \bar{r}))^* \leftrightarrow (S\psi)^* \rightarrow \bar{r}$ , etc.), then we can prove the following:

- (i)  $\mathbb{I}(f_\varphi \rightarrow r) = \mathbb{S}(f_\varphi) \rightarrow r$ . Indeed,  $\mathbb{I}(f_\varphi \rightarrow r) = \mathbb{I}(f_{\varphi \rightarrow \bar{r}}) = v((I(\varphi \rightarrow \bar{r}))^*) = v((S\varphi)^* \rightarrow \bar{r}) = v((S\varphi)^*) \rightarrow \bar{r} = \mathbb{S}(f_\varphi) \rightarrow r$ .
- (ii)  $\mathbb{I}(\min(f_\varphi, r)) = \mathbb{I}(f_{\varphi \wedge \bar{r}}) = v((I(\varphi \wedge \bar{r}))^*) = v((I\varphi)^* \wedge \bar{r}) = \min(v((I\varphi)^*), r) = \min(\mathbb{I}(f_\varphi), r)$ .

- (iii) Let  $\varphi$  and  $\psi$  be such that  $f_\varphi$  and  $f_\psi$  are comonotonic. Then by the rule (RMin), the formula  $(I(\varphi \wedge \psi))^* \leftrightarrow (I\varphi)^* \wedge (I\psi)^*$  belongs to  $Ax^*$ , and hence,  $\mathbb{I}(f_{\varphi \wedge \psi}) = \min(\mathbb{I}(f_\varphi), \mathbb{I}(f_\psi))$ . Similarly, by the rule (RMax) the formula  $(S(\varphi \vee \psi))^* \leftrightarrow (S\varphi)^* \vee (S\psi)^*$  belongs to  $Ax^*$  as well and  $\mathbb{S}(f_{\varphi \vee \psi}) = \max(\mathbb{S}(f_\varphi), \mathbb{S}(f_\psi))$ .
- (iv)  $\mathbb{I}(\underline{1}) = \mathbb{I}(f_\top) = v((I\top)^*) = 1$ ;  $\mathbb{I}(\underline{0}) = \mathbb{I}(f_\perp) = v((I\perp)^*) = 0$   
 $\mathbb{S}(\underline{1}) = \mathbb{S}(f_\top) = v((S\top)^*) = 1$ ;  $\mathbb{S}(\underline{0}) = \mathbb{S}(f_\perp) = v((S\perp)^*) = 0$

Note that for any function  $g : \Omega \rightarrow C_n$  there exists a non-modal formula  $\psi$  such that  $f_\psi = g$ . Effectively, let  $\psi = \bigvee_{\sigma \in \Omega} \chi_\sigma \wedge g(\sigma)$ . Taking into account that  $\omega(\chi_\sigma) = 1$  when  $\omega = \sigma$  and  $\omega(\chi_\sigma) = 0$  otherwise, it is easy to check that  $f_\psi = g$ . Indeed,  $f_\psi(\omega) = \omega(\bigvee_{\sigma \in \Omega} \chi_\sigma \wedge g(\sigma)) = \omega(\chi_\omega \wedge g(\omega)) = \min(1, g(\omega)) = g(\omega)$ . This shows that the set of functions  $f_\varphi : \Omega \rightarrow C_n$  for  $\varphi$  being a non-modal formula is the whose set of functions  $(C_n)^\Omega$ .

Therefore the functionals  $\mathbb{I}$  and  $\mathbb{S}$  are actually mappings  $\mathbb{I}, \mathbb{S} : (C_n)^\Omega \rightarrow C_n$  such that they fulfill the conditions (c1)-(c5) of Theorem 5. It follows then that there exists a capacity  $\mu$  on  $2^\Omega$  (defined as  $\mu^c(A) = v((S(\chi_A))^*) = \mathbb{S}(f_{\chi_A})$ , where  $\chi_A = \bigvee_{\omega \in A} \chi_\omega$ ) such that  $\mathbb{I}$  and  $\mathbb{S}$  are respectively the Gödel integral  $I_\mu^G$  and the Sugeno integral  $S_{\mu^c}$  on  $(C_n)^\Omega$ . Hence, we finally have, for every  $\varphi$ ,  $v((I\varphi)^*) = \mathbb{I}(f_\varphi) = I_\mu^G(f_\varphi)$  and  $v((S\varphi)^*) = \mathbb{S}(f_\varphi) = S_{\mu^c}(f_\varphi)$ , as claimed. As a consequence, we have that the capacity model  $\mathcal{M}^0 = (\Omega, \mu)$  is such that  $\|\Theta\|_{\mathcal{M}^0, v} = v(\Theta^*)$  for all purely modal formula  $\Theta$ . In particular, by hypothesis,  $\|\Theta\|_{\mathcal{M}^0, v}(\Psi) = v(\Psi^*) = 1$  for all  $\Psi \in \Gamma$  but  $\|\Phi\|_{\mathcal{M}^0, v} = v(\Phi^*) < 1$ . In other words,  $\Gamma \not\models \Phi$ .  $\square$

#### 5.4 A logic for the reciprocal of Gödel implication-based integral

A logical formalization, very similar to the one developed in the previous subsections, can be provided to capture reasoning with the reciprocal of Gödel integral.

The idea is to consider a simplified second layer of the logic  $\text{IRG}_n^\sim$  where we get rid of the modal operator  $S$ . Then, given a capacity model  $\mathcal{M} = \langle \Omega, \mu \rangle$ , define the evaluation in  $\mathcal{M}$  of an atomic modal formula  $I\varphi$ , where  $\varphi$  is propositional formula, as

$$\|I\varphi\|_{\mathcal{M}} = I_\mu^{rG}(f_\varphi),$$

and in the axiomatics do the following:

- (i) discard axioms (I3), (I4), (I5) and (RMax),  
 (ii) add the following axiom:  
 (I6)  $I(\sim \bar{r} \vee \varphi) \leftrightarrow (\bar{r} \rightarrow I(\varphi))$ ,  
 (iii) and discard the rule (NecS).

Finally, call the resulting logic  $\text{rIRG}_n^\sim$ . Then, based on the characterisation Theorem 3, and by an analogous proof to that of Theorem 6, one can show that  $\text{rIRG}_n^\sim$  is sound and complete with respect to the intended semantics based on the integral based on the reciprocal of Gödel implication. Due to similarity we omit details.

## 6 Conclusions

In this paper we have focused our attention on Sugeno-like implication-based qualitative integrals, in particular the ones defined by Gödel implication and its reciprocal, providing new characterizations for them. We have also provided a logical formalization of them, namely we have defined corresponding sound and complete two-layer modal many-valued logics, based on an extension of finite-valued Gödel logic.

The qualitative integrals we have considered in this paper use various operations that are definable, via different constructions, from the meet operation (minimum) in the finite totally ordered scale of values and from an order-reversing mapping. For instance, Gödel implication  $\rightarrow_G$  is the residuum of the minimum operation and the reciprocal of Gödel implication  $\rightarrow_{rG}$  is defined by contraposition from the former. These operations and constructions (and more) appear in the following commutative diagram presented in [10]:

$$\begin{array}{ccccc}
 \min(a, b) & \xrightarrow{Res} & a \rightarrow_G b & \xrightarrow{\mathcal{C}} & (1 - b) \rightarrow_G (1 - a) \\
 \uparrow \mathcal{S} \downarrow \mathcal{S} & & \uparrow \mathcal{S} \downarrow \mathcal{S} & & \uparrow \mathcal{S} \downarrow \mathcal{S} \quad Res \\
 \max(1 - a, b) & \xleftarrow{Res} & a \otimes_G b & \xrightarrow{\mathcal{E}} & b \otimes_G a
 \end{array}$$

In this diagram yet another operation appears, the so-called Gödel conjunction  $\otimes_G$  is defined by semi-duality from  $\rightarrow_G$ :  $u \otimes_G v = 1 - (u \rightarrow_G 1 - v)$ , which is a non-commutative and non-associative operation. As future work, we plan to study and characterise Sugeno-like integrals based on  $\otimes_G$ , which looks a challenging task.

## Acknowledgments

The authors thank to the anonymous reviewers for their comments. Lluís Godo acknowledges support by the Spanish project LINEXSYS (PID2022-139835NB-C21) funded by MCIU/AEI/10.13039/501100011033.

## References

1. Ciucci, D., Dubois, D.: A capacity-based framework encompassing Belnap–Dunn logic for reasoning about multisource information. *International Journal of Approximate Reasoning* **106**, 107–127 (2019)
2. Dellunde, P., Godo, L., Marchioni, E.: Extending possibilistic logic over Gödel logic. *International Journal of Approximate Reasoning* **52**(1), 63–75 (2011)
3. Dubois, D., Prade, H., Rico, A.: Residuated variants of Sugeno integrals: towards new weighting schemes for qualitative aggregation methods. *Information Sciences* **329**, 765–781 (2016)
4. Dubois, D., Prade, H., Rico, A., Teheux, B.: Generalized Sugeno integrals. *Information Sciences* **415–416**, 429–445 (2017)

5. Dubois, D., Faux, F., Prade, H., Rico, A.: Qualitative capacities: Basic notions and potential applications. *International Journal of Approximate Reasoning* **148**, 253–290 (2022)
6. Dubois, D., Prade, H., Schockaert, S.: Generalized possibilistic logic: Foundations and applications to qualitative reasoning about uncertainty. *Artificial Intelligence* **252**, 139–174 (2017). <https://doi.org/10.1016/j.artint.2017.08.001>
7. Dvořák, A., Holčapek, M.: Fuzzy measures and integrals defined on algebras of fuzzy subsets over complete residuated lattices. *Inf. Sci.* **185**(1), 205–229 (2012)
8. Esteva, F., Godo, L., Hájek, P., Navara, M.: Residuated fuzzy logics with an involutive negation. *Arch. Math. Log.* **39**(2), 103–124 (2000)
9. Flaminio, T., Godo, L., Marchioni, E.: Reasoning about uncertainty of fuzzy events: an overview. In: Cintula, P., et al. (eds.) *Understanding Vagueness - Logical, Philosophical, and Linguistic Perspectives*, pp. 367–400. College Publications (2011)
10. Fodor, J.C.: On fuzzy implication operators. *Fuzzy Sets and Systems* **42**(3), 293–300 (1991)
11. Grabisch, M.: Set Functions Games and Capacities in Decision Making. In: *Game theory, Social Choice, Decision Theory and Optimization*. No. C 46 in *Theory and Decision Library*, Springer (2016)
12. Hájek, P.: *Metamathematics of Fuzzy Logic*. Trends in Logic, Kluwer Academic (1998)