On some relationships between the WOWA operator and the Choquet integral

Vicenç Torra
Departament d’Enginyeria Informàtica (ETSE), Universitat Rovira i Virgili
Carretera de Salou s/n, E-43006 Tarragona (Catalunya, Spain)
E-mail: vtorra@etse.urv.es

Abstract
In this paper we compare the WOWA operator (an aggregation operator that generalizes both the weighted mean and the OWA operator) with the Choquet integral. We show that the Sugeno λ-measure and the decomposable fuzzy measures are particular cases of the measure implicit in the WOWA operator.

Keywords: WOWA Operator, OWA Operator, Choquet integral, fuzzy measures

1. Introduction

Recently, several aggregation operators have been defined to be applied in different fields of Artificial Intelligence. Among them, in [11] it was defined the WOWA operator that generalizes the weighted mean (characterized in [1]) and the OWA operator (defined in [15] and characterized in [2]).

In this paper, we study an open problem of the WOWA operator: its relation with the Choquet integral. We show that the former operator is a particular case of the latter. As the Choquet integral is a WOWA operator only for some fuzzy measures, we study whether some of the measures defined in the literature (Sugeno λ-measures, and decomposable fuzzy measures) can be used with WOWA operators.

After this introduction, the paper begins with some definitions and results used later on, and then it follows (section 3), with the relationship between the WOWA operator and the Choquet integral. The paper finishes with the conclusions.

2. Preliminaries

In this section we review some results that are used later on. We review the representation theorem of t-conorms as these connectives are used later when considering decomposable fuzzy measures. We review also fuzzy measures and, in particular, the Sugeno λ-measures and the decomposable ones. At the end of the section we review the OWA and the WOWA operators, the Choquet integral and its relation with the OWA operator.

Definition 1. A binary operator $S: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-conorm if it satisfies for all $a,b,c \in [0,1]$ the following axioms: (i) boundary condition: $S(a,0)=a$; (ii) monotonicity: $b \leq c$ implies $S(a,b) \leq S(a,c)$; (iii) commutativity: $S(a,b) = S(b,a)$; (iv) associativity $S(a,S(b,c))=S(S(a,b),c)$.

Definition 2. A t-conorm $S$ is an Archimedean t-conorm if $S(a,a)>a$ for all $a$ in $(0,1)$.

Proposition 1. [8, 7, 3] A t-conorm is continuous and Archimedean if and only if there exists a strictly increasing and continuous function $g: [0,1] \rightarrow [0,\infty]$ with $g(0)=0$ such that

\[ S(a,b) = g^{-1}(g(a)+g(b)) \]  \[ = \begin{cases} 0 & \text{if } a \leq 0 \\ g^{-1}(a) & \text{if } 0 \leq a \leq g(1) \\ 1 & \text{otherwise} \end{cases} \]

Now we consider fuzzy measures and the aggregation operators. We will assume in the following that the set of all possible elements is a finite set $X=\{a_1, ..., a_N\}$ and that $N$ is the cardinality of this set (i.e., $N=|X|$). Each element $a_i$ is evaluated by means of a function $f: X \rightarrow [0,1]$, thus $f(a_i)$ is the value that corresponds to element $a_i$. With the function $p: X \rightarrow [0,1]$ we denote the weight of each element (its importance). For the sake of simplicity $p_i$ stands for $p(a_i)$. In the following, a weighting vector $r$ is a vector of dimension $N$ where $r_i \in [0,1]$ and $\sum_{i=1}^N r_i = 1$. It is assumed that $(p_1, ..., p_N)$ is a weighting vector.

We define now fuzzy measures [9, 18]. As the set of alternatives is finite, we consider fuzzy measures defined over the power set of $X$ (represented by $\mathcal{P}(X)$).

Definition 3. [4] A function $\mu: \mathcal{P}(X) \rightarrow [0,1]$ is a fuzzy measure if and only if it satisfies the following axioms: (i) $\mu(\emptyset)=0$ and $\mu(X)=1$; (ii) monotonicity: $B_1 \subseteq B_2 \subseteq X$ implies $\mu(B_1) \leq \mu(B_2)$.

Definition 4. A fuzzy measure $\mu$ is a Sugeno λ-measure if $B_1 \cap B_2 = \emptyset$ implies that

\[ \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) + \lambda \mu(B_1) \mu(B_2) \]

for some fixed $\lambda > -1$. 

Definition 5. [4] A function $\varphi: X \rightarrow [0,1]$ is a fuzzy measure if and only if it satisfies the following axioms: (i) $\varphi(\emptyset)=0$ and $\varphi(X)=1$; (ii) monotonicity: $B_1 \subseteq B_2 \subseteq X$ implies $\mu(B_1) \leq \mu(B_2)$.
Proposition 2. [18] Let $\mu$ be a Sugeno $\lambda$-measure, then for any disjoint class $\{E_1, ..., E_r\}$ of subsets of $X$ the value of $\mu(C)$ where $C=E_1 \cup E_2 \cup ... \cup E_r$ is determined by:

$$
\frac{1}{\lambda} \prod_{i=1}^{r} \left[1+\frac{\mu(E_i)}{\lambda}\right] - 1
$$

when $\lambda \neq 0$

$$
\sum_{i=1}^{\lambda} \mu(E_i)
$$

when $\lambda = 0$.

Proposition 3 [10, 6]. Let $\mu$ be a Sugeno $\lambda$-measure, then for a fixed set of $\mu(a_i)$, $0<\mu(a_i)<1$ there exists a unique $\lambda \in (-1, +\infty)$ and $\lambda \neq 0$, which satisfies

$$\lambda + 1 = \prod_{i=1}^{N} (1 + \lambda \mu(\{a_i\}))$$

The interest of this proposition is that given $\mu(\{a_i\})$ for all $a_i \in X$, a Sugeno $\lambda$-measure is completely determined. Now, we consider the aggregation operators.

Definition 5 [15, 16]. Let $w$ be a weighting vector, then the OWA (Ordered Weighted Averaging) operator is defined as:

$$\text{OWA}_w(X) = \sum_{i=1}^{N} w_i \omega(\sigma(i))$$

where $\{\sigma(1), ..., \sigma(N)\}$ is a permutation of $\{1, ..., N\}$ such that $f(\omega(1)) \geq f(\omega(2)) \geq ... \geq f(\omega(N))$.

Definition 6 [16]. A function $Q:[0,1] \rightarrow [0,1]$ such that $Q(0)=0$, $Q(1)=1$ and if $x>y$ then $Q(x) \geq Q(y)$ is a regular monotonically non-decreasing quantifier.

Definition 7 [16]. Let $Q$ be a regular monotonically non-decreasing quantifier, then the OWA operator w.r.t. $Q$ is defined as:

$$\text{OWA}_Q(X) = \sum_{i=1}^{N} w_i f(\omega(\sigma(i)))$$

where $w_i=Q(i/N) - Q((i-1)/N)$, and $f(\omega(\sigma(i)))$ is as in definition 5.

Definitions 5 and 6 are equivalent in the sense that from a fuzzy quantifier $Q$ it is possible to extract a set of weights $w_i$ and apply the former definition, or from a set of weights it is possible to interpolate a function and apply the latter one. In [12] is given an interpolation algorithm that is adequate for this purpose.

Definition 8 [11]. Let $p$ be a weighting vector and $Q$ be a regular non-decreasing quantifier, then the WOWA (Weighted OWA) operator w.r.t. $p$ and $Q$ is defined as:

$$\text{WOWA}_Q, p(X) = \sum_{i=1}^{N} \omega_i f(\omega(\sigma(i)))$$

where $\omega_i$ is defined as $\omega_i = Q(\sum_{j<i} p(\sigma(j))) - Q(\sum_{j<i} p(\sigma(j)))$

We define here the WOWA operator using $Q$ instead of using another weighting vector $w$ (as in the OWA definition above). Both definitions are equivalent as reported in [13]. The definition of the WOWA with $p$ and $Q$ is equivalent to the OWA operator considering importance introduced by R. R. Yager in [17].

Definition 9 [4]. Given a fuzzy measure $\mu$, the Choquet integral with respect to $\mu$ is defined by:

$$C_\mu(\{a_1\}, ..., f(\{a_N\})) = \sum_{i=1}^{N} (f(a_{\sigma(i)}) - f(a_{\sigma(i-1)})) \mu(A_{\sigma(i)})$$

where $f(a_{\sigma(i)})$ indicates that the indices have been permutated so that $0 \leq f(a_{\sigma(1)}) \leq ... \leq f(a_{\sigma(N)}) \leq 1$, $A_{\sigma(i)} = \{a_{\sigma(i)}, ..., a_{\sigma(N)}\}$ and $f(a_{\sigma(0)})=0$.

Alternatively, we can express the Choquet integral as:

Proposition 4. Given a fuzzy measure $\mu$ the Choquet integral with respect to $\mu$ can be expressed as:

$$C_\mu(\{a_1\}, ..., f(\{a_N\})) = \sum_{k=1}^{N} f(\omega(\sigma(k))) \mu(A_{\sigma(k)})$$

where $\{\sigma(1), ..., \sigma(N)\}$ is a permutation of $\{1, ..., N\}$ such that $f(a_{\sigma(1)}) \geq f(a_{\sigma(2)}) \geq ... \geq f(a_{\sigma(N)})$, $A_{\sigma(k)} = \{a_{\sigma(j)} \mid j \leq k\}$ (therefore $A_{\sigma(0)} = \{a_{\sigma(1)}, ..., a_{\sigma(r)}\}$ when $r=1$ and $A_{\sigma(0)}=\emptyset$).

We consider now the relation between the OWA operator and the Choquet integral on the sake for completeness. These results are relevant to the WOWA operator because the OWA is a particular case [11] of the WOWA operator when $p_i=1/N$.

Theorem 1 [2, 4]. For every weighting vector $w$, we have $\text{OWA}_w = C_\mu$ with $\mu$ defined by:

$$\mu(B) = \sum_{i=1}^{N} \mu(B \cap a_i)$$

for all $B \subseteq X$.

Proof. See [2] Note that in [2, 4, 5], the OWA operator is defined using the permutation $s(i)$ instead of $\sigma(i)$ and thus their formula for $\mu(B)$ is different than the previous one (although equivalent).

When the OWA operator is defined by means of a fuzzy quantifier, the following proposition holds:

Proposition 5. For every monotonically increasing fuzzy quantifier $Q$, we have $\text{OWA}_Q = C_\mu$ when $\mu$ is defined by

$$\mu(B) = Q(|B| / |X|)$$

for all $B \subseteq X$.
Proof. As weights are extracted from a quantifier Q defining \( w_i = Q(i/N) - Q((i-1)/N) \) we can apply the previous theorem obtaining:
\[
\mu(B) = \sum_{i=1}^{\lvert B \rvert} \left[ (Q(i/N) - Q((i-1)/N)) \right] = Q(|B|/N) - Q(0) = Q(|B|/|X|)
\]
So, this proposition is proven. \( \square \)

The inverse, that a Choquet integral with any fuzzy measure can be expressed as an OWA operator is not true. We have to restrict fuzzy measures to those that for all \( B_1, B_2 \subseteq X \) such that \( |B_1| = |B_2| \) implies \( \mu(B_1) = \mu(B_2) \). This is established in the following propositions:

Theorem 2 [4]. A Choquet integral \( C_\mu \) is commutative (i.e., \( C_\mu(x_1, ..., x_N) = C_\mu(x_{\pi(1)}, ..., x_{\pi(N)}) \) for any permutation \( \{\pi(1), ..., \pi(N)\} \) if and only if it satisfies for all \( B_1, B_2 \subseteq X \) such that \( |B_1| = |B_2| \) then \( \mu(B_1) = \mu(B_2) \).

Theorem 3 [4]. Any commutative Choquet integral \( C_\mu \) is an OWA operator whose weights are \( w_i = \mu(A_i) - \mu(A_{i-1}) \) \( i=1, ..., N \) where \( A_i \) is any subset of \( X \) with \( |A_i| = i \) (note that in particular we can use \( A_{\sigma(i)} \) as \( A_i \) because \( |A_{\sigma(i)}| = i \)).

3. Fuzzy measures and the WOWA operator

In this section we consider the relationship between the WOWA operator and the Choquet integral.

Theorem 4. For every regular monotonically non-decreasing fuzzy quantifier \( Q \) we have WOWA \( C_\mu \) with \( \mu \) defined by:
\[
\mu_Q(p)(B) = Q\left( \sum_{a \in B} p_a \right)
\]
(5)

Proof. Let us consider expression (4) with \( x_i = f(a_i) \) and \( x_{\sigma(i)} = f(a_{\sigma(i)}) \):
\[
C_\mu(x_1, ..., x_N) = \sum_{k=1}^{N} x_{\sigma(k)} \sum_{A_{\sigma(k)}} \mu(A_{\sigma(k)}) - \sum_{A_{\sigma(k)}} \mu(A_{\sigma(k)-1})
\]
(6)

We have proven here that all WOWA operators are Choquet integrals, however the reversal is not always true. To be true, all fuzzy measures have to be decomposed into a weighting vector \( p \) and a monotonically increasing fuzzy quantifier \( Q \) such that for any \( B \subseteq X \) we have that
\[
\mu_Q(p)(B) = Q\left( \sum_{a \in B} p_a \right)
\]
(6)

When a Choquet integral is used with a fuzzy measure of this form, there is an equivalent WOWA operator. We call a fuzzy measure of these characteristics a Q-p-decomposable fuzzy measure:

Definition 10 A Q-p-decomposable fuzzy measure \( \mu \) (a Q-p-measure for short) is a fuzzy measure \( \mu \) that can be decomposed into a weighting vector \( p = (p_1, ..., p_N) \) and a monotonically increasing fuzzy quantifier \( Q \) so that for all \( B \subseteq X \) it is satisfied:
\[
\mu_Q(p)(B) = Q\left( \sum_{a \in B} p_a \right)
\]
(6)

Proposition 6. Any Choquet integral \( C_\mu \) with a Q-p-decomposable fuzzy measure is a WOWA operator.

We show below with an example that not all fuzzy measures are Q-p-measures and we study some measures introduced in the literature showing that they are Q-p-measures.

Example. Let us consider the measure defined in [4], [5]. Let \( \mu \) be the fuzzy measure on \( X = \{M, P, L\} \) defined as:
\[
\mu(M) = 0.45, \quad \mu(L) = 0.3
\]
\[
\mu({M,L}) = 0.9, \quad \mu({M,P}) = 0.5
\]
\[
\mu({P}) = 0.3
\]
It is easy to see that this measure is not a Q-p-measure. If \( \mu \) were a Q-p-measure, then it would hold:
\[
\mu({L}) = Q(p(L)) < Q(p(M)) = \mu(M)
\]
Therefore, \( p(L) < p(M) \) and thus \( p(L)+p(P) < p(M)+p(P) \). This implies:
\[
\mu({L,P}) = Q(p(L)+p(P)) < Q(p(M)+p(P)) = \mu({M,P})
\]
However, this is in contradiction with the measure given above. So, the measure is not a Q-p-measure. \( \square \)

Let us consider now the family of decomposable fuzzy measures. We prove below that they are Q-p-decomposable for continuous archimedean t-conorms (Theorem 5) and for the t-conorm maximum (proposition 7).

Theorem 5. Any fuzzy measure that is decomposable by means of a continuous archimedean t-conorm is a Q-p-decomposable fuzzy measure.
Proof. Assume that \( \mu \) is decomposable by means of a t-conorm \( S \), then
\[
\mu(A \cup B) = S(\mu(A), \mu(B)) \quad \text{when } A \cap B = \emptyset
\]
for any set \( B = \{b_1, \ldots, b_m\} \), \( B \subseteq X \) we can write, taking into account that \( S \) is associative:
\[
\mu(B) = S(\mu\{b_1\}, \ldots, \mu\{b_m\})
\]
Now, as according to proposition 1 all archimedean t-conorms are of the form \( (\cdot)^w \), we can rewrite \( \mu(B) \) as:
\[
\mu_w(B) = g \left( \sum_{i \in B} \mu\{a_i\}^w \right)^{1/w}
\]

We consider below three examples of decomposable fuzzy measures with Yager's family of t-conorms [14].

Example. Let \( \mu_1 \), \( \mu_2 \) and \( \mu_3 \) be defined according to table 1. These measures are decomposable with a t-conorm of the form \( S_w(a, b) = \min(1, (a^w + b^w)^{1/w}) \). As the generator of this family is \( g(x) = x^w \) and its quasi-inverse (see (2)) is:
\[
g^{-1}(x) = \begin{cases} 
0 & a \in (-\infty, 0) \\
\frac{\lambda}{\lambda - 1} x & a \in (1, +\infty) \\
1 & a \in (0, 1)
\end{cases}
\]

Table 1. Decomposable fuzzy measures

<table>
<thead>
<tr>
<th>B \subseteq X</th>
<th>( \mu_1 ), ( w=2 )</th>
<th>( \mu_2 ), ( w=1/2 )</th>
<th>( \mu_3 ), ( w=1/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{x_1}</td>
<td>0.6</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>{x_2}</td>
<td>0.7</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>{x_3}</td>
<td>0.8</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>{x_1, x_2}</td>
<td>0.92</td>
<td>1.0</td>
<td>0.58</td>
</tr>
<tr>
<td>{x_1, x_3}</td>
<td>1.0</td>
<td>1.0</td>
<td>0.75</td>
</tr>
<tr>
<td>{x_2, x_3}</td>
<td>1.0</td>
<td>1.0</td>
<td>0.99</td>
</tr>
<tr>
<td>{x_1, x_2, x_3}</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Let \( Q(x) = g^{-1}(x) \) and \( q_i = g(\mu\{a_i\})^w \), then note that \( q_i \) is a positive number, and that \( Q' \) is an increasing function. However, as \( p_i \) have to satisfy \( \sum_{i=1}^{n} p_i = 1 \) and \( Q(x) \) has to be defined in \([0,1]\) and with \( Q(1) = 1 \), we define
\[
Q(x) = Q'(x/K) \quad \text{and} \quad p_i = q_i / K
\]
with \( K = \sum_{a_i \in X} q_i \).

We can conclude using the previous theorem that
\[
\mu_w(B) = \left( \sum_{i \in B} \mu\{a_i\}^w \right)^{1/w}
\]
where \( K = \sum_{i=1}^{3} p_i \).

We consider below three examples of decomposable fuzzy measures with Yager's family of t-conorms [14].

Table 2. Weighting vectors and constant K

<table>
<thead>
<tr>
<th>( \mu_1 )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>p_1</td>
<td>0.242</td>
<td>0.290</td>
</tr>
<tr>
<td>p_2</td>
<td>0.329</td>
<td>0.335</td>
</tr>
<tr>
<td>p_3</td>
<td>0.429</td>
<td>0.375</td>
</tr>
<tr>
<td>K</td>
<td>1.49</td>
<td>1.887</td>
</tr>
</tbody>
</table>

Due to the fact that Sugeno \( \lambda \)-measures are decomposable by means of a continuous archimedean t-conorm, we have the following corollary:

Corollary 1. Any Sugeno \( \lambda \)-measure is a Q-p-decomposable fuzzy measure.

The weights \( p_i \) and the quantifier \( Q \) are defined for Sugeno \( \lambda \)-measures as:

Case \( \lambda = 0 \). \( Q(x) = x \) and \( p_i = \mu\{a_i\} \).

Case \( \lambda \neq 0 \). \( p_i = \frac{\ln(1 + \lambda \mu\{a_i\})}{\ln(1 + \lambda)} \) \( Q(x) = e^{x \ln(1 + \lambda)} \)

Note that in these definitions we use \( Q'(x) = g^{-1}(x) \), \( q_i = g(\mu\{a_i\}) \), \( Q(x) = Q'(x/K) \) and \( p_i = q_i / K \) from theorem 5 with \( K=1 \), and \( g(x) \) and \( g^{-1}(x) \) defined as:
\[
g_\lambda(x) = \ln(1 + \lambda x) / \ln(1 + \lambda)
\]
\[
g^{-1}_\lambda(x) = (e^{x \ln(1 + \lambda)})^{-1}
\]
\( K=1 \) follows from the fact that this equality holds:
\[
\sum_{a_i \in X} \ln(1 + \lambda \mu\{a_i\}) = \ln(1 + \lambda)
\]
This is so because \( \mu(X) = 1 \) and in this case (according to proposition 2):
\[
1 = \mu(X) = \prod_{i=1}^{N} \frac{1}{\lambda} \left( 1 + \lambda \mu(E_i) \right) - 1
\]
Therefore
\[
\prod_{a_i \in X} \left( 1 + \lambda \mu\{a_i\} \right) = 1 + \lambda
\]
From this, it is easy to see that
\[
\sum_{a_i \in X} \ln(1 + \lambda \mu\{a_i\}) = \ln(1 + \lambda)
\]
We give now an example that shows how from a normalized Sugeno $\lambda$-measure, we can define a $Q$-$p$-decomposable fuzzy measure. The example uses the data of example 1 in [10].

**Example.** Let $\mu$ be the Sugeno $\lambda$-measure defined by $\lambda = 3.109$ and by:

$$
\begin{align*}
\mu(x_1) &= 0.1 \\
\mu(x_2) &= 0.3 \\
\mu(x_3) &= 0.2
\end{align*}
$$

The complete measure is:

$$
\begin{align*}
\mu(\emptyset) &= 0.0 \\
\mu(x_1) &= 0.1 \\
\mu(x_2) &= 0.3 \\
\mu(x_3) &= 0.2 \\
\mu(x_1, x_2) &= 0.493 \\
\mu(x_1, x_3) &= 0.362 \\
\mu(x_2, x_3) &= 0.687 \\
\mu(x_1, x_2, x_3) &= 1.0
\end{align*}
$$

Using the previous definitions, and as $\lambda \neq 0$ we can determine the weighting vector $p$. We have $\log(1+\lambda) = 1.4131$ and $p$ is:

$$
\begin{align*}
p_1 &= 0.1915 \\
p_2 &= 0.4622 \\
p_3 &= 0.3421
\end{align*}
$$

Similarly, the function $Q(x)$ is:

$$
Q(x) = \lambda^{-1} \left( e^{x \ln (1+\lambda)} - 1 \right) = 0.322 \left( e^{1.4131 x} - 1 \right)
$$

This function together with the weighting vector given above, permits to construct the fuzzy measure of the example using expression (6):

$$
\mu_{Q,p}(B) = Q \left( \sum_{a_i \in B} p_i \right)
$$

For simplicity $\mu_{r(i)}$ stands for $\mu(\{ a_{r(i)} \})$. Using the permutation $r$ we can express $\mu(B)$ as:

$$
\mu(B) = \max_{a_{r(i)} \in B} \mu(\{ a_{r(i)} \})
$$

To prove the proposition we define a quantifier $Q$ and a set of weights $p_{r(i)}$ and we show that they are equivalent to the measure defined by $\mu_{r(i)}$ and the maximum. These are the weights and the quantifier:

$$
\begin{align*}
p_{r(i)} &= \frac{1}{2^{N-i+1}} & \text{when } i \in \{1, \ldots, N-1\} \\
p_{r(N)} &= \frac{1}{2} + \frac{1}{2^{N+1}} & \text{when } i = N
\end{align*}
$$

$$
Q(x) = \begin{cases} 
0 & x \in [0, 1/2^{N+1}) \\
\mu_{r(i)} & x \in [1/2^{N-i+1}, 1/2^{N-i}) \text{ for } i = 1, \ldots, N \\
1 & x = 1
\end{cases}
$$

Note that $\sum_{i=1}^{N} p_{r(i)} = 1$ and that $Q$ is a regular non-decreasing quantifier.

Note also that in the definition of $Q(x)$ when $i=N$ we have that $\mu_{r(i)}$ as $\mu(X)$ should be equal to one. This is so because $1=\mu(X) = \max\{\mu_{r(1)}', \ldots, \mu_{r(N)}\}$ and thus it should exist at least one $\mu_{r(i)}=1$. Also as $\mu_{r(N)}$ is the greatest one, it should also be one. From this, it can be deduced that for all $x \in [0.5, 1]$ we have $Q(x) = 1$.

We show now that the measure induced by these definitions is the one obtained by (7). We consider first the simplest cases $|B|=0$ and $|B|=1$ and later $|B|>1$.

1) $\mu(\emptyset) = Q(0) = 0$

2) $\mu(\{a_{r(i)}\}) = Q(p_{r(i)}) = Q(1/2^{N-i+1}) = \mu_{r(i)}$

3) $\mu(\{a_{r(i)}, \ldots, a_{r(N)}\}) = Q(p_{r(i)}) = Q(1/2 + (1-1/2^{N})) = 1 = \mu_{r(i)}$

4) Let us consider $\mu(B)$ when the cardinality of $B$ is greater than one. In this case, let $B=\{a_{r(i)}\}_{i \in I}$ and let $y = \max I$. Under these conditions $\mu(B)$ can be rewritten as:

$$
\mu(B) = Q \left( \sum_{i \in I} p_{r(i)} \right)
$$

Now, we consider two cases according to whether $y \in I$ or not.

a) Case $y \notin I$. In this case

$$
\mu(B) = Q \left( \sum_{i \in I} p_{r(i)} \right) = Q \left( \sum_{i=1}^{N} \frac{1}{2^{N-i+1}} \right)
$$
To finish this case, we need to prove that the following holds:

\[
\sum_{i=1}^{\lfloor y \rfloor} \frac{1}{2^{N-i+1}} < \frac{1}{2^{N-y}}
\]

(8)

The left hand side is trivial as \(y = \max I\) and thus \(y \in I\). To prove the right hand side we consider the worst case that is to have \(I\) with all the elements between 1 and \(y\). This is \(I = \{1, \ldots, y\}\). In this case, we have that

\[
\sum_{i=1}^{\lfloor y \rfloor} \frac{1}{2^{N-i+1}} = \frac{1}{2^N} + \frac{1}{2^{N-1}} + \ldots + \frac{1}{2^{N-y+1}} = \frac{1}{2^{N-y}} - \frac{1}{2^N} < \frac{1}{2^{N-y}}
\]

using the well known equality:

\[
\sum_{i=1}^{m} \frac{1}{2^i} = 1 - \frac{1}{2^m}
\]

Once that (8) is proven we have, using the definition of \(Q\), that

\[
\mu(B) = Q\left(\sum_{i \in I} \frac{1}{2^{N-i+1}}\right) = \mu_{r(y)}
\]

Moreover, as \(y = \max I\) and \(\mu_{r(i)} = \mu_{r(j)}\) if \(i \leq j\) we have that

\[
\mu_{r(y)} = \max_{i \in I} \mu_{r(i)} = \max_{a_{i} \in B} \mu_{r(i)}
\]

and therefore

\[
\mu(B) = \max_{a_{i} \in B} \mu_{r(i)}
\]

that corresponds to (7).

Example. Let \(\mu\) be the fuzzy measure on \(X = \{a, b, c\}\) defined as:

\[
\mu(\emptyset) = 0, \quad \mu(\{a\}) = 0, \\
\mu(\{b\}) = 0, \quad \mu(\{c\}) = 0, \\
\mu(\{a, b\}) = 0.2, \quad \mu(\{a, c\}) = 0.4, \\
\mu(\{b, c\}) = 0.4, \quad \mu(\{a, b, c\}) = 1
\]

This measure is not a decomposable fuzzy measure because in order to be, it should exist a t-conorm \(S\) such that

\[
0.2 = \mu(\{a, b\}) = S(\mu(\{a\}), \mu(\{b\})) = S(0, 0)
\]

However, all t-conorms \(S\) satisfy \(S(0, 0) = 0\) and this is in contradiction with \(\mu(\{a, b\}) = 0.2\).

This fuzzy measure is, however, a Q-p-decomposable fuzzy one with \(p(a) = 0.3, p(b) = 0.3, p(c) = 0.4\) and \(Q(x) = \max (0, 2x - 1)\).

Although in general Q-p-measures are not decomposable fuzzy measures we have proven that they are of this form when the quantifier \(Q\) is continuous and strictly increasing.

Theorem 6. Any fuzzy measure that is Q-p-decomposable by means of a continuous and strictly increasing fuzzy quantifier \(Q\) is decomposable by means of a continuous archimedean t-conorm.

Proof. Let us consider the continuous and strictly increasing fuzzy quantifier \(Q\). As \(Q\) is invertible, we can define the function \(g\) as \(g(x) = Q^{-1}(x)\). This function \(g(x)\) is continuous, strictly increasing and with \(g(0) = 0\) and \(g(1) = 1\). From \(g(x)\) (and the fact that \(g^{-1}(x) = Q(x)\)) we can define its pseudo-inverse using (2):

\[
g^{-1}(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
Q(x) & \text{if } 0 \leq x \leq 1 \\
1 & \text{if } x \geq 1 
\end{cases}
\]

Now, using \(p\) and \(Q\), we have the measure for the singletons as \(\mu(\{a_i\}) = Q(p_i)\). From this definition and the fact that \(Q(x)\) is invertible we can express \(p_i\) as

\[
p_i = Q^{-1}(\mu(\{a_i\})) = g(\mu(\{a_i\}))
\]

Finally, let us consider \(\mu(B)\) for \(|B| \geq 1\). According to the definition of \(\mu\) we have that:

\[
\mu(B) = Q(\sum_{a_i \in B} p_i)
\]

As, \(Q = g^{-1}(x)\) and \(p_i = g(\mu(\{a_i\}))\) we have

\[
\mu(B) = g^{-1}\left(\sum_{a_i \in B} g(\mu(\{a_i\}))\right)
\]
As, according to proposition 1, when \( B = \{ b_1, \ldots, b_m \} \) this expression is equivalent to
\[
\mu(B) = S(\mu(b_1), \ldots, \mu(b_m))
\]
for the t-conorm \( S(a, b) = g^{-1}(g(a) + g(b)) \), the theorem is proven. Note that \( S \) is a t-conorm because \( g: [0,1] \to [0,1] \) is strictly increasing and continuous with \( g(0) = 0 \). 

**Conclusions**

In this paper we have established that the WOWA operator is a particular case of the Choquet integral when measures are Q-p-decomposable, and we have shown that Sugeno \( \lambda \)-measures and decomposable fuzzy measures by means of archimedean t-conorms are Q-p-decomposable. This latter results means that the fuzzy measure introduced with the WOWA operator is more general than decomposable fuzzy measures and therefore, these decomposable measures can be understood on the light of the one in the WOWA operator: to consider importance for each alternative (represented by the weights \( p_k \)) and importance of the values (represented by \( Q \)). We have also studied when Q-p-decomposable fuzzy measures are decomposable ones.

**Acknowledgements**

The support of the European Community, under the contract VIM: CHRX-CT93-0401 and the CICYT, project SMASH: TIC-96-1038-C04-04, is acknowledged.

**References**


