A Concise Function Representation for Faster Exact MPE and Constrained Optimisation in Graphical Models

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Abstract—We propose a novel concise function representation for graphical models, a central theoretical framework that provides the basis for many reasoning tasks. We then show how we exploit our concise representation based on deterministic finite state automata within Bucket Elimination (BE), a general approach based on the concept of variable elimination that accommodates many inference and optimisation tasks such as *most probable explanation* and *constrained optimisation*. We denote our version of BE as FABE. By using our concise representation within FABE, we dramatically improve the performance of BE in terms of runtime and memory requirements. Results on standard benchmarks obtained using an established experimental methodology show that FABE often outperforms the best available approach (RBFAOO), leading to significant runtime improvements (up to 2 orders of magnitude in our tests).

Index Terms—graphical models, most probable explanation, constrained optimisation, deterministic finite state automata

I. INTRODUCTION

Graphical models are a central theoretical framework that provides the basis for many reasoning tasks with probabilistic or deterministic information [1] in real-world scenarios such as sensor networks [2] and gene networks reconstruction [3]. These models employ graphs to concisely represent the structure of the problem and the relations among variables [4] to solve fundamental tasks such as providing a plausible explanation given the observed evidence, namely *most probable explanation* (MPE), or minimise the sum of a given set of objective functions, namely *constrained optimisation*.

One of the most important algorithms for exactly solving these reasoning tasks on graphical models is Bucket Elimination (BE) proposed by Dechter [5], [1], a general approach based on the concept of variable elimination that accommodates many inference and optimisation tasks. BE is also a fundamental component-Mini-Bucket Elimination (MBE), the approximate version of BE [6], is used to compute the initial heuristic that guides the search-of all the algorithms by Marinescu et al. [7], [8], [9], [10] that represent the state of the art for exact MPE inference. On the other hand, BE is characterised by memory requirements that grow exponentially with respect to the *induced width* of the primal graph associated to the graphical model [1], severely hindering its applicability to large exact reasoning tasks. As a consequence, several works have tried to mitigate this drawback [6], [11], but none of these approaches really managed to overcome such a limitation. The main reason for such memory requirements is the fact that the functions employed during BE's execution are usually represented as *tables*, whose size is the product of the domains of the variables in the scope, regardless of the actual values of such functions. This can lead to storing many repeated values in the same table, causing a potential waste of computational resources.¹

Against this background, in this paper we propose a novel function representation specifically devised for exact MPE inference and constrained optimisation that, instead of the traditional mapping variable assignment \rightarrow value, adopts a radical new approach that maps each value v to the minimal *finite state automaton* [12] representing all the variable assignments that are associated to v. We then exploit our representation within FABE, our version of BE that exactly solves the considered tasks. By representing each value only once, and by exploiting the well-known capabilities of automata of compactly representing sets of strings (with a reduction that can be *up to exponential* with respect to a full table), we dramatically improve the performance of BE in terms of runtime and memory requirements. In more detail, this paper advances the state of the art in the following ways:

- We propose a novel function representation for exact MPE inference and constrained optimisation based on finite state automata, which we exploit within FABE.
- Results on standard benchmark datasets show that FABE often outperforms the best available exact approach (RBFAOO), with improvements of up to 2 orders of magnitude in our tests.
- Results also show that FABE outperforms the *structured message passing* (SMP) approach by Gogate and Domingos [13], in virtue of the capability of automata of natively representing non-binary variables present in the considered benchmarks (in contrast with SMP).
- Our concise function representation can be directly employed within MBE to approximately solve the abovementioned reasoning tasks. In virtue of this fact, our work paves the way for a significantly better version of MBE as a key component of AND/OR search algorithms, in which the computation of the initial heuristic can represent a bottleneck, as discussed by Kishimoto *et al.* [10].

¹This is also true for all the above-mentioned AND/OR search algorithms, which also adopt a tabular function representation.

The rest of this paper is structured as follows. Section II provides the necessary background on graphical models and deterministic finite state automata. Section III discusses related work and positions our approach wrt existing literature. Section IV presents our function representation and how we exploit it within FABE. Section V presents our experimental evaluation on standard benchmark datasets, in which we compare FABE against state of the art algorithms for exact inference on graphical models. Section VI concludes the paper and outlines future research directions.

II. BACKGROUND

A. Graphical Models

Graphical models (e.g., Bayesian Networks [14], Markov Random Fields [15], or Cost Networks [1]) capture the factorisation structure of a distribution over a set of n variables.

A graphical model is a tuple $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F} \rangle$, where $\mathbf{X} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F} \rangle$ $\{X_i : i \in V\}$ is a set of variables indexed by set V and $\mathbf{D} = \{D_i : i \in V\}$ is the set of their finite domains of values. $\mathbf{F} = \{\psi_{\alpha} : \alpha \in F\}$ is a set of discrete local functions defined on subsets of variables, where $F \subseteq 2^V$ is a set of variable subsets. We use $\alpha \subseteq V$ and $\mathbf{X}_{\alpha} \subseteq \mathbf{X}$ to indicate the *scope* of function ψ_{α} , i.e., $\mathbf{X}_{\alpha} = var(\psi_{\alpha}) = \{X_i : i \in \alpha\}$. The function scopes yield a *primal graph G* whose vertices are the variables and whose edges connect any two variables that appear in the scope of the same function.

An important inference task that appears in many real-world applications is MPE. MPE finds a complete assignment to the variables that has the highest probability (i.e., a mode of the joint probability), namely: $\mathbf{x}^* = \arg \max_{\mathbf{x}} \prod_{\alpha \in F} \psi_{\alpha}(\mathbf{X}_{\alpha})$. The task is NP-hard to solve [14].

Another important task over deterministic graphical models (e.g., Cost Networks) is the optimisation task of finding an assignment or a configuration to all the variables that minimises the sum of the local functions, namely: \mathbf{x}^* = $\arg \min_{\mathbf{x}} \sum_{\alpha \in F} \psi_{\alpha}(\mathbf{X}_{\alpha})$. This is the task that has to be solved in Weighted Constraint Satisfaction Problems (WCSPs). The task is NP-hard to solve [1].

Al	lgorithm	1	Bucket	Elimination	[1]	
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Input: A graphical model $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F} \rangle$, an ordering d. Output: A max probability (resp. min cost) assignment.

- 1: Partition functions into buckets according to d.
- 2: Define ψ_i as the \otimes of *bucket_i* associated with X_i .
- 3: for $p \leftarrow n$ down to 1 do
- for ψ_p and messages h_1, h_2, \ldots, h_j in bucket_p do 4:
- 5:
- $h_p \leftarrow \Downarrow_{X_p}(\psi_p \otimes \bigotimes_{i=1}^j h_i).$ Place h_p into the largest index variable in its scope. 6:
- 7: Assign maximising (resp. minimising) values in ordering d, consulting functions in each bucket.
- 8: return Optimal solution value and assignment.

To solve the above-mentioned tasks we consider the BE algorithm as discussed by Dechter [1] (Algorithm 1). BE is a general algorithm that can accommodate several exact inference and optimisation tasks over graphical models. In this paper we focus on the version that can optimally solve the abovementioned MPE and optimisation tasks. BE operates on the basis of a *variable ordering d*, which is used to partition the set of functions into sets called buckets, each associated with one variable of the graphical model. Each function is placed in the bucket associated with the last bucket that is associated with a variable in its scope. Then, buckets are processed from last to first by means of two fundamental operations, i.e., combination $(\otimes \in \{\prod, \sum\})$ and projection $(\Downarrow \in \{\max, \min\})$. All the functions in $bucket_p$, i.e., the current bucket, are composed with the \otimes operation, and the result is the input of a \Downarrow operation. Such an operation removes X_p from the scope, producing a new function h_p that does not involve X_p , which is then placed in the last bucket that is associated with a variable appearing in the scope of the new function. To solve the MPE (resp. optimisation) task, $\otimes = \prod$ (resp. \sum) and $\Downarrow = \max$ (resp. min) operators are used.

The computational complexity of the BE algorithm is directly determined by the ordering d. Formally, BE's time and space complexity are $\mathcal{O}(r \cdot k^{w^*(d)+1})$ and $\mathcal{O}(n \cdot k^{w^*(d)})$ respectively, where k bounds the domain size, and $w^*(d)$ is the induced width of its primal graph along d [1].

B. Deterministic Finite State Automata

Let Σ denote a finite alphabet of characters and Σ^* denote the set of all strings over Σ . The size $|\Sigma|$ of Σ is the number of characters in Σ . A language over Σ is any subset of Σ^* . A Deterministic Finite State Automaton (DFSA) [12] δ is specified by a tuple $\langle Q, \Sigma, t, s, F \rangle$, where Q is a finite set of states, Σ is an input alphabet, $t: Q \times \Sigma \to 2^Q$ is a transition function, $s \in Q$ is the start state and $F \subseteq Q$ is a set of final states. A string x over Σ is accepted (or recognised) by δ if there is a labelled path from s to a final state in F such that this path spells out the string x. Thus, the language L_{δ} of a DFSA δ is the set of all strings that are spelled out by paths from s to a final state in F. It is well known that a general DFSA can accept an *infinite* language (i.e., a infinite set of strings) [12]. In this paper we focus on Deterministic Acyclic Finite State Automata (DAFSA), i.e., DFSA whose corresponding graph is a directed acyclic graph. In contrast with general DFSA, DAFSA only accept finite languages [16].

III. RELATED WORK

In recent years, a strand of literature has investigated the use of different algorithms on AND/OR search spaces (i.e., branch-and-bound [8], best-first [7], recursive best-first [9] and parallel recursive best-first [10]), progressively showing the effectiveness of these approaches for exact MPE inference and constrained optimisation. To the best of our knowledge, all the above-mentioned approaches use the standard tabular representation to store functions in memory. In the context of constrained optimisation, the only notable approach that tries to reduce the size of tables in memory is the one by Bistaffa et al. [11], which avoids representing unfeasible assignments for WCSPs.

The task of concisely representing functions for inference has been treated in several works [13], [17] by means of Binary Decision Diagrams (BDDs) [18]. Gogate and Domingos [13] proposed the use of Algebraic Decision Diagrams (ADDs) to tackle redundancy as part of the so-called structured message passing (SMP) algorithm. In [17] the authors proposed a variable elimination algorithm based on Probabilistic Sentential Decision Diagrams (PSDDs) [19]. While conceptually related to DAFSA, BDDs can only represent Boolean functions. In contrast, DAFSA can natively represent any non-Boolean function and, thus, they are inherently more general than BDDs. As a consequence, approaches employing BDDs require to encode non-binary variables as multiple binary ones (e.g., by means of one-hot encoding). In Section V we further investigate the overhead due to the additional number of variables by comparing our approach with SMP (i.e., the most closely related among the above-cited works), showing that it has a significant impact on the runtime.

Mateescu *et al.* [20] also investigated the use of Multivalued Decision Diagrams (MDDs) [21] within the abovediscussed AND/OR search scheme to overcome said limitation of BDDs. While MDDs share similarities with DAFSA (i.e., both can be seen as *decision diagrams* with a branching factor higher than 2), MDDs have never been applied within variable elimination algorithms (such as BE) with the explicit objective of reducing the redundancy inherent in the representation of functions, as we do in this paper. Since several AND/OR search algorithms have been developed over the years (see discussion above), in Section V we only compare with the most recent and best performing ones in such a strand of literature, namely (SP)RBFAOO.²

Lifted probabilistic inference (LPI) [22] is also concerned with reducing redundancy within probabilistic inference. Specifically, LPI tackles redundancy between different factors, whereas we tackle redundancy inside the same factor. Assessing the effectiveness of the combined approach wrt to the separate ones is a non-trivial research question, which will be considered in future work.

IV. A NOVEL DAFSA-BASED VERSION OF BE

All the datasets commonly used as benchmarks for MPE [10] and constrained optimisation [8] are characterised by a very high *redundancy*, i.e., many different variable assignments are associated to the same value in the local functions. Figure 1 shows that the value of redundancy for local functions (defined as $1 - \frac{\text{number of unique values}}{\text{total number of values}}$) for all MPE and WCSP instances is always above 80% (except for smaller grid instances).

Furthermore, in probabilistic graphical models, local functions represent probabilities with values in the interval [0, 1], which, in theory, contains *infinite* real values. In practice, such values are represented by *floating point numbers* that can only represent a *finite* amount of values. Thus, while a table ψ has an arbitrarily large size that is the product of the domains of the variables in its scope, in practice the maximum number of unique values in ψ is bounded by a parameter that depends on the numerical representation. These remarks motivate the study of a novel concise representation that exploits such a

²Notice that we cannot directly compare FABE with the approach by Mateescu *et al.* [20] also because its implementation is not publicly available.

In this paper we propose a way to represent functions by means of DAFSA, as shown in the example in Figure 2. In the traditional way of representing functions as tables, rows are indexed using variable assignments as *keys* (Figure 2, left). In contrast, here we propose a novel approach that uses *values* as keys (Figure 2, right). Formally,

of the variables in the scope.

Definition 1. Given a function ψ that maps each possible assignment of the variables in its scope to a value $v \in \mathbb{R} \cup \{\infty\}$,³ we denote as $D(\psi)$ its corresponding representation in terms of DAFSA. Formally, $D(\psi) = \{(v, \delta)\}$, where v is a value in ψ and δ is the minimal DAFSA that accepts all the strings corresponding to the variable assignments that were mapped to v in ψ . For the sake of simplicity, we do not represent the scope of the function ψ in $D(\psi)$, as we assume it is equal to $var(\psi)$. We label a transition that accepts all the values of a variable's domain as *. Notice that each δ is acyclic because it accepts a finite language [16].

Remark 1. Given that values are employed as keys in our function representation, it is crucial to ensure the absence of duplicates in such a set of keys, i.e., we must be able to correctly determine whether two values v_1 and v_2 are equal. While this is a trivial task in theory, in practice it can be very tricky when v_1 and v_2 are floating point numbers representing real values. Indeed, even if v_1 and v_2 are theoretically equal, their floating point representations can differ due to numerical errors implicit in floating point arithmetic, especially if v_1 and v_2 are the result of a series of operations whose numerical errors have accumulated. To mitigate this aspect, we use a well-known technique for comparing floating point numbers known as ϵ -comparison, i.e., v_1 and v_2 are considered equal if they differ by a quantity smaller than a small ϵ . While there exist more advanced techniques of tackling numerical issues related to floating point numbers and their arithmetic [23], they are well beyond the scope of this paper. This should not be considered as an approximation, rather as a standard method to avoid the propagation of numerical errors.

A crucial property of DAFSA is that one path can accept multiple strings, or, in our case, represent multiple variable assignments. In the example in Figure 2, the DAFSA corresponding to v_3 contains only one path, but it represents both $\langle 1, 0, 0 \rangle$ and $\langle 1, 1, 0 \rangle$. By exploiting this property, our representation can reach a reduction in terms of memory that is, in the best case, *up to* exponential *wrt* the traditional table representation. We remark that memory is the main bottleneck that limits the scalability of BE, hence reducing its memory requirements is crucial, leading to significant improvements as shown by our results in the experimental section. Finally, our representation allows one to trivially avoid representing unfeasible assignments, similarly to [11].

 $^{^{3}}$ We allow ∞ as a possible value, since it can used to represent variable assignments that violate some hard constraint in WCSPs.



Total number of values in local functions

Fig. 1: Redundancy in MPE and WCSPs instances. Best viewed in colours.



Fig. 2: Standard table (left) and corresponding DAFSA-based representation (right). All variables are binary. Best viewed in colours.

Predicting the space complexity (e.g., the number of states) of a minimal DAFSA accepting a given set of strings remains, to the best of our knowledge, an open problem, since it depends on the common prefixes/suffixes of the input set.

A minimal DAFSA can be efficiently constructed from a set of assignments by using the algorithm described by Daciuk [16]. Since all the strings accepted by each DAFSA are of the same length (equal to the cardinality of the scope of the function), so are all the paths in the DAFSA. Thus, there is a mapping between each edge at depth i in each path and the ith variable in the scope (see Figure 2). Without loss of generality, our representation always maintains the variables in the scope ordered wrt their natural ordering.

Having discussed our representation, we now discuss our DAFSA-based version of BE, and specifically its core operations \otimes and \Downarrow .

A. A DAFSA-Based Version of \otimes

In order to better discuss our DAFSA-based version the \otimes operation, let us first recall how this operation works for traditional tabular functions with an example (Figure 3). The result of the \otimes operation is a new function whose scope is the union of the scopes of the input functions, and in which the value of each variable assignment is the $\otimes \in \{\cdot, +\}$ of the values of the corresponding assignments (i.e., with the same assignments

of the corresponding variables) in the input functions. For example, the assignment $\langle X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0 \rangle$ in the result table corresponds to $\langle X_1 = 0, X_2 = 1, X_4 = 0 \rangle$ and $\langle X_3 = 1, X_4 = 0 \rangle$ in the input tables, hence its value is $v_2 \otimes v_3$. The \otimes operation is closely related to the *inner join* operation of relational algebra [1].

To efficiently implement $D(\psi_1) \otimes D(\psi_2)$ we will make use of the *intersection* operation on automata [12]. Intuitively, the intersection of two automata accepting respectively L_1 and L_2 is an automaton that accepts $L_1 \cap L_2$, i.e., all the strings appearing both in L_1 and L_2 . In our case, we will exploit the intersection operation to identify all the corresponding variable assignments in $D(\psi_1)$ and $D(\psi_2)$. To make this possible, we first have to make sure that both functions have the same scope, so that corresponding levels in $D(\psi_1)$ and $D(\psi_2)$ correspond to the same variables. We achieve this by means of the ADDLEVELS operation. Figure 4 shows an example of ADDLEVELS.

Definition 2. Given two functions $D(\psi_1)$ and $D(\psi_2)$, the ADDLEVELS operation inserts (i) one or more levels labelled with * in each DAFSA and (ii) one or more variables in the respective scopes, in a way that the resulting scope is $var(\psi_1) \cup var(\psi_2)$. Each level and variable is added so as to maintain the scope ordered wrt the variable ordering.



Fig. 4: The result of the ADDLEVELS operation on $D(\psi_1)$ and $D(\psi_2)$, where ψ_1 and ψ_2 are the tables in Figure 3. Added levels and variables are denoted with dotted lines and $^+$ superscript.

Proposition 1. The operation of adding one level to a DAFSA δ has a linear complexity wrt the number of paths in δ . Within ADDLEVELS $(D(\psi_1), D(\psi_2))$ this operation has to be executed a total of $|D(\psi_1)| \cdot |var(\psi_2) \setminus var(\psi_1)| + |D(\psi_2)| \cdot |var(\psi_1) \setminus var(\psi_2)|$ times, i.e., the number of values in each function times the number of variables that have to be added to the scope of each function to reach the scope of $D(\psi_1) \otimes D(\psi_2)$.

Our DAFSA-based \otimes operation is implemented by Algorithm 2. Intuitively, for each couple of values (v_i, v_j) , where v_i and v_j are values in $D(\psi_1)$ and $D(\psi_2)$ respectively, we compute the variable assignments associated to their \otimes by computing the intersection $\delta_i \cap \delta_j$ between the corresponding DAFSA δ_i and δ_j . The result is then associated to the value $v_i \otimes v_j$.

Notice that we maintain only one entry for each value $v_i \otimes v_j$ (see Remark 1 in this respect) by accumulating (i.e., taking the union of) all the DAFSA that are associated to the same value (Line 4). Union and intersection over DAFSA have a time complexity of $\mathcal{O}(nm)$ [24], where *n* and *m* are the number of states of the input automata. Depending on their implementations, such operations may not directly produce a minimal DAFSA. Nonetheless, DAFSA can be minimised in linear time *wrt* the number of states with the algorithm by Bubenzer [25].

B. A DAFSA-Based Version of \Downarrow

The $\Downarrow \in \{\max, \min\}$ operation effectively realises *variable elimination* within the BE algorithm. Specifically, $\Downarrow_{X_i} \psi$ removes X_i from the scope of ψ , and, from all the rows

Algorithm 2 $D(\psi_1) \otimes D(\psi_2)$ 1: $(D(\psi_1)', D(\psi_2)') = \text{ADDLEVELS}(D(\psi_1), D(\psi_2)).$ 2: for all $(v_i, \delta_i) \in D(\psi_1)', (v_j, \delta_j) \in D(\psi_2)'$ do3: if $\exists (v_i \otimes v_j, \delta_k) \in D(\psi_1) \otimes D(\psi_2)$ then4: $\delta_k = \delta_k \cup (\delta_i \cap \delta_j).$ 5: else6: Add $\{(v_i \otimes v_j, \delta_i \cap \delta_j)\}$ to $D(\psi_1) \otimes D(\psi_2).$ 7: return $D(\psi_1) \otimes D(\psi_2).$

that possibly have equal variable assignments as a result of the elimination of the column associated to X_i , it only maintains the one with the max (in the case of MPE, or min in the case of optimisation) value. Like \otimes , \Downarrow is also related to a relational algebra operation, i.e., the *project* operation. In terms of SQL, $\Downarrow_{X_i} \psi$ is equivalent to SELECT $var(\psi) \setminus X_i$, max $(\psi(\cdot))$ FROM ψ GROUP BY $var(\psi) \setminus X_i$, in the case of max.

We realise the elimination of the column associated to X_i with the REMOVELEVEL operation, which can be thought of as the inverse of ADDLEVELS. REMOVELEVEL $(D(\psi), X_i)$ removes X_i from the scope of $D(\psi)$ and collapses all the edges at the level associated to X_i from all the DAFSA in $D(\psi)$.

Proposition 2. The operation of removing one level from a DAFSA δ has a linear complexity wrt the number of paths in δ . Within REMOVE LEVEL $(D(\psi), X_i)$ this operation has to be executed a total of $|D(\psi)|$ times, i.e., once for each value of ψ . Notice that removing a level from a DAFSA could result in a non-deterministic automaton if the removal happens in

protein	1duw	1hcz	lfny	2hft	1ad2	latg	lqre	1qhv
FABE RBFAOO SMP	21.36 > 2 h > 2 h	10.33 749.39 > 2 h	6.60 > 2 h 2036.29	322.33 1765.22 6569.95	25.28 1654.75 > 2 h	3.28 1697.87 4098.89	3.47 734.85 1721.50	16.54 > 2 h 4376.94
pedigree	25	30	39	18	31	34	51	9
FABE RBFAOO SMP	28.82 6.32 197.89	7.23 61.34 40.86	3.21 22.46 17.06	7.42 20.11 40.89	910.46 > 2 h 5881.66	8.83 > 2 h 60.78	132.92 > 2 h 789.65	473.94 100.19 2040.32
grid	90-26-5	90-25-5	90-24-5	75-23-5	90-23-5	75-22-5	90-22-5	75-21-5
FABE RBFAOO SMP	3192.15 925.47 > 2 h	> 2 h 902.19 > 2 h	5112.09 1758.19 > 2 h	> 2 h 791.70 > 2 h	508.75 158.17 2453.45	> 2 h 816.87 > 2 h	4883.60 20.37 > 2 h	> 2 h 4.71 > 2 h

TABLE I: Runtime results (in seconds) on 8 largest MPE instances.

correspondence of a branching. Our implementation takes this into account by employing a determinisation algorithm [12]. In general, determinising an automaton could produce a growth (up to exponential, in the worst case) of the number of states.

On the other hand, in all our experiments such a worst-case never happens and the growth factor due to determinisation is, on average, only around 10%. Our results confirm that such a small growth does not affect the overall performance of our approach, which is able to outperform the competitors as described in Section V.

We then implement the maximisation (resp. minimisation) of the values as follows. Without loss of generality, we assume that the values $v_1, \ldots, v_{|D(\psi)|}$ are in decreasing (resp. increasing) order. For each $(v_i, \delta_i) \in D(\psi)$, we subtract from δ_i all δ_j such that v_j precedes v_i in the above-mentioned ordering (i.e., $v_j \ge v_i$, resp. \le). In this way, we remove all duplicate variable assignments and we ensure that each assignment is *only* associated to the maximum (resp. minimum) value, correctly implementing the \Downarrow operation. Subtraction over DAFSA has a time complexity of $\mathcal{O}(nm)$ [24], where *n* and *m* are the number of states of the input automata. Algorithm 3 details our \Downarrow implementation.

Algorithm 3 $\Downarrow_{X_i} D(\psi)$
1: $D(\psi)' = \text{REMOVELEVEL}(D(\psi), X_i).$
2: for all $(v_i, \delta_i) \in D(\psi)'$ with decr. (resp. incr.) v_i do
3: $\delta_i = \delta_i \setminus \delta_{prec}.$
4: $\delta_{prec} = \delta_{prec} \cup \delta_i.$
5: return $D(\psi)'$.

Both our versions of \otimes and \Downarrow entirely operate on our concise representation, never expanding any function to a full table. We directly employ our \otimes and \Downarrow operations within Algorithm 1. We call our DAFSA-based version of BE "Finite state Automata Bucket Elimination" (FABE).

Since the results of our \otimes and \Downarrow operations are equivalent to the original ones, it follows that, as BE, FABE is also an exact algorithm. Finally, we remark that our \otimes and \Downarrow operations can directly be used within the approximated version of BE, i.e., MBE [6].

V. EXPERIMENTAL EVALUATION

We empirically evaluate FABE by comparing it against the RBFAOO algorithm [9]. We consider RBFAOO as a competitor since it has been empirically shown that it is superior to other sequential algorithms for exact MPE inference, namely AOBB [8] and AOBF [7]. We cannot directly compare against the parallel version of RBFAOO, i.e., SPRBFAOO [10], because its implementation has not been made public. We discarded the option of re-implementing SPRBFAOO, as it would have probably led to an unfair comparison due to a sub-optimal implementation. Nonetheless, since RBFAOO is also used as baseline for speed-up calculation in [10], in Table III we compare our values of speed-up with the ones reported for SPRBFAOO by its authors. We also compare FABE against the SMP approach by Gogate and Domingos [13] (see associated discussion in Section III). Since SMP relies on ADDs (which cannot represent non-binary variables natively), we encode nonbinary variables using one-hot encoding, following a standard practice. We do not show results comparing FABE against the standard version of BE with tabular functions [5], since the latter runs out of memory on most of the instances due to its exponential memory requirements.

We evaluate all algorithms on standard benchmark datasets for exact MPE inference [9], [10], i.e., protein, pedigree, grid. In addition, we also consider standard WCSP benchmark datasets [8], i.e., spot5, mastermind, iscas89.⁴ For WCSPs we also compare FABE against toulbar2 [26], a standard solver used for exact optimisation of cost networks.

Since both FABE and RBFAOO require to compute the same variable ordering d before execution, we consider this as a pre-processing phase and we do not include its runtime in the reported results, also because it is negligible *wrt* the runtime of the solution phase. For each problem instance, we compute

⁴Online at: www.ics.uci.edu/~dechter/softwares/benchmarks.

MPE	protein	pedigree	grid
i	$\{2, 4\}$	$\{6, 14\}$	$\{6, 14\}$
WCSP	spot5	mastermind	iscas89
i	$\{8, \ldots, 18\}$	$\{8, \dots, 18\}$	$\{8, \dots, 18\}$

TABLE II: RBFAOO *i* parameters for each dataset.

Dataset	protein	pedigree	grid
Average Redundancy	96%	85%	64%
FABE speed-up wrt RBFAOO	58.6 (1%, 11%)	5.5 (0%, 32%)	0.1 (43%, 0%)
FABE speed-up wrt SMP	1006.5 (1%, 38%)	6.8 (0%, 5%)	4.0 (43%, 70%)
SPRBFAOO speed-up wrt RBFAOO	~7	~7	~5
Dataset	spot5	mastermind	iscas89
Average Redundancy	85%	85%	87%
FABE speed-up wrt RBFAOO	36.9 (0%, 0%)	6.2 (0%, 0%)	0.4 (0%, 0%)
FABE speed-up wrt SMP	5.8 (0%, 0%)	2.5 (0%, 0%)	5.1 (0%, 0%)
FABE speed-up wrt toulbar2	10615.5 (0%, 50%)	0.4 (0%, 0%)	0.1 (0%, 0%)

TABLE III: Average speed-up results for MPE (top) and WCSP (bottom) instances. For SPRBFAOO we report the same speed-up values reported by the authors [10]. Values in parentheses indicate the percentages of instances unsolved by first and second algorithm.

d using a weighted MIN-FILL heuristic [1], and we use the same d for both algorithms. We execute RBFAOO with the parameters detailed in authors' previous work [8], [9], [10], including cache size and i parameter (see Table II).

Following [10], we set a time limit of 2 hours. We exclude from our analysis all instances that could not be solved by any algorithm in the considered time limit. FABE and SMP are implemented in C++.⁵ We employ the implementations of RBFAOO and toulbar2 provided by the authors. All implementations have been compiled with the same options. All experiments have been run on a cluster whose computing nodes have 2.50GHz CPUs and 384 GBytes of RAM. As for Remark 1, for FABE we consider $\epsilon = 10^{-10}$. Given the large number of instances in MPE datasets, in Table I we only report the runtimes on the 8 largest instances wrt the number of variables. Full experimental results on MPE datasets are reported in A. In Table III we report the aggregated results of the speed-up achieved by FABE wrt other approaches. Each speed-up is calculated only considering the instances where both algorithms terminate within the time limit. Information about unsolved instances is also reported in Table III.

Results confirm that FABE's performance is correlated with the value of redundancy. FABE obtains good performance on the protein and pedigree datasets, achieving speedups of \sim 1-2 orders of magnitude, and solving a total of 34 instances that RBFAOO could not solve. As expected, RBFAOO is superior on the grid dataset, which is characterised by low

⁵Our source code is available at https://github.com/filippobistaffa/FABE.

spot5	42b	505b	408b	29	503	54
FABE RBFAOO SMP toulbar2	0.26 13.37 1.62 > 2 h	0.26 10.27 1.80 > 2 h	0.29 9.97 1.60 > 2 h	0.09 5.61 0.72 0.10	0.05 1.37 0.27 1957.80	0.07 1.36 0.30 0.09
mastermind	3-8-5	10-8-3	4-8-4	3-8-4	4-8-3	3-8-3
FABE RBFAOO SMP toulbar2	247.27 4.93 659.42 0.18	69.30 3.01 185.45 0.09	0.36 2.96 0.95 0.09	0.22 1.96 0.43 0.12	0.10 0.85 0.29 0.05	0.06 0.68 0.11 0.06
iscas89	s1238	c880	s1196	s953	s1494	s1488
FABE RBFAOO SMP toulbar2	38.50 1.47 229.64 0.04	25.36 1.17 146.43 0.06	73.54 0.61 410.96 0.04	286.43 0.54 1464.98 0.04	1.42 0.41 9.78 0.04	1.12 0.39 6.02 0.07

TABLE IV: Runtime results (in seconds) on WCSP instances.

redundancy. Results also show that, despite not employing parallelism, FABE's speed-up on the protein dataset is much higher than the one reported for SPRBFAOO, while it is comparable on the pedigree datasets.

As for WCSPs (Table IV), FABE outperforms both RBFAOO and toulbar2 on the spot5 dataset. On the mastermind dataset, FABE is comparable with toulbar2 (since both compute solutions in tenths of seconds) but superior to RB-FAOO, except for 3-8-5 and 10-8-3 instances. toulbar2 is superior on the iscas89 dataset.

Finally, FABE consistently outperforms SMP using onehot encoding, confirming that the use of additional encodings (required by the presence of non-binary variables that cannot be represented by ADDs) introduces a significant overhead compared to our representation using DAFSA, which can natively represent non-binary variables. Such an impact is more pronounced on datasets with larger variable domains, which require more binary variables to be represented by ADDs. Indeed, FABE obtains a speed-up of 3 orders of magnitude on the protein dataset, where variables reach a domain of 81.

VI. CONCLUSIONS

We proposed FABE, an algorithm for exact MPE inference and constrained optimisation that exploits our concise function representation based on DAFSA. Results achieved by comparing FABE with state of the art approaches following an established experimental methodology confirm the efficacy of our concise function representation.

Future research directions include extending FABE to other exact inference tasks and integrating FABE (in its alreadyavailable Mini-Bucket version) to compute the initial heuristic for AND/OR search algorithms, which, at the moment, use the table-based implementation of BE. We deem this research direction very relevant since the computation of the MBE heuristic for AND/OR search algorithms can represent a bottleneck for high values of i, forcing one to resort to values of i that correspond to weaker heuristics, as acknowledged in [10]. A faster version of MBE could represent an important contribution for this family of algorithms.

ACKNOWLEDGMENT

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T	Variables	D - 4 4	Runtime (seconds)			
Instance	variables	Redundancy	FABE	RBFAOO	SMP	
pdb1duw	241	96.61%	21.36	> 2 h	> 2 h	
pdb1hcz	211	96.22%	10.33	749.39	> 2 h	
pdblfny	199	94.37%	6.60	> 2 h	2036.29	
pdb2hft	190	96.31%	322.33	1765.22	6569.95	
pdb1ad2	177	96.74%	25.28	1654.75	> 2 h	
pdblatg	175	95.47%	3.28	1697.87	4098.89	
pdblqre	175	95.43%	3.47	734.85	1721.50	
pdblqhv	173	95.86%	16.54	> 2 h	4376.94	
pdb1pbv	170	96.50%	234.70	1543.11	> 2 h	
pdb1g3p	165	96.04%	1.21	5080.64	580.33	
pdb2fcb	158	95.49%	3.80	1677.44	6780.26	
pdbleuo	157	96.75%	39.81	> 2 h	13/4.51	
pdblfnl	157	95.79%	8.43	> 2 h	> 2 h	
pdblkgd	156	96.43%	28.05	> 2 h	> 2 h	
pdbld16	154	96.23%	> 2 h	1802.38	> 2 h	
pablkia	153	96.69%	4343.93	> 2 n	> 2 n	
pabinuw	152	96.41%	155.20	4102.90	> 2 n	
pdblwba	151	96.79%	770.00	210.38	43/0.2/	
	130	90.09%	594.44	2004.97	> 2 II > 2 h	
publka0	140	90.70%	> 2 II 781 32	> 2 II > 2 h	> 2 II > 2 h	
pdb1131	147	90.30%	214.80	> 2 II > 2 h	> 2 II > 2 h	
publase pdblalu	144	97.05%	5/3/ 08	> 2 II > 2 h	> 2 II > 2 h	
publalu pdb2fcr	1/13	96.03%	5 03	> 2 h	> 2 h	
pdb21C1	143	96.05%	3.50	1125.04	6111.28	
pdb2e2C	142	96 72%	5313.47	3024.08	> 2 h	
publics]	140	96 56%	3277.05	> 2 h	> 2 h	
pdb1c31	140	96 56%	15 28	1979 26	> 2 h	
pdb1ciw	139	95.85%	926.69	> 2 h	> 2 h	
pdb1e3b	139	96.79%	52.45	> 2 h	> 2 h	
pdb1ek0	139	96.83%	69.81	18.65	6451.68	
pdblamx	137	95.46%	26.24	> 2 h	> 2 h	
pdb1r16	137	96.71%	39.74	34.01	4533.46	
pdb1bv1	134	95.78%	4494.03	5492.51	> 2 h	
pdb1b8e	133	96.00%	43.60	> 2 h	> 2 h	
pdb1j98	133	96.15%	36.24	> 2 h	> 2 h	
pdb1b2v	132	90.14%	0.54	0.22	45.15	
pdblqnt	132	96.35%	95.04	451.55	6957.60	
pdb1dg6	131	96.05%	162.50	> 2 h	> 2 h	
pdb1bgc	130	96.29%	3625.70	1801.50	6944.96	
pdb1dk8	130	96.36%	6.71	929.46	3595.40	
pdbldvo	127	96.97%	226.02	15.44	> 2 h	
pdb1buu	126	96.51%	1.07	272.80	5946.49	
pdblexr	125	96.23%	15.22	4.57	6364.23	
pdb1h6h	125	96.68%	467.49	187.14	> 2 h	
pdb1cbs	123	96.71%	43.71	1446.61	> 2 h	
pdbldbu	123	96.95%	1819.33	3052.28	> 2 h	
pdblrcy	123	95.63%	1.30	1.59	1854.50	
pdblaly	122	95.71%	190.04	100.19	> 2 h	
pablatu	122	90.18%	15.21	17.81	$> 2 \Pi$	
papit]	122	93.02%	1.1/	155.94	1745 44	
	122	90.03%	1.9/	> 2 II 1282.00	1/43.44	
papilyp	122	90.92% 06.92%	1909.23	4362.00	> 2 n \ 2 h	
pub2111D	122	90.83% 06 75%	17.00	20.0U 2017.61	> 2 II > 2 h	
pdb1c÷0	121	90.13%	17.04	2017.01	> 4 II 1244 12	
publej8 pdb1+fc	121	90.38% 06 55%	705.74 5116.41	6105.95	1244.13	
pubilize	121	90.33%	126 50	1137 45	$> 2 \Pi$ > 2 h	
pubitsy	119	96.80%	3.04	> 2 h	5484 35	
pubivis pdb5pul	119	95.88%	18 25	> 2 h > 2 h	> 2 h	
Publint	117	15.00 /0	10.23	/ 2 11	/ 2 11	

	Runtime (seconds)			nds)		
Instance	Variables	Redundancy				
			FABE	RBFAOO	SMP	
pdb1dqg	118	96.41%	101.99	1895.29	> 2 h	
pdb1e29	117	96.37%	1.92	23.98	2825.05	
pdb1f1e	116	96.66%	576.79	1690.88	> 2 h	
pdbljb3	115	96.41%	> 2 h	4293.90	> 2 h	
pdbllit	115	96.42%	9.60	1/01.41	2805.14	
papicsm pdb1f4p	115	95.45%	1.11	1040.27	0/0.35	
pdb114p	112	95.95%	54 54	60.41	2104.30	
pdb2e11 pdb1bkb	112	90.29 % 97 34%	23.61	402.04	1534 33	
pdb1tn3	111	96.15%	5.16	> 2 h	> 2 h	
pdb1doi	109	95.52%	90.39	801.83	> 2 h	
pdb1hbv	109	97.12%	527.68	1088.08	2731.80	
pdb1buo	108	95.41%	2.01	199.56	> 2 h	
pdb1c9x	108	96.61%	3.01	799.14	> 2 h	
pdb1h9k	108	96.28%	16.88	1.24	445.85	
pdb1fit	107	97.05%	127.17	80.24	> 2 h	
pdblijt	107	96.73%	> 2 h	18.79	> 2 h	
pdb1msc	106	96.41%	2.32	18.38	> 2 h	
pdblqhq	106	94.48%	4.62	34.91	2543.03	
pdb1qjp	106	96.90%	27.79	1499.44	> 2 h	
pdbla62	105	96.18%	6.46	1.74	6021.26	
pdblcuo	104	96.59%	2.84	13.42	2104.38	
pabldix	103	95.21%	3.46	13.11	815.00	
pablekg	103	96.12%	18.37	2/1.55	> 2 n > 2 h	
pdb1 jse	103	90.34%	0.57	2000.50	> 2 II 282 35	
pdb1neu pdb1rfs	102	90.42 %	29.76	14.85	1737.63	
pdb1115	101	96.05%	37.20	762.38	> 2 h	
pdb1ypi	101	96.39%	2.87	99.93	692.77	
pdb1whi	101	97.48%	1405.80	148.15	> 2 h	
pdb3nul	101	96.62%	4.28	76.57	2670.80	
pdb1dly	100	96.53%	2351.17	4806.69	> 2 h	
pdb2tgi	100	94.97%	2.30	8.55	> 2 h	
pdb1c52	99	97.17%	51.26	5.76	3547.11	
pdblsfp	99	95.48%	48.75	51.88	3984.85	
pdb1c44	98	96.44%	9.25	3.69	> 2 h	
pdb1b0b	97	95.74%	7.15	273.85	> 2 h	
pdb1bkr	97	95.70%	31.93	> 2 h	> 2 h	
pdblqto	97	96.12%	1.86	2.55	4201.69	
pdblthx	97	96.17%	285.03	1.61	> 2 h	
papitmy	97	90.31%	102.86	> 2 n 770 02	> 2 n 1704.06	
pap2npg	97	90.02% 07 20%	1.90	118.80 126.06	1/04.00	
pubiceW	90 96	91.20%	5.20 73.00	420.00	> 2 h	
ndh1 ior	96	95 46%	2 95	130.02	1412 22	
pdb2cv3	96	96 64%	0.35	1 69	304 60	
pdb7fd1	96	95.10%	4.16	9.85	4736.83	
pdb1a1x	95	96.43%	159.48	879.45	> 2 h	
pdb1bea	95	96.83%	68.10	1352.07	877.64	
pdb1bqk	94	96.80%	2.27	11.12	1031.58	
pdb1npl	94	96.33%	0.84	0.84	1567.54	
pdb2pii	94	97.06%	412.83	241.40	4279.84	
pdb4rhn	94	96.41%	2.07	473.25	3103.93	
pdb1cxc	93	97.11%	3.95	1.80	7118.94	
pdb1rro	93	96.25%	20.87	405.46	> 2 h	
pdblaiu	92	95.99%	2.19	2.13	953.85	
pdb1btn	92	96.88%	63.43	16.04	> 2 h	
pdb1ew4	92	95.92%	28.44	> 2 h	3701.17	
pabipuc	92	97.36%	2.27	102.72	155/.91	
papisk7	92	91.2.5%	8.44	50.14	> 2 n	

TABLE V: Runtime results on protein instances (1-60).

TABLE VI: Runtime results on protein instances (61-120).

.	37 . 11	D 1 1	Runtime (second		nds)
Instance	variables	Redundancy	FABE	RBFAOO	SMP
pdb1t1d	92	97.21%	57.12	2076.69	> 2 h
pdb2rhe	92	95.42%	0.60	0.61	763.51
pdb3kvt	92	96.63%	13.17	> 2 h	2261.06
pdb3vub	92	96.84%	831.03	427.18	5546.07
pdblcqy	91	95.64%	17.76	274.04	> 2 h
pdbldzo	91	96.85%	106.69	4.04	4866.95
pdble85	91	97.08%	2.33	050.66	> 2 n
papigad	91	90.20%	2.03	939.00	1/39.49
publwau	91	90.98%	0.48	2.52	481.62
pdb3cy1	90	96.15%	0.40	122.87	243.67
pdb1vcc	89	96.74%	43.92	> 2 h	> 2 h
pdb3c2c	89	96.79%	16.31	744.95	> 2 h
pdb1bxe	88	97.11%	6.89	17.90	6925.76
pdb1jhq	88	96.60%	14.11	9.27	6608.15
pdb1ncg	88	96.51%	2.44	2.88	3299.30
pdb1ris	88	97.20%	2223.82	22.10	> 2 h
pdb2rta	88	95.40%	0.49	2.72	786.18
pdb3ezm	88	95.64%	0.18	0.43	127.30
pdblopc	87	96.94%	54.91	1041.76	> 2 h
pdb2tir	87	94.66%	0.76	203.02	1426.95
pdb1co6	86	97.14%	7.02	26.46	3056.25
pdblgmx	86	95.44%	12.77	2.10	> 2 h
publaac	63 85	90.85%	12.77	52.97	> 2 II
publibilio pdb1i8o	85 85	90.80%	3.02	1.88	> 2 II > 2 h
pdb1100	85	96.06%	0.84	5.64	671.15
pdb2cdv	85	97.29%	1.62	17.63	1200.34
pdblcpq	84	96.52%	0.46	5.44	465.32
pdblhxi	84	95.73%	6.87	1.27	2066.29
pdb3cao	84	96.19%	0.51	0.87	770.46
pdbldlw	83	95.35%	2.19	79.62	1009.88
pdb1g2r	83	97.16%	226.21	53.32	> 2 h
pdblhbk	83	97.13%	53.65	19.57	> 2 h
pdblqt9	83	93.84%	1.74	44.33	6303.49
pdblplc	82	94.54%	0.29	0.56	102.99
pab2nts	81	90.84%	4/.30	807.30	2014.12
publica pdb1evv	80	94.44%	16.15	1164 53	3128.25
pdb13vy	80	95.66%	1 16	3.18	1477.93
pdb2mcm	80	91.85%	0.06	0.09	4.67
pdb2pvb	80	96.94%	13.68	3838.69	6442.13
pdb1mho	79	96.43%	274.90	6.05	> 2 h
pdb1cyo	78	96.83%	6.26	129.71	1381.34
pdb2cbp	77	96.60%	1.99	0.35	4771.07
pdblaba	76	96.29%	3.01	2.67	4263.71
pdblgvp	76	96.19%	1.98	1.26	1357.80
pdbltig	76	97.38%	551.97	5.26	> 2 h
pdblcei	75	96.02%	3446.15	1.46	5/54.94
pdblina	/5 75	96.46%	0.15	0.17	64.35
pdb1bvv	73 74	90.89%	111.00	1000.09	2164 16
pdb1cxv	69	95.41%	0.61	4 34	2510 95
pdb1h98	68	96.16%	0.95	4.76	> 2 h
pdb1iq5	68	95.97%	348.94	178.90	> 2 h
pdb1i27	67	97.34%	122.06	8.36	5017.91
pdbliqz	67	94.68%	1.30	0.89	211.61
pdb1cc8	66	96.64%	25.51	19.05	1962.13
pdb1fk5	66	96.98%	0.40	2.37	166.39
pdblgdv	66	95.77%	3.89	0.94	3287.50

			Runtime (seconds)			
Instance	Variables	Redundancy ·	FABE	RBFAOO	SMP	
pdb1tif	66	97.22%	15.04	7.17	> 2 h	
pdb1en2	65	96.31%	0.74	0.76	459.84	
pdblrzl	65	96.57%	4.62	1.13	2076.39	
pdblutg	65	96.05%	0.89	0.82	2607.19	
pdb1bt0	64	96.40%	189.50	164.53	3695.00	
pdblh/5	64 64	96.35%	0.20	3.30	> 2 n	
papikpo pabinto	04 64	93.38%	0.09	5.15	111 24	
pabinin	64	95.78%	2.66	0.40	2 h	
pdb3118	63	95 38%	0.43	0.22	524.05	
pdb1vfv	63	96 55%	1 43	1 40	1601 58	
pdb1ail	62	97.29%	26.49	2.22	> 2 h	
pdb451c	62	96.91%	294.31	1.16	> 2 h	
pdb1ctj	61	95.40%	4.37	0.41	1908.81	
pdb1b0y	60	96.11%	43.14	583.95	> 2 h	
pdb1hoe	60	95.83%	0.10	0.17	45.79	
pdb1f94	58	96.24%	2.65	0.98	305.53	
pdb1hpi	57	96.10%	0.58	4.04	268.80	
pdb1k51	57	96.20%	0.14	0.41	100.65	
pdb3ebx	57	97.09%	0.41	0.97	747.96	
pdb1a8o	56	97.10%	4.11	2.68	3267.70	
pdb1hg7	56	95.86%	0.34	0.78	397.53	
pdblypc	56	96.97%	135.69	1.66	> 2 h	
pdblfas	55	97.61%	2.87	3.78	1528.03	
pdb1g2b	55	96.98%	20.44	1.55	1430.55	
pdblaho	54	96.11%	0.21	0.46	146.97	
pdb1hh5	54	96.27%	1.35	1.18	2529.51	
pdblmjc	54	93.72%	0.17	1.70	42.96	
pdb1r69	54 52	97.10%	326.78	3.04	4953.95	
pdbla/W	53 52	96.91%	1.25	1.55	120.12	
pubzsns	55	90.45%	0.15	0.49	516.07	
publul4	51	95.95%	0.52	0.34	22 52	
pdb1h7d	50	96 95%	0.00	17.85	2361.41	
pdb1kth	50	96.77%	0.41	2.85	510.41	
pdb1nkd	50	96.85%	1.92	10.92	> 2 h	
pdb2iqd	50	95.64%	0.21	0.61	496.56	
pdb1be7	48	96.40%	76.01	0.81	> 2 h	
pdb2ovo	48	95.46%	0.16	0.42	47.69	
pdb1c75	47	96.08%	0.94	0.56	340.63	
pdblctf	47	96.48%	0.96	1.09	2158.53	
pdb1mof	46	96.50%	0.20	0.52	1824.54	
pdb1g6x	44	96.35%	0.82	0.43	3554.66	
pdb1rb9	42	95.30%	0.19	0.31	185.82	
pdb2fdn	42	92.84%	0.04	0.05	3.79	
	39 20	97.07%	1.25	1.85	> 2 n 14 22	
publjoe pdb2orl	39	90.07%	0.08	0.14	5 17	
pubzeri pdblajj	32	94.41%	0.05	0.11	53 52	
publajj pdb1pig	29	96 92%	0.00	1.01	2401 71	
pdblaie	26	97.37%	3.88	0.52	> 2 h	
pdb1rh4	21	96.87%	0.12	0.96	341.66	
pdb1pef	17	96.26%	0.08	0.15	104.86	
pdb1akg	14	81.02%	0.01	0.00	0.04	
pdb1pen	13	86.52%	0.01	0.01	0.12	
pdb1not	11	90.15%	0.01	0.01	0.79	
pdbletm	10	86.59%	0.01	0.00	0.08	
pdbletl	9	81.65%	0.00	0.00	0.07	
pdbletn	9	82.81%	0.01	0.00	0.07	
pdb1xy2	7	93.29%	0.01	0.01	0.22	

TABLE VII: Runtime results on protein instances (121-180).

TABLE VIII: Runtime results on protein instances (181-240).

Instance	Variablas	Dadundanar	Runtime (seconds)			
Instance	variables	Redundancy	FABE	RBFAOO	SMP	
pedigree25	1289	79.06%	28.82	6.32	197.89	
pedigree30	1289	81.97%	7.23	61.34	40.86	
pedigree39	1272	78.40%	3.21	22.46	17.06	
pedigree18	1184	82.65%	7.42	20.11	40.89	
pedigree31	1183	87.43%	910.46	> 2 h	5881.66	
pedigree34	1160	80.74%	8.83	> 2 h	60.78	
pedigree51	1152	82.00%	132.92	> 2 h	789.65	
pedigree9	1118	87.06%	473.94	100.19	2040.32	
pedigree13	1077	78.10%	519.08	> 2 h	2011.26	
pedigree7	1068	79.03%	21.79	323.77	136.11	
pedigree41	1062	85.49%	24.06	> 2 h	151.40	
pedigree37	1032	79.69%	6.73	146.23	27.91	
pedigree40	1030	90.36%	808.10	> 2 h	3683.38	
pedigree44	811	84.81%	75.90	765.39	1316.35	
pedigree33	798	79.90%	2.45	19.47	12.62	
pedigree19	793	90.66%	148.93	> 2 h	1398.50	
pedigree38	724	90.25%	37.96	7.88	453.58	
pedigree50	514	93.64%	4.03	17.26	48.83	
pedigree42	448	90.69%	976.28	610.55	4779.61	
pedigree20	437	88.34%	369.82	632.75	> 2 h	
pedigree23	402	86.17%	86.27	150.26	320.63	
pedigree1	334	86.28%	2.77	0.36	16.40	

TABLE IX: Runtime results on pedigree instances.

T	V	De dese des est	Runtime (seconds)			
Instance	variables	Redundancy	FABE	RBFAOO	SMP	
90-26-5	676	77.03%	3192.15	925.47	> 2 h	
90-25-5	625	77.11%	> 2 h	902.19	> 2 h	
90-24-5	576	79.86%	5112.09	1758.19	> 2 h	
75-23-5	529	66.62%	> 2 h	791.70	> 2 h	
90-23-5	529	77.43%	508.75	158.17	2453.45	
75-22-5	484	64.76%	> 2 h	816.87	> 2 h	
90-22-5	484	77.72%	4883.60	20.37	> 2 h	
75-21-5	441	65.70%	> 2 h	4.71	> 2 h	
90-21-5	441	76.00%	142.23	8.22	802.17	
50-20-5	400	41.22%	> 2 h	163.44	> 2 h	
75-20-5	400	64.99%	831.80	19.01	3711.20	
90-20-5	400	77.71%	267.50	10.27	946.07	
50-19-5	361	41.78%	> 2 h	153.02	> 2 h	
75-19-5	361	64.50%	> 2 h	10.74	> 2 h	
50-18-5	324	41.10%	> 2 h	73.03	> 2 h	
75-18-5	324	66.24%	845.21	9.96	4300.02	
50-17-5	289	41.41%	> 2 h	7.76	> 2 h	
75-17-5	289	65.70%	1525.23	3.21	> 2 h	
50-16-5	256	42.40%	> 2 h	3.10	> 2 h	
75-16-5	256	65.30%	6452.75	1.88	> 2 h	
50-15-5	225	43.34%	1335.12	2.71	4098.00	
50-14-5	196	40.54%	1442.72	1.13	> 2 h	
50-12-5	144	37.24%	749.27	0.84	1289.04	

TABLE X: Runtime results on grid instances.

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