Mixed Rational Assessments of Possibility and Probability Measures

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Introduction

The ground many-valued logic $RL_{\Delta}$

The logic $FP\Pi(RL_{\Delta})$

Mixed rational assessments and the coherence problem CMA

Complexity for CMA problem

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Introduction

The ground many-valued logic $R\mathcal{L}_\Delta$

The logic $FP\Pi(R\mathcal{L}_\Delta)$

Mixed rational assessments and the coherence problem CMA

Complexity for CMA problem

Future work
A fuzzy logical approach to probability and possibility was firstly introduced in 1995 by Hájek, Godo and Esteva. The central idea is the following:

- Add a modality $P [\Pi]$ for probable [possible] to the language of $RPL$
- An event $E$ is regarded as an equivalence class of a formula of Classical Logic $\varphi_E$
- The probability [possibility] of an event $E$ can be expressed as $P(\varphi_E) [\Pi(\varphi_E)]$ saying $\varphi_E$ is probable [$\varphi_E$ is possible]
Introduction and motivations (2)

- The logic $FP(RPL)$ for reasoning about probability,
- The logic $F\Pi(RPL)$ for reasoning about possibility (and necessity)

Axioms and rules are those of $RPL$ plus an axiom schema to deal with those of a probability measure (possibility measure) and the rule of $P$-Necessitation: *If we deduce $\varphi$, then $P(\varphi)$* (in an analogous way for $\Pi$)
The ground many-valued logic $\mathcal{RL}_\Delta$

The language of $\mathcal{RL}_\Delta$ consists in a countable set of propositional variables $p_0, p_1, \ldots$, the propositional constant $\bar{0}$, the binary connectives $\oplus$ and $\rightarrow$, unary connectives $\Delta$, and $\delta_n$ (for each $n \in \mathbb{N}$).

Further definable connectives are:

- $\neg \varphi$ stands for $\varphi \rightarrow \bar{0}$
- $\varphi \& \psi$ stands for $\neg(\varphi \rightarrow \neg \psi)$
- $\varphi \equiv \psi$ stands for $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$
- $\varphi \wedge \psi$ stands for $\varphi \&(\varphi \rightarrow \psi)$
- $\varphi \lor \psi$ stands for $\neg(\neg \varphi \land \neg \psi)$
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- $\varphi \& \psi$ stands for $\neg(\varphi \rightarrow \neg \psi)$
- $\varphi \equiv \psi$ stands for $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$
- $\varphi \land \psi$ stands for $\varphi \& (\varphi \rightarrow \psi)$
- $\varphi \lor \psi$ stands for $\neg(\neg \varphi \land \neg \psi)$
The ground many-valued logic $R\Delta$ (2)

Axioms and rules are the following:

(Ł) Those of Łukasiewicz logic,

(Δ) The following axioms for $\Delta$:

1. $\Delta(\varphi \rightarrow \psi) \rightarrow (\Delta \varphi \rightarrow \Delta \psi)$
2. $\Delta \varphi \lor \neg \Delta \varphi$
3. $\Delta \varphi \rightarrow \varphi$
4. $\Delta \varphi \rightarrow \Delta(\Delta \varphi)$
5. $\Delta(\varphi \lor \psi) \equiv (\Delta \varphi \lor \Delta \psi)$

(G) The rule of Generalization $\frac{\varphi}{\Delta \varphi}$

(D) For each $n \in \mathbb{N}$, the following axioms for $\delta_n$

1. $n.\delta_n \varphi \equiv \varphi$
2. $\neg \delta_n \varphi \oplus (n - 1).\neg (\delta_n \varphi)$

where $n.\varphi$ stands for $\underbrace{\varphi \oplus \ldots \oplus \varphi}_{n\text{-times}}$
DMV_Δ-algebras

Example

Let \( A = \langle [0, 1], \oplus, \neg, \Delta, \{\delta_n\}_{n \in \mathbb{N}}, 0, 1 \rangle \) where:

- \([0, 1]\) is the real unit interval,
- For each \( x, y \in [0, 1] \) and for each \( n \in \mathbb{N} \):
  - \( x \oplus y = \min\{1, x + y\} \)
  - \( \neg x = 1 - x \)
  - \( \Delta(x) = 1 \) if \( x = 1 \), \( \Delta(x) = 0 \) otherwise
  - \( \delta_n(x) = x/n \)

Then \( A \) is the so called standard \( DMV_Δ \)-algebra.

Theorem

\( RL_Δ \) is sound and complete w.r.t. the standard \( DMV_Δ \)-algebra.
DMV_Δ-algebras

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Theorem

$R\mathcal{L}_\Delta$ is sound and complete w.r.t. the standard DMV_Δ-algebra.
The logic $FP\Pi(\mathcal{R}_\Delta)$ is so defined:

- The language is that one of $\mathcal{R}_\Delta$ plus the unary modalities $P$ and $\Pi$.
- Formulas are divided into two classes:
  - The class $BF$ of non modal formulas is the smallest class of formulas such that the propositional variables belongs to $BF$ and moreover $BF$ is closed under the classical connectives $\land$ and $\neg$.
  - The class $MF$ of modal formulas is the smallest class of formulas containing all the atomic modal formulas like $P(\varphi)$ and $\Pi(\varphi)$ and being closed under the connectives of $\mathcal{R}_\Delta$. 

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Axioms and rules of $FP\Pi(R\ell_\Delta)$ are those of $R\ell_\Delta$ plus the following:

(P1) $P(\neg \varphi) \equiv \neg P(\varphi)$
(P2) $P(\varphi \rightarrow \psi) \rightarrow (P(\varphi) \rightarrow P(\psi))$
(P3) $P(\varphi \lor \psi) \equiv [(P(\varphi) \rightarrow P(\varphi \land \psi)) \rightarrow P(\psi)]$
(Π1) $\Pi(\varphi \lor \psi) \rightarrow (\Pi(\varphi) \lor \Pi(\psi))$
(Π2) $\neg \Pi(\emptyset)$

Further deduction rules are the following:

- $P$-Necessitation $\frac{\varphi}{P(\varphi)}$
- $\Pi$-Necessitation $\frac{\varphi}{\Pi(\varphi)}$
- $\Pi$-Monotonicity $\frac{\varphi \rightarrow \psi}{\Pi(\varphi) \rightarrow \Pi(\psi)}$
A probabilistic-possibilistic Kripke model (PΠ-Kripke model) for $FPΠ(RLΔ)$ is a system $\mathcal{K} = \langle W, \mathcal{U}, e, \mu, \rho \rangle$ where:

- $W$ is a non-empty set of possible worlds and $\mathcal{U}$ is a Boolean algebra of subset of $W$.
- $e : V \times W \rightarrow \{0, 1\}$ is such that, for each fixed $w \in W$, the function $e(\cdot, w) : V \rightarrow \{0, 1\}$ is a Boolean evaluation. For each $\varphi \in BF$ let $W_\varphi = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\mu : \mathcal{U} \rightarrow [0, 1]$ is a finitely additive probability measure over $\mathcal{U}$ such that $W_\varphi$ is $\mu$-measurable for each $\varphi \in BF$.
- $\rho : \mathcal{U} \rightarrow [0, 1]$ is a possibility measure over $\mathcal{U}$ such that $W_\varphi$ is $\rho$-measurable for each $\varphi \in BF$. 
Let $\mathcal{K} = \langle W, \mathcal{U}, e, \mu, \rho \rangle$ be a $P\Pi$-Kripke model for $FP\Pi(RL_\Delta)$ and let $\Phi$ be a modal formula. Then the truth degree of $\Phi$ in $\mathcal{K}$ ($\|\Phi\|_\mathcal{K}$) is defined as:

- If $\Phi$ is $P(\varphi)$, then $\|P(\varphi)\|_\mathcal{K} = \mu(W\varphi)$
- If $\Phi$ is $\Pi(\varphi)$, then $\|\Pi(\varphi)\|_\mathcal{K} = \rho(W\varphi)$
- If $\Phi$ is a compound formula then $\|\Phi\|_\mathcal{K}$ is computed by evaluating the atomic modal formulas occurring in $\Phi$ and then using the truth functions associated to the connectives in $\Phi$.

**Theorem**

$FP\Pi(RL_\Delta)$ is sound and complete w.r.t. the class of $P\Pi$-Kripke models.
Mixed rational assessments and the coherence problem

Let $\mathcal{E} = \{\varphi_1, \ldots, \varphi_n\}$ be a finite class of events. Let

\[(\chi_P)\quad P(\varphi_i) = \alpha_i \in [0, 1] \cap \mathbb{Q}\]

be an assessment of probability over $\mathcal{E}$, and let

\[(\chi_\Pi)\quad \Pi(\varphi_i) = \beta_i \in [0, 1] \cap \mathbb{Q}\]

be an assessment of possibility over $\mathcal{E}$. Then $(\chi) = (\chi_P) \cup (\chi_\Pi)$ is called a rational mixed assessment.

Definition

A rational mixed assessment $(\chi)$ is said coherent if the following are satisfied:

- $(\chi_P)$ and $(\chi_\Pi)$ are coherent via a probability $\mu$ and a possibility $\rho$ respectively (internal coherence)
- For each $\psi$ in the Boolean algebra $\mathcal{B}$ generated by the events in $\mathcal{E}$, $\mu(\psi) \leq \rho(\psi)$ (external coherence)
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Mixed rational assessments and the coherence problem

**Theorem**

Let $\mathcal{E} = \{\varphi_1, \ldots, \varphi_n\}$ be a finite set of events, let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be the set of atoms generated by $\varphi_1, \ldots, \varphi_n$ and let

$$(\chi): \mu(\varphi_i) = \alpha_i, \rho(\varphi_i) = \beta_i \ (i = 1, \ldots, n)$$

be a rational mixed assessment over $\mathcal{E}$. Then the following are equivalent:
Theorem

(i) $(\chi)$ is coherent,

(ii) There are $n + 1$ atoms $a_1, \ldots, a_{n+1}$ such that the $FP\Pi(R\Lambda)-$theory $T_\chi$ whose proper axioms are:

(A1) $\Delta(\bigwedge_{i=1}^n P(\varphi_i) \equiv \overline{\alpha_i})$

(A2) $\Delta(\bigwedge_{i=1}^n \Pi(\varphi_i) \equiv \overline{\beta_i})$

(A3) $\Delta(\bigwedge_{i=1}^n \Pi(a_i) \rightarrow \Pi(a_{i+1}))$

(A4) $\Delta(\bigwedge_{j=1}^{n+1} \left[ \bigoplus_{i=1}^j P(a_i) \right] \rightarrow \Pi(a_j))$

is logically consistent, i.e. $T_\chi \not\vdash_{FP\Pi} \overline{0}$. 
Now we are going to show that the coherence test for rational mixed assessment is a problem NP-complete.

**Theorem**

1. The satisfiability problem for $R\Delta$ is NP-complete.
2. The satisfiability problem for modal formulas of $FP\Pi(R\Delta)$ is NP-complete.

**Theorem**

*Testing the coherence of a rational mixed assessment ($\chi$) is a problem NP-complete.*
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**Theorem**

Testing the coherence of a rational mixed assessment ($\chi$) is a problem NP-complete.
Complexity for CMA problem

Proof.

(Sketch): By the previous theorem we can reduce the CMA problem can be reduced to the satisfiability problem for $T_\chi$

**Hardness.** Easy: CPA is a subproblem of CMA and CPA is well known to be NP-complete. Thus CMA is NP-hard.

**Membership.** The NP algorithm works as follows:

1. Randomly generate $n + 1$ atoms from the events in $\mathcal{E}$
2. Build the theory $T_\chi$
3. Test the satisfiability of $T_\chi$

Notice that the algorithm is NP given that $T_\chi$ contains a number of atoms which is polynomial in the number $n$ of the events in $\mathcal{E}$.
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Complexity for CMA problem

Proof.

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Mixed Rational Assessments
Future work

Our future work will deal with conditional measures to deal with the conditional version of the CMA problem in a way allowing to prove that also this problem is NP-complete.
A general way to prove that the conditional version of CPA is NP-complete, is to decompose a conditional assessment into a class of simple assessments $P_1, \ldots, P_n$. In an analogous way it can be done for $\Pi$.
In order to define a nice logic for this proposal we have to know, starting form a conditional mixed assessment, how many simple modalities $P_1, \ldots, P_n$ and $\Pi_1, \ldots, \Pi_m$ we have to introduce or otherwise how to proceed if we are totally ignorant on it.