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Indicator of inclusion grade for interval-valued fuzzy sets. Application to approximate reasoning based on interval-valued fuzzy sets

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This paper is dedicated to my mother, who died on December 21, 1998.

Abstract

We begin the paper studying the axioms that the indicators of the grade of inclusion of a fuzzy set in another fuzzy set must satisfy. Next, we present an expression of such indicator, first for fuzzy sets and then for interval-valued fuzzy sets, analyzing in both cases their main properties. Then, we suggest an expression for the similarity measure between interval-valued fuzzy sets. Besides, we study two methods for inference in approximate reasoning based on interval-valued fuzzy sets, the inclusion grade indicator and the similarity measure. Afterwards, we expose some of the most important properties of the methods of inference presented and we compare these methods to Gorzalczany's. Lastly, we use the indicator of the grade of inclusion for interval-valued fuzzy sets as an element that selects from the different methods of inference studied, the one that will be executed in each case.

Keywords: Interval-valued fuzzy set; Approximate reasoning; Inclusion grade indicator; Similarity measure; Fuzzy complement

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1. Introduction

Since Zadeh introduced interval-valued fuzzy sets [74,75] (i.e., a fuzzy set with an interval-valued membership function) many authors have used them in different fields of science; for example Sambuc [52] in medical diagnosis in thyroidian pathology, Gorzalczany [29] in Approximate reasoning, Turksen in Interval-valued logic, etc., these works and others [2,15,45] show the importance of these sets.

In this paper we propose an expression for measuring the degree of truth of the proposition 'A is a subset of B', with interval-valued fuzzy sets A and B on the same referential X. From this expression we present in the first place, a similarity measure between interval-valued fuzzy sets, and in the second, an algorithm that allows us to obtain the conclusion of the generalized modus ponens and a system of rules in the following way:

$$\begin{array}{l} \text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \\ \text{If } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \\ \qquad\qquad \vdots \\ \text{If } x \text{ is } A_m \text{ then } y \text{ is } B_m \\ \quad x \text{ is } A' \\ \hline \quad y \text{ is } B' \end{array}$$

when working in both cases with interval-valued fuzzy sets.

The same as Sinha and Dougherty did for fuzzy sets, we begin the paper presenting the axioms that we believe every indicator of the grade of inclusion of a fuzzy set in another fuzzy set that is also interval-valued fuzzy must satisfy. Then, based on the works of these authors [53–55] and of De Baetes and Kerre [17,67], we propose an expression for such indicator, which does not coincide exactly with the one proposed by these authors for the following reasons:

1. we work with interval-valued fuzzy sets;
2. we are not going to apply the inclusion grade indicator to fuzzy mathematical morphology as Sinha and Dougherty do, so our indicator is not obligated to satisfy certain axioms that these authors impose;
3. our expression of the grade of inclusion must satisfy some specific axioms that are imposed by the use of such indicator in the method of inference in approximate reasoning based on interval-valued sets that we will present in the last sections of the paper.

Among the objectives of this paper we can point out two:

(a) To justify, as concisely as possible, the different axioms that the indicators of the grade of inclusion for interval-valued fuzzy sets must satisfy. This objective has lead us on the one hand to study a similarity measure between interval-valued fuzzy sets and on the other, to study a method of inference in approximate reasoning based on these sets, the inclusion grade indicator, and on the similarity measure defined from them.

(b) To present some applications of the inclusion grade indicators for interval-valued fuzzy sets. In this sense we first study the above-mentioned method of inference and secondly, we analyze the way of using the indicator as an element which decides the starting off of one method or another.

We have organized the paper in the following way:

In the preliminaries we recall the definitions of fuzzy complements, t-norms, t-conorms, and interval-valued fuzzy sets, along with the properties of these concepts that we will use in this paper. In this section we also present a geometric interpretation of interval-valued fuzzy sets.

We then establish the axioms that the indicators of the grade of inclusion for interval-valued fuzzy sets must verify, always justifying the choice of axioms, although some of them will stand fully justified in the sections dedicated to the achievement of the generalized modus ponens conclusion (Sections 5 and 7).

Afterwards, we take an expression for such indicator and we study its main properties, (first for fuzzy sets and then for interval-valued fuzzy sets). In this section we also justify the choice of this expression.

Subsequently, we present an expression for the similarity measure between interval-valued fuzzy sets in accordance with the inclusion grade indicator previously defined and we study its properties.

Next we propose a method for inference in approximate reasoning based on interval-valued fuzzy sets, the inclusion grade indicator and in the similarity measure.

In Section 8 we compare the results obtained with the algorithm when working with the inclusion grade indicator, and the results obtained when working with the similarity measure, in both cases the results are also compared with the ones obtained through Gorzalczany's [29] indicator. Lastly, we generalize the algorithms in terms of any t-norm and any t-conorm. It is important to point out that the algorithm studied allows us to justify some of the axioms required of the inclusion grade indicator when we work with interval-valued fuzzy sets.

Finally, in Section 9 we present an alternative to the method of inference studied in the previous sections, such alternative allows us to resolve some of the problems with this method. It is precisely in this section where we use the indicator of the grade of inclusion for interval-valued fuzzy sets as an element that selects the execution of one method of inference or another.

Due to the length of some of the proofs of the theorems, corollaries, and propositions, these proofs can be found in Appendix A.

2. Preliminaries

The desire to describe the axioms that must characterize the inclusion grade indicator for interval-valued fuzzy sets, as well as the study of the main properties of these indicators, leads us to recall in this section the definitions of fuzzy complement, triangular norm, triangular conorm in [0,1], (taking into account that as non-classical connectives, they do not satisfy the boolean standard identities), and the concept of interval-valued fuzzy set. We will also point out the properties that we will later use of each of these concepts.

2.1. Definitions

Let $c : [0,1] \to [0,1]$. c is a *fuzzy complement* iff:
(i) $c(0) = 1$ and $c(1) = 0$,
(ii) $c(x) \leqslant c(y)$ if $x \geqslant y$ (monotonicity).
A fuzzy complement is *strict* iff:
(iii) $c(x)$ is *continuous,*
(iv) $c(x) < c(y)$ for $x > y$ for all $x, y \in [0,1]$.
A strict fuzzy complement is *involutive* iff:
(v) $c(c(x)) = x$ for all $x \in [0,1]$.

We know that $c(x) = 1 - x$ is the classical fuzzy complement. One class of involutive fuzzy complements is Sugeno's class [58,59] defined by $c_\lambda(x) = (1-x)/(1+\lambda x)$, where $\lambda > -1$. Another example of a class of involutive fuzzy complements is defined by $c_w(x) = (1 - a^w)^{1/w}$, where $0 < w$, let us refer to it as Yager's class [72,73] of fuzzy complements.

Different approaches to the study of fuzzy complements were used by Lowen [38], Esteva et al. [22], and Ovchinnikov [43,44], Higashi and Klir [32], Yager investigated fuzzy complements for the purpose of developing useful measures of fuzzyness [72,73].

Trillas [60,61] proved that it is possible to represent all involutive and strict fuzzy complement $c : [0,1] \to [0,1]$ in the following way: $c(x) = g^{-1}(1 - g(x))$, being $g : [0,1] \to [0,1]$ an increasing bijection, such that $g(0) = 0$ and $g(1) = 1$. Note that in the conditions above: $g(c(x)) = 1 - g(x)$.

We will use this result hereinafter. From now on we will always take the following functions:

$$g : [0,1] \to [0,1]$$

continuous and strictly increasing such that $g(0) = 0$ and $g(1) = 1$.

We will call t-norm in [0,1] every mapping

$$T : [0,1] \times [0,1] \to [0,1]$$

satisfying the properties:
(i) boundary conditions, $T(x, 1) = x$ for all $x \in [0,1]$,
(ii) monotony, $T(x, y) \leqslant T(z, u)$ if $x \leqslant z$ and $y \leqslant u$,

(iii) commutative, $T(x,y) = T(y,x)\ \forall x,y \in [0,1]$,
(iv) associative, $T(T(x,y),z) = T(x,T(y,z))$ for all $x,y,z \in [0,1]$.
We will call t-conorm in [0,1] every mapping

$$S : [0,1] \times [0,1] \to [0,1]$$

satisfying the properties:
(i) boundary conditions, $S(x,0) = x$ for all $x \in [0,1]$,
(ii) monotony, $S(x,y) \leqslant S(z,u)$ if $x \leqslant z$ and $y \leqslant u$,
(iii) commutative, $S(x,y) = S(y,x) \forall x,y \in [0,1]$,
(iv) associative, $S(S(x,y),z) = S(x,S(y,z))$ for all $x,y,z \in [0,1]$.

We say that a t-norm T and a t-conorm S are *dual* with respect to a fuzzy complement c iff $c(T(a,b)) = S(c(a),c(b))$ and $c(S(a,b)) = T(c(a),c(b))$.

Let the triple $\langle T,S,c\rangle$ denote that T and S are dual with respect to c. Besides we know that $\langle \wedge,\vee,c\rangle$ is dual with respect to any fuzzy complement c, being $\vee = \max$ and $\wedge = \min$.

The most important properties of t-norms and t-conorms can be found in [1,7,18,27,28,30,31,35,36,50,69].

In this paper, unless it is indicated otherwise, we will designate the t-norms and t-conorms en general with the Greek letters α, β and λ.

The triangular norms and conorms are binary operations [34,39,44,62]. However, their successive application allows them to be considered as n-ary operations using the inductive expression

$$\alpha(x_1,x_2,\ldots,x_n) = \alpha(\alpha(x_1,x_2,\ldots,x_{n-1}),x_n),$$

the properties of boundary, monotony, symmetric, and associativity being valid for these inductive expressions.

Let I be a finite family of indexes and $\{a_i\}_{i\in I}$, $\{b_i\}_{i\in I}$ number collections of [0,1]. For every α t-norm or t-conorm and for every λ t-norm or t-conorm

$$\begin{aligned} &\underset{i}{\alpha}(a_i \vee b_i) \geqslant \underset{i}{\alpha}(a_i) \vee \underset{i}{\alpha}(b_i), \\ &\underset{i}{\lambda}(a_i \wedge b_i) \leqslant \underset{i}{\lambda}(a_i) \wedge \underset{i}{\lambda}(b_i) \end{aligned} \tag{1}$$

are verified [9].

With this result and with the result given by Fung and Ku [26] relative to the fact that α is an idempotent t-conorm (idempotent t-norm) if and only if $\alpha = \vee$ ($\alpha = \wedge$), we get the following theorem.

Theorem 0. *Let α,λ be not null t-norms or t-conorms. For all $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$*
(i) $\underset{i}{\alpha}(a_i \vee b_i) = \underset{i}{\alpha}(a_i) \vee \underset{i}{\alpha}(b_i)$ *if and only if* $\alpha = \vee$,
(ii) $\underset{i}{\lambda}(a_i \wedge b_i) = \underset{i}{\lambda}(a_i) \wedge \underset{i}{\lambda}(b_i)$ *if and only if* $\lambda = \wedge$
hold.

The proof of this theorem can be found in [11–13].

All these properties will be used in the proofs of the different theorems, corollaries and propositions herein.

2.2. Interval-valued fuzzy sets

Next we recall the notion of interval-valued fuzzy set or Φ-fuzzy set introduced by Zadeh [74,75] and Sambuc [52]. Such sets are being used at present by different authors, among which we can point out the works of Ponsard [47], Turksen [63–66] and Gorzalczany [29] on approximate reasoning, these works will be used in the last section of this paper. We will also recall the most important properties of these sets. We begin by presenting the notation we are going to use:

Let $D[0,1]$ be the set of closed subintervals of the interval [0,1]; we will represent the elements of the set with capital letters $M, N, \ldots$ It is known that $M = [M_L, M_U]$, where M_L and M_U are the lower and the upper extreme respectively. We will call $W_M = M_U - M_L$ *amplitude* of the interval M.

Let us represent the interval-ordering on $\mathbb{R}$ as $\leqslant$, i.e., $M \leqslant N$ if $M_L \leqslant N_L$ and $M_U \leqslant N_U$, This relation is transitive, reflexive and antisymmetric and expresses that M lies weakly left from N, i.e., for every point $x \in M$ there exists a point $y \in N$ such that $y \geqslant x$. We must point out that in interval-valued fuzzy set literature there have also been other orders [12,33,47,52], however, the relation that we present here is the most usual and it is the one that we will use hereinafter. (Nevertheless, to touch upon this fact, we will represent the indicator of the grade of inclusion of an interval-valued fuzzy set in another, when we work with the relation '$\leqslant$' as $\Upsilon_{\leqslant}$.)

We know that [12] $M = N$ if and only if $M_L = N_L$ and $M_U = N_U$. We will designate $c(M) = M_c$ as the *complementary* of M, that is, $c(M) = [c(M_U), c(M_L)]$. Besides [12,33], if α is a t-norm or a t-conorm, then $\alpha(M,N) = [\alpha(M_L,N_L), \alpha(M_U,N_U)]$ for all $M,N \in D[0,1]$.

Let $X \neq \phi$ be a given set [10,12,47,52,63,70]. An interval-valued fuzzy set in X is an expression A given by

$$A = \{\langle x, M_A(x)\rangle | x \in X\},$$

where the function

$$\begin{aligned} M_A &: X \to D[0,1] \\ &\quad x \to M_A(x) = [M_{AL}(x), M_{AU}(x)] \end{aligned}$$

defines the degree of membership of an element x to A. We will represent as $IVFSs(X)$ the set of all interval valued fuzzy sets on the same X. *We should insist on the fact that hereinafter X will be finite and empty*, *so that* $Cardinal(X) = n$.

We shall say that an interval-valued fuzzy set A is *normal* if there is at least one $x \in X$ such that $M_A(x) = [1, 1]$. We will designate $FSs(X)$ as the set of all fuzzy sets on the same X and $P(X)$ is the class of all crisp sets of X.

The following expressions are defined in [10,12,48,49,63] for all $A, B \in IVFSs$

1. $A \leqslant B$ if and only if $M_{AL}(x) \leqslant M_{BL}(x)$ and $M_{AU}(x) \leqslant M_{BU}(x)$ $\forall x \in X$,
2. $B \preceq A$ if and only if $M_{AL}(x) \leqslant M_{BL}(x)$ and $M_{AU}(x) \geqslant M_{BU}(x)$ for all $x \in X$,
3. $A \sqsubseteq B$ if and only if $M_{AU}(x) \leqslant M_{BL}(x)$ $\forall x \in X$,
4. $A = B$ if and only if $M_{AL}(x) = M_{BL}(x)$ and $M_{AU}(x) = M_{BU}(x)$ $\forall x \in X$,
5. $A_c = \{\langle x, c(M_A(x))\rangle | x \in X\} = \{\langle x, [c(M_{AU}(x)), c(M_{AL}(x))]\rangle | x \in X\}$.

Theorem 1 [12]. *Let β and α be t-norm and t-conorm respectively, we define*

$$\beta(A, B) \equiv \{\langle x, [\beta(M_{AL}(x), M_{BL}(x)), \beta(M_{AU}(x), M_{BU}(x))]\rangle | x \in X\},$$
$$\alpha(A, B) \equiv \{\langle x, [\alpha(M_{AL}(x), M_{BL}(x)), \alpha(M_{AU}(x), M_{BU}(x))]\rangle | x \in X\}$$

for all $A, B \in IVFSs$. Then, it is verified that:

(a) *If $\beta = \wedge$ and $\alpha = \vee$ then $\{IVFS(X), \wedge, \vee\}$ is a distributive lattice, which is bounded, not complemented and satifies Morgan's laws.*

(b) *For any β and α (α dual of β), the conmutative, associative properties and $\beta(A_c, B_c) = (\alpha(A, B))_c, \alpha(A_c, B_c) = (\beta(A, B))_c$ are satisfied.*

In 1987 Gorzalczany [29] defined the *degree of compatibility* between two interval-valued fuzzy sets A and B on the same referential X in the following way:

The compatibility degree $\Gamma(A, B)$ of an interval-valued fuzzy set A (such that there is at least one $x \in X$ with $M_{AL}(x) \neq 0$) with an interval-valued fuzzy set B is an element of the family D[0,1], where

$$\Gamma(A, B) = \left[\min\left(\frac{\max_{x\in X}\{\min(M_{AL}(x), M_{BL}(x))\}}{\max_{x\in X} M_{AL}(x)}, \frac{\max_{x\in X}\{\min(M_{AU}(x), M_{BU}(x))\}}{\max_{x\in X} M_{AU}(x)} \right) \right.$$
$$\left. \max\left(\frac{\max_{x\in X}\{\min(M_{AL}(x), M_{BL}(x))\}}{\max_{x\in X} M_{AL}(x)}, \frac{\max_{x\in X}\{\min(M_{AU}(x), M_{BU}(x))\}}{\max_{x\in X} M_{AU}(x)} \right) \right].$$

The main properties of the degree of compatibility between two $IVFSs$ are the following:

(i) for all $A \in IVFSs(X), \Gamma(A, A) = [1, 1]$;
(ii) $\Gamma(A, B) = [0, 0]$ if and only if $A \wedge B = \{\langle x, M_{A\wedge B}(x) = [0, 0]\rangle | x \in X\}$;
(iii) in general $\Gamma(A, B) \neq \Gamma(B, A)$.

The definition of the degree of compatibility, and above all its property (ii) will be of great importance in the Section 9 of the paper, dedicated to the use of the inclusion grade indicator between interval-valued fuzzy sets as an element that selects one out of the methods existing in the literature for obtaining of the conclusion of the generalized modus ponens. Evidently, one of the possible methods that can be selected is the one developed by Gorzalczany in [29].

2.3. Geometric interpretation

A geometric interpretation of interval-valued fuzzy sets is presented in Fig. 1. Basically, it is interpreted as follows. Since the lower, the upper extreme and amplitude of all the intervals are numbers from [0,1], we can imagine a unit cube with three edges given by these parameters. Since $M_L(x) \leqslant M_U(x)$ and $W(x) = M_U(x) - M_L(x)$ for all $x \in X$, the values of the parameters characterizing an interval-valued fuzzy set can belong to the triangle ACB only. So an interval-valued fuzzy set is mapped from X into the triangle ACB in that each element of X corresponds to an element of ACB, as an example, a point $x' \in ACB$ corresponding to $x \in X$ is marked (the values of $M_L(x), M_U(x), W(x)$).

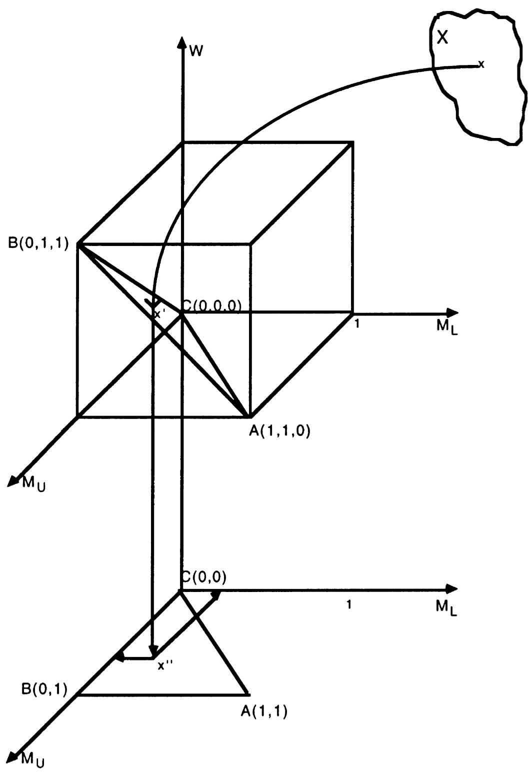


Fig. 1.

When $W(x) = 0$, then $M_L(x) = M_U(x)$ in Fig. 1, this condition is fulfilled only on the segment CA. Segment CA may by therefore viewed to represent a fuzzy set.

The orthogonal projection of the triangle ACB gives the representation of an interval-valued fuzzy set on the plane, on this plane the interior of the triangle ACB is the area where $W > 0$.

3. Characterization of the indicator for *IVFSs*(X)

For fuzzy sets an indicator for fuzzified set inclusion Υ is defined in such a way that $\Upsilon(A, B)$ measures belief in the proposition *A is a subset of B* with A and B as fuzzy sets.

In 1993, Sinha and Dougherty [54] in their paper *Fuzzification of Set Inclusion*: *Theory and Applications*, studied the desirable properties an indicator should have for fuzzified inclusion when working with fuzzy sets. The authors proposed a set of axioms that the inclusion grade indicators must satisfy, indicating that many of the indicators that exist in the literature do not satisfy all of the axioms that they propose.

We shall now present the axioms that in our point of view should characterize every inclusion grade indicator when we work with interval-valued fuzzy sets. For reasons that we have given in the introduction, these axioms do not have to coincide with the ones established by Sinha and Dougherty in [53–55] for fuzzy sets.

For each $A, B \in$ *IVFSs*(X), we will represent as $\Upsilon(A, B)$ the inclusion grade indicator of set A in set B.

The first problem we approach regarding the shape that the inclusion grade indicator should have when we work with interval-valued fuzzy sets is that this indicator must be a number of [0,1], or a pair of numbers of $[0, 1] \times [0, 1]$ or an element of $D[0, 1]$.

Since the relation '$\leqslant$' for *IVFSs* is always defined comparing pairs of extremes, it seems logical to rule out the idea that the inclusion grade is only one number. As for the alternative of it being an element of the set $\{\langle (x_1, x_2) \in [0, 1] \times [0, 1]\rangle | x_1, x_2 \in [0, 1]\}$, on the one hand it has the advantage that each of the elements of the pair will give us specific information about the extremes that we analyze in each case, however, it has the disadvantage that because it is not an interval we cannot use the indicator in Gorzalczany's [29] method of approximate reasoning based on *IVFSs* instead of the compatibility measure between *IVFSs*. This argument has led us to establish the following axiom.

Axiom 1. $\Upsilon(A, B) \in D[0, 1]$, that is, $\Upsilon(A, B) = [\Upsilon_L(A, B), \Upsilon_U(A, B)] \in D[0, 1]$.

Due to this axiom the inclusion grade indicator for *IVFSs* will hereinafter be called *interval-valued indicator of the grade of inclusion for interval-valued fuzzy sets*.

In the Sections 5 and 7 we find other arguments that also justify this axiom.

The two following axioms respond to the desire that if A and B are crisp sets, then $\Upsilon(A,B) \in \{0,1\}$.

Axiom 2. $\Upsilon(A,B) = [1,1]$ if and only if $A \leqslant B$.

A consequence of this axiom is that if

$$A = \{\langle x, [M_{AL}(x), M_{AU}(x)] = [0,0]\rangle | x \in X\}$$

then for any interval-valued fuzzy set $B \in IVFSs(X)$, $\Upsilon(A,B) = [1,1]$. On the other hand if $B = \{\langle x, [M_{BL}(x), M_{BU}(x)] = [0,0]\rangle | x \in X\}$, then for any $A \in IVFSs(X)$, $\Upsilon(A,B)$ represents the degree to which A can be classified as the null set.

Axiom 3. $\Upsilon(A,B) = [0,0]$ if and only if the set

$$\{x | [M_{AL}(x), M_{AU}(x)] = [1,1] \text{ and } [M_{BL}(x), M_{BU}(x)] = [0,0]\} \neq \emptyset$$

being x an any element in X.

Note that Axiom 3 is stronger than the one required by Dubois and Prade in [19] and De Baets and Kerre in [17,67] for fuzzy sets.

In all that we have seen up to now the indicator of the grade of inclusion of an interval-valued fuzzy set in another is characterized by an element in D[0,1], that is, by and interval. On the other hand, it is known [12] that $FSs(X) \subset IVFSs(X)$, that is, the fuzzy sets on a referential X are a particular case of the interval-valued fuzzy sets on the same referential, for when for all $x \in X$, the degree of membership to the set is characterized by an interval with amplitude 0, that is, $M_L(x) = M_U(x)$, the set considered is fuzzy. Besides, taking into account that many of the indicators of the grade of inclusion of an interval-valued fuzzy set in another are characterized by a single number, it seems logical to establish the following axiom.

Axiom 4. If A and B are fuzzy sets on the same referential X, then $\Upsilon_L(A,B) = \Upsilon_U(A,B)$.

We know that for

$$A = \{\langle x, [M_{AL}(x), M_{AU}(x)]\rangle | x \in X\},$$
$$B = \{\langle x, [M_{BL}(x), M_{BU}(x)]\rangle | x \in X\},$$

the relation $A \leqslant B$ iff $M_{AL}(x) \leqslant M_{BL}(x)$ and $M_{AU}(x) \leqslant M_{BU}(x)$ is transitive, reflexive and antisymmetric. Besides, if $M_{AL}(x) \leqslant M_{BL}(x)$ and $M_{AU}(x) \leqslant M_{BU}(x)$, then $c(M_{AL}(x)) \geqslant c(M_{BL}(x))$ and $c(M_{AU}(x)) \geqslant c(M_{BU}(x))$ where c is a fuzzy complement, since $A_c = \{\langle x, [c(M_{AU}(x)), c(M_{AL}(x))]\rangle | x \in X\}$ and $B_c = \{\langle x, [c(M_{BU}(x)), c(M_{BL}(x))]\rangle | x \in X\}$, we have that if $A \leqslant B$, then $A_c \geqslant B_c$, where A_c and B_c are obtained, on the one hand, from complementing the extremes of the interval, and on the other from changing the order, that is, we take as lower extremes $c(M_{AU}(x))$ and $c(M_{BU}(x))$ and as upper extremes $c(M_{AL}(x))$ and $c(M_{BL}(x))$. For this reason we consider that Υ should preserve the relationship between complement set inclusion and the change in extreme, that is, it must verify the following axiom.

Axiom 5. $\Upsilon(A, B) = \Upsilon(B_c, A_c)$.

The origin of both of the following axioms is the transitivity of set inclusion in Zadeh's sense.

Axiom 6. If $B \leqslant C$, then $\Upsilon(A, B) \leqslant \Upsilon(A, C)$.

Axiom 7. If $B \leqslant C$, then $\Upsilon(C, A) \leqslant \Upsilon(B, A)$.

We know that if $A \subseteq B$ or $A \subseteq C$, then $A \subseteq B \cup C$, however A can be a subset of $B \cup C$ without being a subset of either B or C, this fact lead Sinha and Dougherty [54] to establish the following axiom.

Axiom 8. $\Upsilon(A, B \vee C) \geqslant \vee(\Upsilon(A, B), \Upsilon(A, C))$, that is,

$$\begin{cases} \Upsilon_L(A, B \vee C) \geqslant \vee(\Upsilon_L(A, B), \Upsilon_L(A, C)), \\ \Upsilon_U(A, B \vee C) \geqslant \vee(\Upsilon_U(A, B), \Upsilon_U(A, C)). \end{cases}$$

With crisp sets if $B \subseteq A$ and $C \subseteq A$, then $B \cap C \subseteq A$, however the reciprocal is not necessarily true, this fact leads us to establish the following axiom.

Axiom 9. $\Upsilon(B \wedge C, A) \geqslant \vee(\Upsilon(B, A), \Upsilon(C, A))$, that is,

$$\begin{cases} \Upsilon_L(B \wedge C, A) \geqslant \vee(\Upsilon_L(B, A), \Upsilon_L(C, A)), \\ \Upsilon_U(B \wedge C, A) \geqslant \vee(\Upsilon_U(B, A), \Upsilon_U(C, A)). \end{cases}$$

As we said before, there are authors who impose certain axioms that do not coincide with any of the nine presented above on the inclusion grade indicators of a fuzzy set in another also fuzzy set. For example:

1. Baets [16] demands another axiom, it is probably the most controversial, requires the transitivity of the inclusion, and is inspired by the property $A \leqslant B$ and $B \leqslant C$, then $A \leqslant C$ valid for crisp subsets A, B and C of X;
2. Sinha and Dougherty impose three additional axioms when they work in Fuzzy Mathematical Morphology. Taking into account that these authors work with fuzzy sets, the first two additional axioms that they impose will read for interval-valued fuzzy sets as follows:

$$\Upsilon(B \vee C, A) = [\wedge(\Upsilon_L(B,A), \Upsilon_L(C,A)), \wedge(\Upsilon_U(B,A), \Upsilon_U(C,A))], \tag{2}$$

$$\Upsilon(A, B \wedge C) = [\wedge(\Upsilon_L(A,B), \Upsilon_L(A,C)), \wedge(\Upsilon_U(A,B), \Upsilon_U(A,C))]. \tag{3}$$

The third axiom that they add is based on the fact that for crisp sets, the inclusion is invariant under the reflection ($\mu_{-A}(x) = \mu_A(-x)$).

In this paper we generally maintain the nine axioms mentioned above and as Sinha and Dougherty say *it will be the specific applications* (*Fuzzy Mathematical Morphology*, *Approximate Reasoning*, …, *etc.*) *to which the expression of the inclusion grade indicator is destined that will determine the axioms to be satisfied by this indicator.*

We must point out that the axioms that we impose generally coincide with the most common existing in the literature, and prove to be very useful for the algorithm that we present in the last sections of the paper dedicated to inference in approximate reasoning with interval-valued fuzzy sets. We will later see that such algorithm will fully justify the demand for Axiom 1.

4. Interval-valued inclusion grade indicator

In order to justify the expression of the interval-valued inclusion grade indicator that we present en this section, we begin recalling some definitions and theorems of Fuzzy Set Theory.

Next, we present the expression of this indicator for fuzzy sets and we analyze its main properties. We conclude proposing an expression of the interval-valued indicator of the grade of inclusion of an interval-valued fuzzy set in another also interval-valued fuzzy set, and studying its main properties.

4.1. Expression of the inclusion grade indicator for fuzzy sets on account of Sinha and Dougherty

Sinha and Dougherty in [54] propose the following formula for measuring the grade of inclusion of fuzzy set A in set B:

$$\Upsilon(A,B) = \operatorname*{Inf}_{x \in X}\{\wedge(1, \phi(1 - \mu_A(x)) + \phi(\mu_B(x)))\}, \tag{4}$$

where ϕ is a function of [0,1] in [0,1] verifying the following conditions:
(i) $\phi(0) = 0$ and $\phi(1) = 1$;
(ii) for all $p \in [0, 1], \phi(1 - p) + \phi(p) \geqslant 1$;
(iii) ϕ is a non decreasing function;
(iv) $\phi(1 - p) = 0$ has a single solution;
(v) let $p, q \in [0, 1]$ and $\phi(1 - p) = \phi(1 - q) \geqslant 0.5$, then $p = q$.

Noteworthy is the concise and precise work that these authors carry out for the justification of the use in (4) of terms such as '$\mathrm{Inf}_{x \in X}$' and '$\wedge$'. They are precisely the ones that justify the presence of '$\mathrm{Inf}_{x \in X}$' and '$\wedge$' in the expression of the inclusion grade indicator that we propose in Proposition 1 for fuzzy sets and in Theorem 2 for interval-valued fuzzy sets.

4.2. Implication operators

We know [51] that a $[0, 1]^2 \to [0, 1]$ mapping I is called an implication operator if and only if it satisfies the boundary conditions $I(0, 0) = I(0, 1) = I(1, 1) = 1$ and $I(1, 0) = 0$.

The above conditions are of course the least we can expect from an implication operator. Other interesting potential properties of implication operators are listed in [6,17,24,34,36,51,56,60,61].

In 1987, Smets and Magrez [56] establish a collection of nine axioms for the implication operators and prove the following theorem:

Theorem. *A function $I : [0, 1]^2 \to [0, 1]$ satisfies the nine axioms of a fuzzy implication for a particular fuzzy complement c if and only if there exists a strict increasing continuous function $g : [0, 1] \to [0, \infty)$ such that $g(0) = 0$ and $g(1) < \infty$,*

$$I(x, y) = g^{(-1)}((g(1) - g(x) + g(y)) \wedge g(1))$$

for all $x, y \in [0, 1]$, and

$$c(x) = g^{-1}(g(1) - g(x))$$

for all $x \in [0, 1]$.

According to this theorem it is clear that

$$g(I(x, y)) = \wedge(g(1), g(1) - g(x) + g(y)) = \wedge(g(1), g(c(x)) + g(y)).$$

It is important to point out that in the Fuzzy Set Theory there have been many expressions for the implication operators, the choice of one over the rest normally depends on the application in which it is going to be used. Many of these implication operators do not satisfy the nine axioms imposed by Smets and Magrez (for example Godel's, that only satisfies seven of the nine axioms, or Kleene–Dienes' that also satisfies seven but not the same ones as Godel). Evidently, Lukasiewicz's implication operator satisfies the nine axioms.

4.3. Expression of the inclusion grade indicator for fuzzy sets owed to De Baets and Kerre

In order to measure the grade of inclusion of a fuzzy set A in another B (in the sense introduced by Bandler and Kohout [7]) De Baets [16] and De Baets and Kerre [17] present the following definition:

Definition. Consider an implication operator I. The binary fuzzy relation Inc defined by, for any A and B fuzzy sets on X,

$$\mathrm{Inc}(A,B) = \operatorname*{Inf}_{x\in X}(I(\mu_A(x),\mu_B(x))) \tag{5}$$

is a fuzzy inclusion.

A more detailed study of the main properties of this expression can be found in [17,67]. Note that (5) is given with respect to any implication operator. That is, the operators that appear in this expression do not necessarily have to satisfy the nine axioms imposed by Smets and Magrez. To Smets and Magrez, this does not imply a restriction, for (5) satisfies the properties that they demand of the inclusion grade indicator for fuzzy sets.

In any case, (5) will also justify, in a certain way, the choice of the expression of the inclusion grade indicator that we present later on in this section.

4.4. A new expression of the inclusion grade indicator for fuzzy sets

Let $A,B \in \mathit{FSs}(X)$. We propose that the indicator for fuzzified set inclusion be defined as

$$\begin{aligned}\Upsilon(A,B) &= \operatorname*{Inf}_{x\in X}\{\wedge(1, 1 - g(\mu_A(x)) + g(\mu_B(x)))\}\\ &= \operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(\mu_A(x))) + g(\mu_B(x)))\}\end{aligned} \tag{6}$$

so that $g : [0,1] \to [0,1]$ is a continuous and strictly increasing function such that $g(0) = 0, g(1) = 1$ and $c(x) = g^{-1}(1 - g(x))$.

We must point out that this expression does not coincide with the one given by Sinha and Dougherty (4), nor with De Baets and Kerre's (5). We have used them as basis in order to take (6), we only need to observe:

(a) In (6) we maintain '$\mathrm{Inf}_{x\in X}$' and $\wedge$. We must point out that the use of these terms in (4) can found perfectly justified in [54], and they are precisely the same arguments that justify its presence in (6). Besides, (6) is similar to (4) substituting ϕ for g, keeping in mind that g satisfies different properties from the ones ϕ does. It is important to point out that we use any fuzzy complementation, while Sinha and Dougherty always use the standard complementation $c(x) = 1 - x$ for all $x \in X$.

(b) From the theorem Smets and Magrez proved

$$g(I(x,y)) = \wedge(g(1), g(1) - g(x) + g(y))$$

and expression (5), if we take $\mathrm{Inf}_{x\in X}(g(I(\mu_A(x), \mu_B(x))))$ instead of $\mathrm{Inf}_{x\in X}(I(\mu_A(x), \mu_B(x)))$, we have

$$\begin{aligned}\Upsilon(A,B) &= \operatorname*{Inf}_{x\in X}(g(I(\mu_A(x), \mu_B(x)))) \\ &= \operatorname*{Inf}_{x\in X}\{\wedge\,(g(1), g(1) - g(\mu_A(x)) + g(\mu_B(x)))\},\end{aligned}$$

which is the expression that we propose for the inclusion grade indicator for fuzzy sets by just imposing $g(1) = 1$.

(c) De Baets and Kerre's indicator is given with respect to any implication indicator. We, on the other hand, have started from Smets and Magrez's theorem in order to obtain (6), therefore we exclusively take implication operators that satisfy the nine axioms demanded by them. This is not of great importance to us because we have only looked at the expression Smets and Magrez present in their theorem and we operate on it. It is precisely the way of obtaining (6) from this theorem that allows us to establish one of the main properties of this expression, which is the *possibility of being generated by the same functions that generate c, a widely studied problem in fuzzy literature* [22,32,37,60,61,72].

The following propositions prove the main properties of expression (6).

Proposition 1. *Let $A, B \in FSs(X)$ and let $g : [0,1] \to [0,1]$ be a continuous and strictly increasing function such that $g(0) = 0$ and $g(1) = 1$, and let $c : [0,1] \to [0,1]$ be an involutive and strict fuzzy complement such that $c(x) = g^{-1}(1 - g(x))$. In these conditions for the fuzzy inclusion*

$$\begin{aligned}\Upsilon(A,B) &= \operatorname*{Inf}_{x\in X}\{\wedge\,(1, 1 - g(\mu_A(x)) + g(\mu_B(x)))\} \\ &= \operatorname*{Inf}_{x\in X}\{\wedge\,(1, g(c(\mu_A(x))) + g(\mu_B(x)))\},\end{aligned}$$

the following properties hold:

(i) $\Upsilon(A,B) = 1$ *if and only if* $A \leqslant B$;

(ii) $\Upsilon(A,B) = 0$ *if and only if the set* $\{\langle x, \mu_A(x) = 1$ *and* $\mu_B(x) = 0\rangle\} \neq \emptyset$;

(iii) if $B \leqslant C$, *then* $\Upsilon(A,B) \leqslant \Upsilon(A,C)$;

(iv) if $B \leqslant C$, *then* $\Upsilon(C,A) \leqslant \Upsilon(B,A)$;

(v) $\Upsilon(B_c, A_c) = \Upsilon(A,B)$;

(vi) $\Upsilon(B \vee C, A) = \wedge(\Upsilon(B,A), \Upsilon(C,A))$;

(vii) $\Upsilon(A, B \wedge C) = \wedge(\Upsilon(A, B), \Upsilon(A, C))$;
(viii) $\Upsilon(A, B \vee C) \geqslant \vee(\Upsilon(A, B), \Upsilon(A, C))$;
(ix) $\Upsilon(B \wedge C, A) \geqslant \vee(\Upsilon(B, A), \Upsilon(C, A))$.

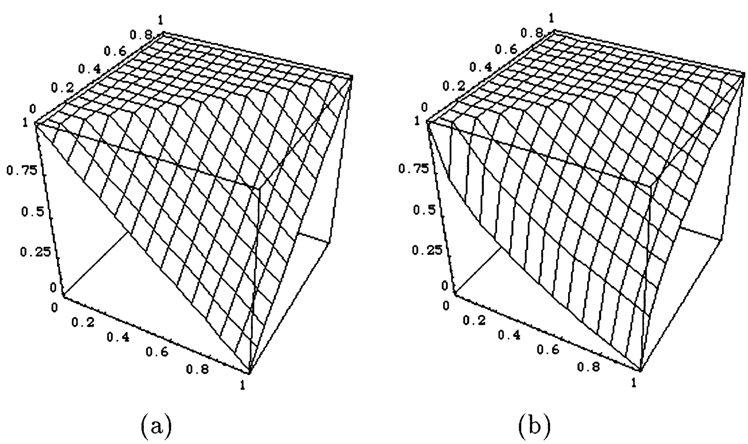
Proposition 2. *In the same conditions as in Proposition* 1 *being* $A_i, B_i, A, B \in FSs(X)$ *with* $i = 1, \ldots, n$,
(i) $\Upsilon(\bigvee_{i=1}^{n} A_i, B) = \bigwedge_{i-1}^{n} (\Upsilon(A_i, B))$;
(ii) $\Upsilon(A, \bigwedge_{i=1}^{n} B_i) = \bigwedge_{i=1}^{n} (\Upsilon(A, B_i))$;
(iii) $\Upsilon(A, \bigvee_{i=1}^{n} B_i) \geqslant \bigvee_{i=1}^{n} (\Upsilon(A, B_i))$;
(iv) $\Upsilon(\bigwedge_{i=1}^{n} A_i, B) \geqslant \bigvee_{i=1}^{n} (\Upsilon(A_i, B))$
holds.

In Fig. 2, $\wedge(1, g(c(\mu_A(x))) + g(\mu_B(x)))$ are represented for the cases: (a) $g(x) = x$ and (b) $g(x) = x^{\omega}$ with $\omega = 0.5$.

4.5. Expression of the interval-valued indicator of the grade of inclusion for IVFSs

In the following theorem we present an expression of the interval-valued inclusion grade indicator for the case when we are working with interval-valued fuzzy sets.

Theorem 2. *Let* $g : [0, 1] \to [0, 1]$ *be a continuous and strictly increasing function such that* $g(0) = 0$ *and* $g(1) = 1$, *and let* $c : [0, 1] \to [0, 1]$ *be an involutive and strict fuzzy complement such that* $c(x) = g^{-1}(1 - g(x))$. *Let us consider*



(a) (b)

Fig. 2.

$$\Upsilon_{\leqslant} : IVFSs(X) \times IVFSs(X) \to D[0,1]$$

given by

$$\Upsilon_{\leqslant}(A,B) = [\Upsilon_{\leqslant_L}(A,B), \Upsilon_{\leqslant_U}(A,B)],$$

where

$$\Upsilon_{\leqslant_L}(A,B) = \operatorname*{Inf}_{x\in X}\{\wedge(1, \wedge(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x))) + g(M_{BU}(x))))\},$$

$$\Upsilon_{\leqslant_U}(A,B) = \operatorname*{Inf}_{x\in X}\{\wedge(1, \vee(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x))) + g(M_{BU}(x))))\}$$

for all $A,B \in IVFSs(X)$. Then $\Upsilon_{\leqslant}(A,B)$ satisfies Axioms 1–9.

The considerations carried out before to justify the choice of (6) are now valid for justifying the choice of $\Upsilon_{\leqslant_L}$ and $\Upsilon_{\leqslant_U}$. The great advantage of the expressions $\Upsilon_{\leqslant_L}$ and $\Upsilon_{\leqslant_U}$ lies in that they can be generated by the same functions that generate fuzzy complementations.

Corollory 1. *In the same conditions as in Theorem* 2. *$A_i, B_i, A, B \in IVFSs(X)$ with $i = 1, \ldots, n$,*

$$\text{(i)} \quad \begin{cases} \Upsilon_{\leqslant_L}\left(\bigvee_{i=1}^{n} A_i, B\right) = \bigwedge_{i=1}^{n}(\Upsilon_{\leqslant_L}(A_i,B)), \\ \Upsilon_{\leqslant_U}\left(\bigvee_{i=1}^{n} A_i, B\right) \leqslant \bigwedge_{i=1}^{n}(\Upsilon_{\leqslant_U}(A_i,B)); \end{cases}$$

$$\text{(ii)} \quad \begin{cases} \Upsilon_{\leqslant_L}\left(A, \bigwedge_{i=1}^{n} B_i\right) = \bigwedge_{i=1}^{n}(\Upsilon_{\leqslant_L}(A,B_i)), \\ \Upsilon_{\leqslant_U}\left(A, \bigwedge_{i=1}^{n} B_i\right) \leqslant \bigwedge_{i=1}^{n}(\Upsilon_{\leqslant_U}(A,B_i)); \end{cases}$$

$$\text{(iii)} \quad \Upsilon_{\leqslant}\left(A, \bigvee_{i=1}^{n} B_i\right) \geqslant \bigvee_{i=1}^{n}(\Upsilon_{\leqslant}(A,B_i));$$

$$\text{(iv)} \quad \Upsilon_{\leqslant}\left(\bigwedge_{i=1}^{n} A_i, B\right) \geqslant \bigvee_{i=1}^{n}(\Upsilon_{\leqslant}(A_i,B))$$

holds.

Note that item (iii) of this corollary is a generalization of Axiom 8 and item (iv) of Axiom 9.

The expression of the inclusion grade indicator presented in Theorem 2 does not satisfy the properties (2) and (3) proposed by Sinha and Dougherty [54] as axioms, (when working with fuzzy sets and in Fuzzy Mathematical Morphology). However, it is clear that if in the corollary above we take $i = 1, 2$, then the lower extremes of $\Upsilon_{\leqslant}$ verify the equalities demanded in (2) and (3), and the upper extremes, without verifying the equalities both satisfy the same type of inequality, in the same conditions as in Corollary 1:

$$\Upsilon_{\leqslant L}(B \vee C, A) = \wedge(\Upsilon_{\leqslant L}(B, A), \Upsilon_{\leqslant L}(C, A)),$$

$$\Upsilon_{\leqslant U}(B \vee C, A) \leqslant \wedge(\Upsilon_{\leqslant U}(B, A), \Upsilon_{\leqslant U}(C, A)),$$

$$\Upsilon_{\leqslant L}(A, B \wedge C) = \wedge(\Upsilon_{\leqslant L}(A, B), \Upsilon_{\leqslant L}(A, C)),$$

$$\Upsilon_{\leqslant U}(A, B \wedge C) \leqslant \wedge(\Upsilon_{\leqslant U}(A, B), \Upsilon_{\leqslant U}(A, C)).$$

Let us apply Theorem 2 to some functions g. (Table 1) presents the corresponding expressions of $g(c(M_{AL}(x))) + g(M_{BL}(x))$ and $g(c(M_{AU}(x)))+$ $g(M_{BU}(x))$ according to the generic functions g that we take in each case.

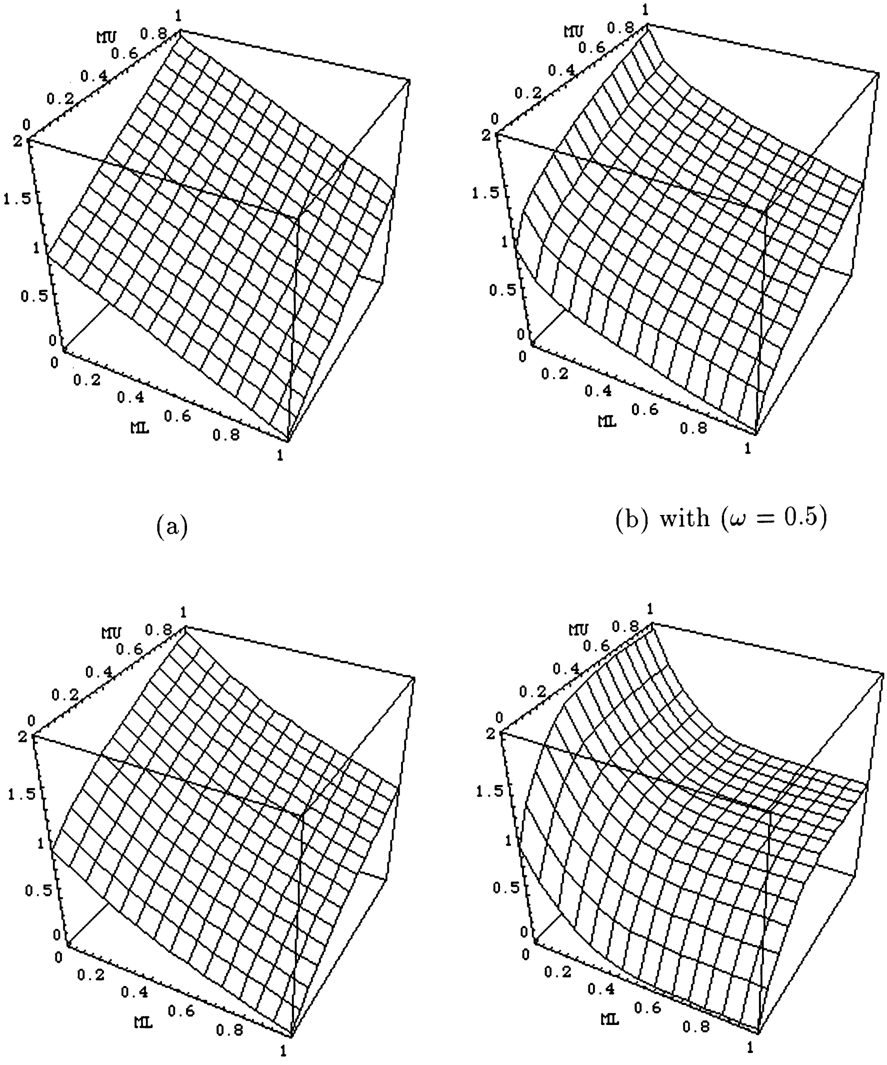
In Fig. 3 the surfaces $g(c(M_{AL})) + g(M_{BL})$ or $g(c(M_{AU})) + g(M_{BU})$ of the examples in Table 1 are found represented.

Example 1. The data in all of the examples in the paper are the ones that appear in [15].

Let X be the universe of discourse, $X = \{x_1, x_2, \ldots, x_{14}\}$, and let A and B be two interval-valued fuzzy sets of X, where

Table 1

		$g(c(M_{AL})) + g(M_{BL})$	$g(c(M_{AU})) + g(M_{BU})$
(a)	$g(x) = x$ $c(x) = 1 - x$	$1 - M_{AL}(x) + M_{BL}(x)$	$1 - M_{AU}(x) + M_{BU}(x)$
(b)	$g(x) = x^w$ $c(x) = (1 - x^w)^{\frac{1}{w}}$ $w > 0$	$1 - M^w_{AL}(x) + M^w_{BL}(x)$	$1 - M^w_{AU}(x) + M^w_{BU}(x)$
(c)	$g(x)\mathrm{Log}_a(1 + (a-1)x)$ $c(x) = \frac{1-x}{1+(a-1)x}$ $a \in (1, \infty)$	$\mathrm{Log}_a\left(\frac{a(1 + (a-1)M_{BL}(x))}{1 + (a-1)M_{AL}(x)}\right)$	$\mathrm{Log}_a\left(\frac{a(1 + (a-1)M_{BU}(x))}{1 + (a-1)M_{AU}(x)}\right)$
(d)	$g(x) = 1 - (1-x)^p$ $c(x) = 1 - (1 - (1-x)^p)^{1/p}$ $p > 1$	$(1 - M_{AL}(x))^p$ $+1 - (1 - M_{BL}(x))^p$	$(1 - M_{AU}(x))^p$ $+1 - (1 - M_{BU}(x))^p$



(a)

(b) with ($\omega = 0.5$)

(c) with ($a = e$)

(b) with ($p = 5$)

Fig. 3.

$$A = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \langle x_6, [0,0]\rangle,$$
$$\langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0.6]\rangle, \langle x_{10}, [0.87,0.92]\rangle, \langle x_{11}, [1,1]\rangle,$$
$$\langle x_{12}, [0.87,0.92]\rangle, \langle x_{13}, [0,0.6]\rangle, \langle x_{14}, [0,0]\rangle\},$$

$$B = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \langle x_6, [0,0]\rangle, \langle x_7, [0,0.5]\rangle, \langle x_8, [0.74,0.82]\rangle, \langle x_9, [0.94,0.95]\rangle, \langle x_{10}, [1,1]\rangle, \langle x_{11}, [0.94,0.95]\rangle, \langle x_{12}, [0.74,0.82]\rangle, \langle x_{13}, [0,0.5]\rangle, \langle x_{14}, [0,0]\rangle\}.$$

The membership function curves of these interval-valued fuzzy sets are shown in Fig. 4.

With these sets A and B the interval-valued inclusion grade indicator $\Upsilon_{\leqslant}(A,B)$ for the different functions g on Table 1 takes the following values of Table 2.

We have said in the preliminaries that in the whole paper we are going to consider the order relation '$\leqslant$' and for this reason we will represent the expression of the interval-valued inclusion grade indicator presented in Theorem 2 as $\Upsilon_{\leqslant}$. Nevertheless, it seems logical to study the behavior of $\Upsilon_{\leqslant}$ when applied to sets $A, B \in IVFSs(X)$ such that $A \preceq B$ or $A \sqsubseteq B$. In this sense the following theorem establishes the expression of $\Upsilon_{\leqslant}(A,B)$, first when the interval-valued fuzzy set B is settled in interval-valued fuzzy set A, that is, when for all $x \in X$, $M_{AL}(x) \leqslant M_{BL}(x) \leqslant M_{BU}(x) \leqslant M_{AU}(x)$, and second, when $A \sqsubseteq B$, that is when $M_{AU}(x) \leqslant M_{BL}(x)$ for all $x \in X$.

Theorem 3. *In the same conditions as in Theorem* 2

(i) *if* $A \preceq B$, *then* $\Upsilon_{\leqslant}(A,B) = \left[\operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(M_{AU}(x))) + g(M_{BU}(x)))\}, 1\right]$;

(ii) *if* $A \sqsubseteq B$, *then* $\Upsilon_{\leqslant}(A,B) = [1,1]$;

holds.

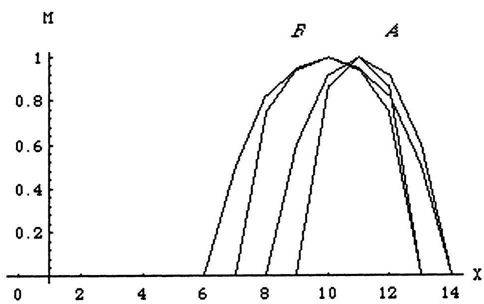


Fig. 4.

Table 2

	$\Upsilon_{\leqslant}(A,B)$
(a)	[0.870, 0.900]
(b) and $\omega = 0.5$	[0.930, 0.950]
(c) and $a = e$	[0.910, 0.930]
(d) and $p = 5$	[0.980, 0.999]

We know that one of the main advantages of the expression presented in Theorem 2 is that from any continuous and strictly increasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$, (functions that generate fuzzy complementations like $c(x) = g^{(-1)}(1 - g(x))$) we can obtain interval-valued inclusion grade indicators for interval-valued fuzzy sets. On the other hand it is known in the Fuzzy Set Theory that fuzzy complementations (involutive) can also be generated by continuous and strictly decreasing functions $f : [0,1] \to [0,1]$ such that $f(0) = 1$ and $f(1) = 0$. It seems logical to think that the interval-valued inclusion grade indicators for interval-valued fuzzy sets may also be generated by these functions. In this sense we present the following theorem:

Theorem 4. *Let $f : [0,1] \to [0,1]$ be a continuous and strictly decreasing function such that $f(0) = 1$ and $f(1) = 0$, and let $c : [0,1] \to [0,1]$ be an involutive and strict fuzzy complement such that $c(x) = f^{-1}(1 - f(x))$. Let us consider*

$$\Upsilon_{\leqslant} : IVFSs(X) \times IVFSs(X) \to D[0,1]$$

given by

$$\Upsilon_{\leqslant}(A,B) = [\Upsilon_{\leqslant_L}(A,B), \Upsilon_{\leqslant_U}(A,B)],$$

where

$$\Upsilon_{\leqslant_L}(A,B) = \operatorname*{Inf}_{x \in X}\{\wedge(1, \wedge(f(c(M_{BL}(x))) + f(M_{AL}(x)), f(c(M_{BU}(x))) + f(M_{AU}(x))))\},$$

$$\Upsilon_{\leqslant_U}(A,B) = \operatorname*{Inf}_{x \in X}\{\wedge(1, \vee(f(c(M_{BL}(x))) + f(M_{AL}(x)), f(c(M_{BU}(x))) + f(M_{AU}(x))))\}$$

for all $A, B \in IVFSs(X)$. Then $\Upsilon_{\leqslant}(A,B)$ satisfies Axioms 1–7.

Evidently if we take $f(x) = 1 - x$, the fuzzy complement generated by f is given by $c(x) = 1 - x$, so that the expression that we obtain for the inclusion grade indicator from Theorem 4 coincides with the example (a) on Table 1. If $f(x) = 1 - x^w$ with $w > 0$, the expression obtained from Theorem 4 coincides with (b). If $f(x) = \mathrm{Log}_a(a/1 + (a-1)x)$ with $a \in (1, \infty)$ we have the expression (c) on Table 1, etc.

In the following corollary we present the main properties of the interval-valued inclusion grade indicator in Theorem 2 (or Theorem 5). Such properties wll be used en successive sections in the paper.

Corollary 2. *In the same conditions as in Theorem* 2 (*or Theorem* 4)

(i) *if* $\Upsilon_{\leqslant}(A,B) = [0,1]$, *then*

$$\{x|(M_A(x) = [1,1] \text{ and } M_B(x) = [0,1]) \text{ or } (M_A(x) = [0,1] \text{ and } M_B(x) = [0,0])\} \neq \emptyset;$$

(ii) *if* $(\{x|M_A(x) = [1,1] \text{ and } M_B(x) = [0,1]\} \neq \emptyset \text{ and } M_{AU}(x) \leqslant M_{BU}(x) \forall x \in X)$ *or* $(\{x|M_A(x) = [0,1] \text{ and } M_B(x) = [0,0]\} \neq \emptyset \text{ and } M_{AL}(x) \leqslant M_{BL}(x) \forall\, x \in X)$, *then* $\Upsilon_{\leqslant}(A,B) = [0,1]$;

(iii) $\Upsilon_{\leqslant_L}(A,A_c) = \Upsilon_{\leqslant_U}(A,A_c) = \operatorname{Inf}_{x\in X}\{\wedge(1, g(c(M_{AL}(x))) + g(c(M_{AU}(x))))\}$;

(iv) *if* $A \in IVFSs(X)$ *is a normal set*, *then* $\Upsilon_{\leqslant}(A,A_c) = [0,0]$;

(v) $\Upsilon_{\leqslant_L}(A_c,A) = \Upsilon_{\leqslant_U}(A_c,A) = \operatorname{Inf}_{x\in X}\{\wedge(1, g(M_{AL}(x)) + g(M_{AU}(x)))\}$;

(vi) *if* $A_c \in IVFSs(X)$ *is a normal set*, *then* $\Upsilon_{\leqslant}(A_c,A) = [0,0]$;

(vii) *if* $B = \{\langle x, M_B(x) = [0,0]\rangle | x \in X\}$, *then for each* $A \in IVFSs(X)$

$$\Upsilon_{\leqslant}(A,B) = \left[\operatorname{Inf}_{x\in X}(g(c(M_{AU}(x)))), \operatorname{Inf}_{x\in X}(g(c(M_{AL}(x))))\right]$$

holds.

5. Method for inference in approximate reasoning based on *IVFSs* and interval-valued indicator

Approximate reasoning is, informally speaking, as Turksen says in [64], the process or processes by which a possible imprecise conclusion is deduced from a collection of imprecise premises.

The classical modus ponens is expressed by

$$\begin{array}{l} A \to B \\ \underline{A \qquad\quad} \\ \qquad B \end{array}$$

this means that if both:

A implies B and A are true, then
B is also true.

This reasoning scheme was extended to fuzzy reasoning by Zadeh [74,75] as follows:

(A) The implication $A \to B$ is replaced by the fuzzy inference rule

if x is A then y is B,

where A and B may be fuzzy sets, A in a universe of discourse X and B in a universe of discourse Y; and x is a variable which takes values in X, and y is a variable which takes values in Y. The fuzzy rule represents a relation between two variables x and y.

(B) Similarly, the premise A is replaced by a fuzzy premise:

$$x \ \text{ is } A',$$

where A' is a fuzzy set, expressing the knowledge we have about the value of x. A' is a fuzzy set in a universe of discourse X.

Combining the rule and the premise, it is possible to deduce a new piece of information, written

$$y \ \text{ is } B',$$

where B' is a fuzzy set in a universe of discourse Y.

Therefore, exact reasoning modus ponens can be extended to approximate reasoning which deals with the inherent vagueness of human language. Thus, the generalized modus ponens (GMP) is introduced to reach a conclusion from fuzzy premises. This rule can be expressed in standard as follows:

$$\begin{array}{l} \text{If } x \ \text{ is } A \ \text{ then } y \text{ is } B \\ \quad x \ \text{ is } A' \\ \hline \quad y \ \text{ is } B' \qquad \text{(GMP)} \end{array}$$

The main advantage of this extension to fuzzy reasoning is being able to deduce new information, even when the knowledge is not exactly identical to the condition of the rule or when the information we consider is not crisp. It is noted that in (GMP) when $A = A'$, then the generalized modus ponens reduces to the case of the modus ponens.

For the fuzzy inference rules, different methods have been suggested by various authors such as Zadeh [74,75], Fukami [25], Mizumoto and Zimmermann [40,41], Ezawa and Kandel [23]. In 1980 Fukami et al. [25] suggested the following set of axioms for the generalized modus ponens:

(F1) If $A' = A$, then $B' = B$ (coincidence with classical modus ponens),

(F2)

$$\text{either}\begin{cases} \text{(i) If } A' = A^2, \text{ then } B' = B \\ \qquad\qquad \text{or} \\ \text{(ii) If } A' = A^2, \text{ then } B' = B^2, \end{cases}$$

(F3) If $A' = A^{1/2}$, then $B' = B^{1/2}$,

(F4)

$$\text{either}\begin{cases} \text{(i) If } A' = A_c, \text{ then } B' = Y \\ \qquad\qquad \text{or} \\ \text{(ii) If } A' = A_c, \text{ then } B' = B_c. \end{cases}$$

Almost simultaneously, Baldwin and Pilsworth [5] establish another set of six axioms for generalized modus ponens, some of them in complete contradiction with some of the ones imposed by Fukami et al. For example, Baldwin and Pisworth demand that $B' \geqslant B$, which is impossible if (F2)(ii) is demanded at the same time. Baldwin and Pisworth are guided by classical logic and in their desire to avoid sorites paradox in fuzzy logic proposed their six axioms. Fukammi it al. are rather guided by common sense interpretations of fuzzy rules, in order to establish (F1)–(F4).

We will pay attention to axioms given by Fukami et al. and use them in Section 9 of this paper to propose a new application of the interval-valued inclusion grade indicator for *IVFSs*.

In this section and in Section 7 we are going to study the generalized modus ponens using interval-valued fuzzy sets following a different reasoning from the one used by the above-named authors. The basis will be the ideas established by Baldwin [4,5], Nafarie [42], Gorzalczany [29], etc., which can be summarized in the two following steps:

(1) first relate A to A',

(2) build the consequence B' using the result of the comparison above and B.

We will carry out step (1) using the inclusion grade indicator of the interval-valued fuzzy set A' in A.

We will see in Section 9 that it is not always advisable to follow the methodology of the steps (1) and (2). Sometimes, it is very hard to relate A' with A, (step 1), in theses cases it is preferable to use Zadeh's compositional rule. Unless otherwise indicated, we will follow steps (1) and (2).

We begin studying an algorithm for the GMP and then another for a system of m rules as follows:

$$\begin{array}{l} \text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \\ \text{If } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \\ \qquad\qquad \vdots \\ \text{If } x \text{ is } A_m \text{ then } y \text{ is } B_m \\ \underline{\quad x \text{ is } A' \qquad\qquad\qquad} \\ \quad y \text{ is } B' \end{array}$$

where $A', A_i \in \mathit{IVFSs}(X)$ and $B_i \in \mathit{IVFSs}(Y)$, with $i = 1, \ldots, m$.

5.1. Method for GMP

In this subsection we present an algorithm for obtaining the conclusion of the GMP:

$$\begin{array}{l} \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \underline{\quad x \text{ is } A' \qquad\qquad\qquad} \\ \quad y \text{ is } B' \qquad (\text{GMP}) \end{array}$$

that is, for obtaining B' from the interval-valued inclusion grade indicator for $\mathit{IVFSs}(X)$, $\Upsilon_{\leqslant}(A', A)$.

Firstly we will present a method of construction of interval-valued fuzzy sets on the referential Y from the interval-valued indicator.

For each $A', A \in IVFSs(X)$, we calculate $\Upsilon_{\leqslant}(A', A)$. In these conditions we construct

$$\Phi_{\Upsilon_{\leqslant}(A',A)} = \{\langle y, M_{\Phi_{\Upsilon_{\leqslant}(A',A)}}(y) = [M_{\Phi_{\Upsilon_{\leqslant}(A',A)},L}(y), M_{\Phi_{\Upsilon_{\leqslant}(A',A)},U}(y)]\rangle | y \in Y\},$$

where

$$M_{\Phi_{\Upsilon_{\leqslant}(A',A)}L}(y) = \Upsilon_{\leqslant L}(A', A),$$
$$M_{\Phi_{\Upsilon_{\leqslant}(A',A)}U}(y) = \Upsilon_{\leqslant U}(A', A).$$

Evidently $\Phi_{\Upsilon_{\leqslant}(A',A)}$ is an interval-valued fuzzy set on Y. We can point out the set $\Phi_{\Upsilon_{\leqslant}(A',A)}$ constructed in this way is such that $M_{\Phi_{\Upsilon_{\leqslant}(A',A)}}(y) = \text{const}$ for all $y \in Y$, besides if $A' = A_c$, in accordance with item (v) in Corollary 2, then $\Phi_{\Upsilon_{\leqslant}(A',A)}$ is a fuzzy set on Y.

In these conditions, the algorithm we present for obtaining the conclusion of the GMP is expressed in the following three items:

(i) The interval-valued inclusion grade indicator of A' in A is determined, that is, $\Upsilon_{\leqslant}(A', A)$.

(ii) Next we construct the set $\Phi_{\Upsilon_{\leqslant}(A',A)}$.

(iii) The conclusion is generated in the following way: $B'_{\Phi_{\Upsilon_{\leqslant}(A',A)}} = \wedge\big(\Phi_{\Upsilon_{\leqslant}(A',A)}, B\big)$.

Note that item (ii) of this algorithm justifies Axiom 1 demanded of all interval-valued inclusion grade indicators when working with interval-valued fuzzy sets.

As for the notation we must say that $B'_{\Upsilon_{\leqslant}(A',A)}$ indicates that the conclusion is generated from the interval-valued inclusion grade indicator applied to the sets A', A, in this order.

5.2. Characteristics of this algorithm

Proposition 3. *In the same conditions of the algorithm above,*

- $(GMP\Upsilon_{\leqslant}1)$ *if* $A' = A$, *then* $B'_{\Upsilon_{\leqslant}(A',A)} = B$;
- $(GMP\Upsilon_{\leqslant}2)$ *if* $A' < A$, *then* $B'_{\Upsilon_{\leqslant}(A',A)} = B$;
- $(GMP\Upsilon_{\leqslant}3)$ *for all* $A', A \in IVFSs(X)$, $B'_{\Upsilon_{\leqslant}(A',A)} \leqslant B$;
- $(GMP\Upsilon_{\leqslant}4)$ *if* $A'_1 \leqslant A'_2$, *then* $B'_{\Upsilon_{\leqslant}(A'_1,A)} \geqslant B'_{\Upsilon_{\leqslant}(A'_2,A)}$;
- $(GMP\Upsilon_{\leqslant}5)$ *if* A' *is a normal interval-valued fuzzy set* $A' = A_c$, *then*

$$B'_{\Upsilon_{\leqslant}(A',A)} = \{\langle y, M_{B'_{\Upsilon_{\leqslant}(A',A)}}(y) = [0, 0]\rangle | y \in Y\};$$

- $(GMP\Upsilon_{\leqslant}6)$ *if* $\Upsilon_{\leqslant}(A',A)=[0,0]$, *then* $B'_{\Upsilon_{\leqslant}(A',A)}=\{\langle y, M_{B'_{\Upsilon_{\leqslant}(A',A)}}(y)=[0,0]\rangle \mid y\in Y\}$;
- $(GMP\Upsilon_{\leqslant}7)$ *if* $A'=A_c$ *and for all* $y\in Y$ *it holds that* $\Upsilon_{\leqslant U}(A',A)\leqslant M_{BL}(y)$, *then* $B'_{\Upsilon_{\leqslant}(A',A)}=\Phi_{\Upsilon_{\leqslant}(A',A)}$, *this conclusion being a normal fuzzy set*;
- $(GMP\Upsilon_{\leqslant}8)$ *if* $A'\preceq A$, *then*

$$B'_{\Upsilon_{\leqslant}(A',A)}=\left\{\left\langle y, M_{B'_{\Upsilon_{\leqslant}(A',A)}}(y)=\left[\wedge\left(\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(M_{A'U}(x)))+g(M_{AU}(x)))\}, M_{BL}(y)\right), M_{BU}(y)\right]\right\rangle\right\}$$

holds.

Remark 1. According to the property $(GMP\Upsilon_{\leqslant}1)$ if $A'=A$ with the algorithm presented we recover the usual modus ponens, therefore $(GMP\Upsilon_{\leqslant}1)$ allows us to assure that such algorithm satisfies (F1). The problem of recovering the usual modus ponens in the fuzzy inference has motivated many works, for example, Trillas and Valverde [62], Dubois and Prade [20,21], etc.

The property $(GMP\Upsilon_{\leqslant}2)$ establishes that the theorem above satisfies $(F2)(i)$. If $A'=A^2$, (that is, if $M_{A'L}(x)=M^2_{AL}(x)$ and $M_{A'U}(x)=M^2_{AU}(x)$), then $A'\leqslant A$, in these conditions $B_{\Upsilon_{\leqslant}(A',A)}=B$. *The desire that this algorithm satisfies* $(F2)(i)$ *has led us to use* $\Upsilon_{\leqslant}(A',A)$ *instead of* $\Upsilon_{\leqslant}(A,A')$. We must point out that $(GMP\Upsilon_{\leqslant}2)$ is even stronger than (F2)(i). To many authors having to satisfy (F2)(i) implies a very strong restriction. This fact has led us to develop the two following sections in the paper.

From $(GMP\Upsilon_{\leqslant}3)$ and $(GMP\Upsilon_{\leqslant}4)$ we deduce that our algorithm does not satisfy the axioms established by Baldwin and Pilsworth. We said before that due to properties such as $(GMP\Upsilon_{\leqslant}3)$ there are contradictions between the Baldwin and Pilsworth's axioms and those of Fukami et al.

A consequence of the property $(GMP\Upsilon_{\leqslant}3)$ is that the interval-valued inclusion grade indicator $B'_{\Upsilon_{\leqslant}(A',A)}$ in the set B is always [1,1], that is, $\Upsilon_{\leqslant}(B'_{\Upsilon_{\leqslant}(A',A)},B)=[1,1]$.

From the property $(GMP\Upsilon_{\leqslant}5)$ we deduce that our algorithm does not satisfy the axiom (F4). In the conditions indicated by this property, we have $B'_{\Upsilon_{\leqslant}(A',A)}=Y_c$ instead of $B'_{\Upsilon_{\leqslant}(A',A)}=Y$. Situations like this and the non-fulfillment of the Axiom (F4) have lead us to develop Section 9 of this paper.

Example 2. Let us consider the referential sets

$$X=Y=\{x_1,x_2,\ldots,x_{14}\}.$$

Let A and B be two interval-valued fuzzy sets of X, where

$$A = \{\langle x_1,[0,0]\rangle,\langle x_2,[0,0]\rangle,\langle x_3,[0,0]\rangle,\langle x_4,[0,0]\rangle,\langle x_5,[0,0]\rangle,\langle x_6,[0,0]\rangle,\\ \langle x_7,[0,0.5]\rangle,\langle x_8,[0.74,0.82]\rangle,\langle x_9,[0.94,0.95]\rangle,\langle x_{10},[1,1]\rangle,\\ \langle x_{11},[0.94,0.95]\rangle,\langle x_{12},[0.74,0.82]\rangle,\langle x_{13},[0,0.5]\rangle,\langle x_{14},[0,0]\rangle\},$$

$$B = \{\langle x_1,[0,0]\rangle,\langle x_2,[0,0]\rangle,\langle x_3,[0.90,0.95]\rangle,\langle x_4,[1,1]\rangle,\langle x_5,[0.90,0.95]\rangle,\\ \langle x_6,[0,0.8]\rangle,\langle x_7,[0,0]\rangle,\langle x_8,[0,0]\rangle,\langle x_9,[0,0]\rangle,\langle x_{10},[0,0]\rangle,\\ \langle x_{11},[0,0]\rangle,\langle x_{12},[0,0]\rangle,\langle x_{13},[0,0]\rangle,\langle x_{14},[0,0]\rangle\}.$$

The membership function curves of these interval-valued fuzzy sets are shown in Fig. 5. Let A' be the following set:

$$A' = \{\langle x_1,[0,0]\rangle,\langle x_2,[0,0]\rangle,\langle x_3,[0,0]\rangle,\langle x_4,[0,0]\rangle,\langle x_5,[0,0]\rangle,\\ \langle x_6,[0,0]\rangle,\langle x_7,[0,0]\rangle,\langle x_8,[0,0]\rangle,\langle x_9,[0,0.6]\rangle,\langle x_{10},[0.87,0.92]\rangle,\\ \langle x_{11},[1,1]\rangle,\langle x_{12},[0.87,0.92]\rangle,\langle x_{13},[0,0.6]\rangle,\langle x_{14},[0,0]\rangle\}$$

it is necessary to point out that said sets A, B and A' are similar to the ones taken in [15].

We know from example one that:

(a) The interval-valued inclusion grade indicator between A and A' for the example (a) in Table 1, that is, for $g(x)=x$ and $c(x)=1-x$ is $\Upsilon_{\leqslant}(A',A)=[0.87,0.90]$ (Fig. 6).

In these conditions by item (ii) in the method above we have

$$\Phi_{\Upsilon_{\leqslant}(A',A)} = \{\langle x_1,[0.87,0.90]\rangle,\langle x_2,[0.87,0.90]\rangle,\langle x_3,[0.87,0.90]\rangle,\\ \langle x_4,[0.87,0.90]\rangle,\langle x_5,[0.87,0.90]\rangle,\langle x_6,[0.87,0.90]\rangle,\\ \langle x_7,[0.87,0.90]\rangle,\langle x_8,[0.87,0.90]\rangle,\langle x_9,[0.87,0.90]\rangle,\\ \langle x_{10},[0.87,0.90]\rangle,\langle x_{11},[0.87,0.90]\rangle,\langle x_{12},[0.87,0.90]\rangle,\\ \langle x_{13},[0.87,0.90]\rangle,\langle x_{14},[0.87,0.90]\rangle\}$$

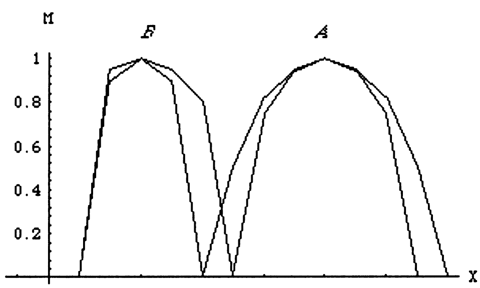


Fig. 5.

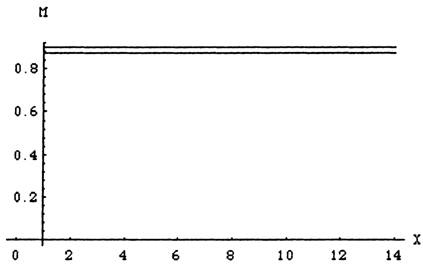


Fig. 6.

obtaining by item (iii) the following conclusion for the generalized modus ponens, (the representation of the membership functions of the conclusion, as well as of set A' are given in Fig. 7):

$$B'_{\Phi_{\Upsilon_{\leqslant}(A',A)}} = \{\langle x_1,[0,0]\rangle, \langle x_2,[0,0]\rangle, \langle x_3,[0.87,0.90]\rangle, \langle x_4,[0.87,0.90]\rangle,$$
$$\langle x_5,[0.87,0.90]\rangle, \langle x_6,[0,0.8]\rangle, \langle x_7,[0,0]\rangle, \langle x_8,[0,0]\rangle, \langle x_9,[0,0]\rangle,$$
$$\langle x_{10},[0,0]\rangle, \langle x_{11},[0,0]\rangle, \langle x_{12},[0,0]\rangle, \langle x_{13},[0,0]\rangle, \langle x_{14},[0,0]\rangle\}.$$

(b) The interval-valued inclusion grade indicator between A and B for (b) in Table 1 (with $\omega = 0.5$) is: $\Upsilon_{\leqslant}(A',A) = [0.93,0.95]$, in these conditions the conclusion of the generalized modus ponens (represented in Fig. 8 next to the representation of A') is

$$B'_{\Phi_{\Upsilon_{\leqslant}(A',A)}} = \{\langle x_1,[0,0]\rangle, \langle x_2,[0,0]\rangle, \langle x_3,[0.90,0.95]\rangle, \langle x_4,[0.93,0.95]\rangle,$$
$$\langle x_5,[0.90,0.95]\rangle, \langle x_6,[0,0.80]\rangle, \langle x_7,[0,0]\rangle, \langle x_8,[0,0]\rangle,$$
$$\langle x_9,[0,0]\rangle, \langle x_{10},[0,0]\rangle, \langle x_{11},[0,0]\rangle, \langle x_{12},[0,0]\rangle, \langle x_{13},[0,0]\rangle,$$
$$\langle x_{14},[0,0]\rangle\}.$$

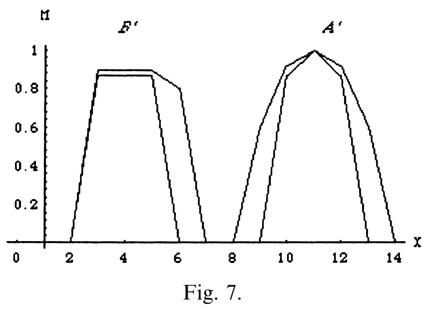


Fig. 7.

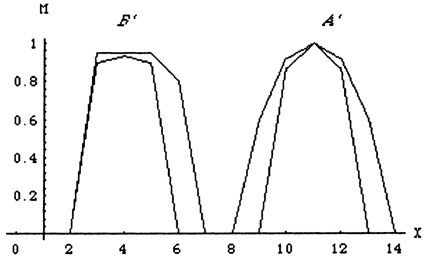


Fig. 8.

(c) For the case (c) in Table 1 with $a = e$ we have $\Upsilon_{\leqslant}(A', A) = [0.91, 0.93]$, following the steps in the algorithm above we have (Fig. 9):

$$B'_{\Phi_{\Upsilon_{\leqslant}(A',A)}} = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0.90, 0.93]\rangle, \langle x_4, [0.91, 0.93]\rangle, \langle x_5, [0.90, 0.93]\rangle, \langle x_6, [0, 0.8]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\}.$$

(d) For (d) with $p = 5$ we have: $\Upsilon_{\leqslant}(A', A) = [0.98, 0.999]$. For item (iii) the conclusion, (represented in Fig. 10 next to the representation of A'), is

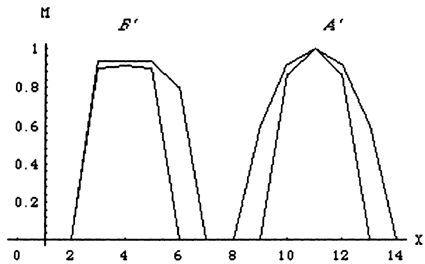


Fig. 9.

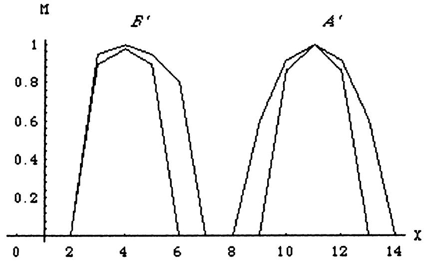


Fig. 10.

$$B'_{\Phi_{\Upsilon_{\leqslant}(A',A)}} = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0.90, 0.95]\rangle, \langle x_4, [0.98, 0.99]\rangle,$$
$$\langle x_5, [0.90, 0.95]\rangle, \langle x_6, [0, 0.8]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle,$$
$$\langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\}.$$

5.3. Method for the multiconditional approximate reasoning

We consider the interval-valued fuzzy sets, $A_i, A' \in IVFSs(X)$ and $B_i \in IVFSs(Y)$, with $i = 1, \dots, m$. The description of the object has the form:

$$\begin{array}{l}
\text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \\
\text{If } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \\
\qquad\qquad \vdots \\
\text{If } x \text{ is } A_m \text{ then } y \text{ is } B_m \\
\underline{\quad x \text{ is } A' \qquad\qquad\qquad} \\
\quad y \text{ is } B'
\end{array}$$

The algorithm that we present for obtaining B' is given by the following steps:

(i) For each $i = 1, \dots, m$, we calculate the interval-valued inclusion grade indicator of A' in A_i, that is, $\Upsilon_{\leqslant}(A', A_i)$.

(ii) Next we construct the sets

$$\Phi^i_{\Upsilon_{\leqslant}(A',A_i)} = \{\langle y, M_{\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}}(y) = [\Upsilon_{\leqslant} L(A', A_i), \Upsilon_{\leqslant} U(A', A_i)]\rangle | y \in Y\},$$

with $i = 1, \dots, m$.

(iii) We generate the conclusion $B'_{\Upsilon_{\leqslant}(A',A_i)}$ in the following way:

$$B'_{\Upsilon_{\leqslant}(A',A_i)} = \bigvee_{i=1}^{m} \Big(\wedge (\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i) \Big).$$

Note that the inclusion grade indicators of A' with each of sets A_i, $i = 1, \dots, m$, are determined consecutively.

In accordance with the construction presented in item (ii) we will represent as $\Phi^{(\vee_i A_i)}_{\Upsilon_{\leqslant}(A',A_i)}$ the interval-valued fuzzy set constructed from the indicator $\Upsilon_{\leqslant}(A', \bigvee_{i=1} A_i)$, that is,

$$\Phi^{(\vee_i A_i)}_{\Upsilon_{\leqslant}(A',A_i)} = \left\{ \left\langle y, M_{\Phi^{(\vee_i A_i)}_{\Upsilon_{\leqslant}(A',A_i)}}(y) = \Upsilon_{\leqslant}\left(A', \bigvee_{i=1}^{m} A_i\right) \right\rangle \middle| y \in Y \right\}.$$

5.4. Characteristics of this algorithm

Proposition 4. *In the conditions of the algorithm above,*

- $(MAR\Upsilon_{\leqslant}1)$ $B'_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \bigvee_{i=1}^m B_i$, *that is,* $\Upsilon_{\leqslant}\left(B'_{\Upsilon_{\leqslant}(A',A_i)}, \bigvee_{i=1}^m B_i\right) = [1,1]$;
- $(MAR\Upsilon_{\leqslant}2)$ *if* $A' \leqslant A_i$ *for each* $i = 1,\ldots,m$, *then* $B'_{\Upsilon_{\leqslant}(A',A_i)} = \bigvee_{i=1}^m B_i$;
- $(MAR\Upsilon_{\leqslant}3)$ *if for a j such that* $1 \leqslant j \leqslant m$, *we have that* $A' = A_jc$ *is a normal interval-valued fuzzy set, then* $B'_{\Upsilon_{\leqslant}(A',A_i)} = \bigvee_{\substack{i=1\\ i\neq j}}^m \left(\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i)\right)$;
- $(MAR\Upsilon_{\leqslant}4)$ *if for a value of* $i = 1,\ldots,m$, *for example for j,*

$$\{x|[M_{A'L}(x), M_{A'U}(x)] = [1,1] \text{ and } [M_{A_jL}(x), M_{A_jU}(x)] = [0,0]\} \neq \emptyset,$$

then $B'_{\Upsilon_{\leqslant}(A',A_i)} = \bigvee_{\substack{i=1\\ i\neq j}}^m \left(\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i)\right)$;

- $(MAR\Upsilon_{\leqslant}5)$ $B'_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \Phi^{(\vee_i A_i)}_{\Upsilon_{\leqslant}(A',A_i)}$;
- $(MAR\Upsilon_{\leqslant}6)$ *j fixed we have*

$$\Upsilon_{\leqslant L}(B'_{\Upsilon_{\leqslant}(A',A_i)}, B_j) = \bigwedge_{\substack{i=1\\ i\neq j}} \left(\Upsilon_{\leqslant L}\left(\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i), B_j\right)\right),$$

$$\Upsilon_{\leqslant U}(B_{\Upsilon_{\leqslant}(A',A_i)}\prime, B_j) \leqslant \bigwedge_{\substack{i=1\\ i\neq j}} \left(\Upsilon_{\leqslant U}\left(\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i), B_j\right)\right);$$

- $(MAR\Upsilon_{\leqslant}7)$ $B'_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \bigvee_{i=1}^m \Phi^i_{\Upsilon_{\leqslant}(A',A_i)}$;
- $(MAR\Upsilon_{\leqslant}8)$ *if* $A_m \leqslant A_{m-1} \leqslant \cdots \leqslant A_2 \leqslant A_1$, *then*

$$B'_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \wedge\left(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, \bigvee_{i=1}^m B_i\right);$$

- $(MAR\Upsilon_{\leqslant}9)$ *if* $\Upsilon_{\leqslant}(A', A_i) = [0,1]$ *for all* $i = 1,\ldots,m$, *then*

$$B'_{\Upsilon_{\leqslant}(A',A_i)} = \left\{\left\langle y, M_{B_{\Upsilon_{\leqslant}(A',A_i)}\prime}(y) = \left[0, \bigvee_{i=1}^m M_{B_iU}(y)\right]\right\rangle \middle| y \in Y\right\}$$

holds.

Remark 2. The property $(MAR\Upsilon_{\leqslant}3)$ can be stated by saying if $A' = A_jc$ is a normal interval-valued fuzzy set, then the respective set B_j is eliminated from point (iii) of the algorithm. As for $(MAR\Upsilon_{\leqslant}4)$ we can say the same, that is, if the hypothesis of these properties are met, then the respective B_j is eliminated from point (iii) of the algorithm.

Note that if in $(MAR\Upsilon_{\leqslant}4)$ we take $i = 1, 2$, then this property states the following:

$$\text{If } \Upsilon_{\leqslant}(A', A_1) = [0,0], \text{ then } B'_{\Upsilon_{\leqslant}(A',A_i)} = \wedge(\Phi^2_{\Upsilon_{\leqslant}(A',A_i)}, B_2)$$

or

$$\text{if } \Upsilon_{\leqslant}(A', A_2) = [0,0], \text{ then } B'_{\Upsilon_{\leqslant}(A',A_i)} = \wedge(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_1).$$

On the other hand, if $i = 1, 2$, then the property $(MAR\Upsilon_{\leqslant}5)$ is written in the following way:

$$B'_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \left\{ \left\langle y, M_{\Phi^{(A_1 \vee A_2)}_{\Upsilon_{\leqslant}(A',A_i)}}(y) = \Upsilon_{\leqslant}(A', A_1 \vee A_2) \right\rangle \middle| y \in Y \right\} = \Phi^{(A_1 \vee A_2)}_{\Upsilon_{\leqslant}(A',A_i)}$$

inequality obtained from Axiom 8.

The property $(MAR\Upsilon_{\leqslant}6)$ expresses the grade of inclusion of the conclusion $B'_{\Upsilon_{\leqslant}(A',A_i)}$ in any set B_j of the collection of sets B_i, $(i = 1, \ldots, m)$. Besides if $i = 1, 2$, then this property takes the following form:

$$\Upsilon_{\leqslant L}(B'_{\Upsilon_{\leqslant}(A',A_i)}, B_1) = \Upsilon_{\leqslant L}\Big(\wedge(\Phi^2_{\Upsilon_{\leqslant}(A',A_i)}, B_2), B_1 \Big),$$
$$\Upsilon_{\leqslant U}(B'_{\Upsilon_{\leqslant}(A',A_i)}, B_1) \leqslant \Upsilon_{\leqslant U}\Big(\wedge(\Phi^2_{\Upsilon_{\leqslant}(A',A_i)}, B_2), B_1 \Big),$$

$$\Upsilon_{\leqslant L}(B'_{\Upsilon_{\leqslant}(A',A_i)}, B_2) = \Upsilon_{\leqslant L}\Big(\wedge(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_1), B_2 \Big),$$
$$\Upsilon_{\leqslant U}(B'_{\Upsilon_{\leqslant}(A',A_i)}, B_2) \leqslant \Upsilon_{\leqslant U}\Big(\wedge(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_1), B_2 \Big),$$

and in these conditions by Axiom 9 we have

$$\Upsilon_{\leqslant L}\Big(\wedge(\Phi^2_{\Upsilon_{\leqslant}(A',A_i)}, B_2), B_1 \Big) \geqslant \vee\Big(\Upsilon_{\leqslant L}(\Phi^2_{\Upsilon_{\leqslant}(A',A_i)}, B_1), \Upsilon_{\leqslant L}(B_2, B_1) \Big),$$
$$\Upsilon_{\leqslant U}\Big(\wedge(\Phi^2_{\Upsilon_{\leqslant}(A',A_i)}, B_2), B_1 \Big) \geqslant \vee\Big(\Upsilon_{\leqslant U}(\Phi^2_{\Upsilon_{\leqslant}(A',A_i)}, B_1), \Upsilon_{\leqslant U}(B_2, B_1) \Big),$$

$$\Upsilon_{\leqslant L}\Big(\wedge(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_1), B_2 \Big) \geqslant \vee\Big(\Upsilon_{\leqslant L}(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_2), \Upsilon_{\leqslant L}(B_1, B_2) \Big),$$
$$\Upsilon_{\leqslant U}\Big(\wedge(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_1), B_2 \Big) \geqslant \vee\Big(\Upsilon_{\leqslant U}(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_2), \Upsilon_{\leqslant U}(B_1, B_2) \Big).$$

If we impose $B_m \leqslant B_{m-1} \leqslant \cdots \leqslant B_2 \leqslant B_1$ on the property $(MAR\Upsilon_{\leqslant}8)$, then

$$B'_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \wedge\Big(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, B_1 \Big).$$

Example 3. Let us consider the following single-input-single-output interval-valued approximate reasoning scheme

$$\begin{array}{l} \text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \\ \text{If } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \\ \text{If } x \text{ is } A_3 \text{ then } y \text{ is } B_3 \\ \text{If } x \text{ is } A_4 \text{ then } y \text{ is } B_4 \\ \text{If } x \text{ is } A_5 \text{ then } y \text{ is } B_5 \\ \quad x \text{ is } A' \\ \hline \quad y \text{ is } B' \end{array}$$

where $A', A_1, \ldots, A_5, B_1, B_2, \ldots, B_5$ are interval-valued fuzzy sets of universe of discourse $X = Y = \{x_1, x_2, \ldots, x_{14}\}$. These interval-valued fuzzy sets are shown as follows:

$$\begin{aligned} A_1 = \{&\langle x_1, [1,1]\rangle, \langle x_2, [1,1]\rangle, \langle x_3, [0.82, 0.95]\rangle, \langle x_4, [0, 0.7]\rangle, \langle x_5, [0,0]\rangle, \\ &\langle x_6, [0,0]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \\ &\langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\}, \end{aligned}$$

$$\begin{aligned} A_2 = \{&\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0, 0.5]\rangle, \langle x_4, [0.75, 0.8]\rangle, \langle x_5, [0.94, 0.95]\rangle, \\ &\langle x_6, [1,1]\rangle, \langle nglex_7, [0.94, 0.95]\rangle, \langle x_8, [0.75, 0.83]\rangle, \langle x_9, [0, 0.5]\rangle, \\ &\langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\}, \end{aligned}$$

$$\begin{aligned} A_3 = \{&\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \langle x_6, [0,0]\rangle, \\ &\langle x_7, [0, 0.6]\rangle, \langle x_8, [0.87, 0.92]\rangle, \langle x_9, [1,1]\rangle, \langle x_{10}, [0.87, 0.92]\rangle, \langle x_{11}, [0, 0.6]\rangle, \\ &\langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\}, \end{aligned}$$

$$\begin{aligned} A_4 = \{&\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \langle x_6, [0,0]\rangle, \\ &\langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0, 0.9]\rangle, \langle x_{10}, [0.87, 0.92]\rangle, \langle x_{11}, [1,1]\rangle, \\ &\langle x_{12}, [0.87, 0.92]\rangle, \langle x_{13}, [0, 0.6]\rangle, \langle x_{14}, [0,0]\rangle\}, \end{aligned}$$

$$\begin{aligned} A_5 = \{&\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \langle x_6, [0,0]\rangle, \\ &\langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0, 0.6]\rangle, \\ &\langle x_{13}, [0.87, 0.92]\rangle, \langle x_{14}, [1,1]\rangle\}. \end{aligned}$$

The membership function curves of these *IVFSs* are shown in Fig. 11

$$\begin{aligned} B_1 = \{&\langle x_1, [1,1]\rangle, \langle x_2, [0.94, 0.96]\rangle, \langle x_3, [0, 0.65]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \\ &\langle x_6, [0,0]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \\ &\langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\}, \end{aligned}$$

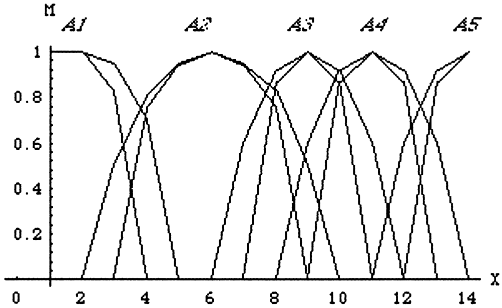


Fig. 11.

$$B_2 = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0.6]\rangle, \langle x_3, [0.87,0.92]\rangle, \langle x_4, [1,1]\rangle, \langle x_5, [0.87,0.92]\rangle, \langle x_6, [0,0.6]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\},$$

$$B_3 = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0.5]\rangle, \langle x_5, [0.74,0.82]\rangle, \langle x_6, [0.94,0.95]\rangle, \langle x_7, [1,1]\rangle, \langle x_8, [0.94,0.95]\rangle, \langle x_9, [0.74,0.82]\rangle, \langle x_{10}, [0,0.5]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\},$$

$$B_4 = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \langle x_6, [0,0]\rangle, \langle x_7, [0,0.5]\rangle, \langle x_8, [0.74,0.82]\rangle, \langle x_9, [0.94,0.95]\rangle, \langle x_{10}, [1,1]\rangle, \langle x_{11}, [0.94,0.95]\rangle, \langle x_{12}, [0.74,0.82]\rangle, \langle x_{13}, [0,0.5]\rangle, \langle x_{14}, [0,0]\rangle\},$$

$$B_5 = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0,0]\rangle, \langle x_4, [0,0]\rangle, \langle x_5, [0,0]\rangle, \langle x_6, [0,0]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0.6]\rangle, \langle x_{12}, [0.87,0.92]\rangle, \langle x_{13}, [1,1]\rangle, \langle x_{14}, [1,1]\rangle\}.$$

The membership function curves of these interval-valued fuzzy sets are shown in Fig. 12.

Assume that given the fact 'x' is A', where

$$A' = \{\langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0.90,0.95]\rangle, \langle x_4, [1,1]\rangle, \langle x_5, [0.90,0.95]\rangle, \langle x_6, [0,0.8]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \langle x_{14}, [0,0]\rangle\}.$$

In these conditions the inclusion grade indicators $\Upsilon_{\leqslant}(A', A_i)$ for $g(x) = x$ and $c(x) = 1 - x$ with $i = 1, \ldots, 5$ are given in Table 3.

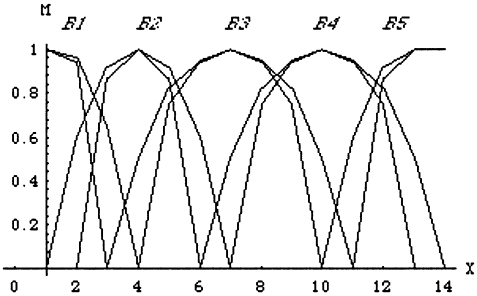


Fig. 12.

Table 3

$\Upsilon_{\leqslant}(A', A_1)$	[0.00, 0.10]
$\Upsilon_{\leqslant}(A', A_2)$	[0.00, 0.55]
$\Upsilon_{\leqslant}(A', A_3)$	[0.00, 0.00]
$\Upsilon_{\leqslant}(A', A_4)$	[0.00, 0.00]
$\Upsilon_{\leqslant}(A', A_5)$	[0.00, 0.00]

According to item (iii) in the algorithm above the conclusion, whose representation of the membership functions as opposed to those of A' can be found in Fig. 13 is given by

$$
\begin{aligned}
B'_{\Phi_{\Upsilon_{\leqslant}(A',A)}} = \{ & \langle x_1, [0, 0.1]\rangle, \langle x_2, [0, 0.55]\rangle, \langle x_3, [0.1, 0.55]\rangle, \langle x_4, [0.1, 0.55]\rangle, \\
& \langle x_5, [0.1, 0.55]\rangle, \langle x_6, [0, 0.55]\rangle, \langle x_7, [0, 0]\rangle, \langle x_8, [0, 0]\rangle, \\
& \langle x_9, [0, 0]\rangle, \langle x_{10}, [0, 0]\rangle, \langle x_{11}, [0, 0]\rangle, \langle x_{12}, [0, 0]\rangle, \langle x_{13}, [0, 0]\rangle, \\
& \langle x_{14}, [0, 0]\rangle \}.
\end{aligned}
$$

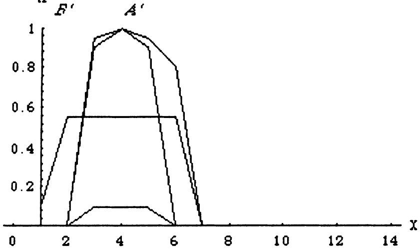


Fig. 13.

6. Interval-valued similarity measures

The similarity measure of two fuzzy sets introduced by Wang [68] and other people is a measure that indicates the similarity between fuzzy sets. Many authors [8,46,57,66,70–72], and in very different ways, have studied the problem of 'distinguishing' between similar fuzzy sets. Basing ourselves on their ideas we proposed a continuation of the similarity measure between interval-valued fuzzy sets.

Definition 1. A function $\mathscr{S} : IVFSs(X) \times IVFSs(X) \to D[0,1]$ is called a normal interval-valued similarity measure, if $\mathscr{S}$ has the following properties:
(i) $\mathscr{S}(A,B) = \mathscr{S}(B,A)$ for all $A, B \in IVFSs(X)$,
(ii) $\mathscr{S}(D, D_c) = [0,0]$ for all $D \in P(X)$,
(iii) $\mathscr{S}(C,C) = [1,1]$ for all $C \in IVFSs(X)$,
(iv) for all $A, B, C \in IVFSs(X)$ if $A \leqslant B \leqslant C$, then

$$\mathscr{S}(A,B) \geqslant \mathscr{S}(A,C)$$

and

$$\mathscr{S}(B,C) \geqslant \mathscr{S}(A,C),$$

(v) if $A, B \in FSs(X)$, then $\mathscr{S}(A,B) \in [0,1]$.

We can prove that (iv) is equivalent to (iv)': for all $A, B, C, D \in IVFSs(X)$, if $A \leqslant B \leqslant C \leqslant D$, then $\mathscr{S}(B,C) \geqslant \mathscr{S}(A,D)$.

Theorem 5. *Let us consider any t-norm β and an interval-valued inclusion grade indicator Υ. For any A and B in $IVFSs(X)$,*

$$\begin{aligned}\mathscr{S}(A,B) &= [\mathscr{S}_L(A,B), \mathscr{S}_U(A,B)] \\ &= [\beta(\Upsilon_L(A,B), \Upsilon_L(B,A)), \beta(\Upsilon_U(A,B), \Upsilon_U(B,A))]\end{aligned}$$

is an interval-valued normal similarity measure.

Proposition 5. *If $A \leqslant B$ and $\{x | M_B(x) = [1,1]$ and $M_A(x) = [0,0]\} \neq \emptyset$, then $\mathscr{S}(A,B) = [0,0]$.*

Note that the theorem and the proposition above are presented for any t-norm β. However, due to the fact that we want the algorithm of the following section to allow us to justify some of the axioms demanded of all interval-valued inclusion grade indicators, and these axioms are expressed with respect to the maximum and the minimum, hereinafter we will take $\beta = \wedge$.

Theorem 6. *In the same conditions as in Theorem* 5:

If $\beta = \wedge$ *and* $\Upsilon = \Upsilon_{\leqslant}$, *then*

(i) $\mathscr{S}_{\leqslant}(A, A_c) = [0, 0]$ *if and only if* A *or* A_c *are interval-valued normal fuzzy sets on the same* X;
(ii) $\mathscr{S}_{\leqslant}(A, B) = [1, 1]$ *if and only if* $A = B$;
(iii) *if* $A \sqsubseteq B$, *then* $\mathscr{S}_{\leqslant}(A, B) = \Upsilon_{\leqslant}(B, A)$;
(iv) *if* $S_{\leqslant}(A, B) = [0, 1]$, *then* $\{x|(M_A(x) = [1, 1]$ *and* $M_B(x) = [0, 1])$ *or* $(M_A(x) = [0, 1]$ *and* $M_B(x) = [0, 0])\} \neq \emptyset$ *holds.*

Example 4. With the same data as the ones considered in Example 1 and Table 1 we have Table 4.

7. Method for inference in approximate reasoning based on *IVFSs* and interval-valued similarity measures

The algorithm presented in Section 5 has the property that if $A' \leqslant A$, then $B' = B$, nevertheless it is often interesting to weaken this property in the sense that if $A' < A$, $B' = B$ not always happens. The objective of having algorithms for which the conditions above are verified has led us to develop this section.

7.1. Method for GMP

The algorithm for the GMP that we present below uses interval-valued similarity measures between interval-valued fuzzy sets.

We begin presenting a method of construction of interval-valued fuzzy sets on the referential Y form the interval -valued similarity measure.

For each $A', A \in IVFSs(X)$, we calculate $\mathscr{S}_{\leqslant}(A', A) = \wedge(\Upsilon_{\leqslant}(A', A), \Upsilon_{\leqslant}(A, A'))$. In these conditions we construct

$$\Phi_{\mathscr{S}_{\leqslant}}(A', A) = \{\langle y, M_{\Phi_{\mathscr{S}_{\leqslant}}(A',A)}(y) = [M_{\Phi_{\mathscr{S}_{\leqslant}}(A',A),L}(y), M_{\Phi_{\mathscr{S}_{\leqslant}}(A',A),U}(y)]\rangle | y \in Y\},$$

where

$$M_{\Phi_{\mathscr{S}_{\leqslant}}(A',A)L}(y) = \mathscr{S}_{\leqslant}L(A', A)$$

Table 4

	$\mathscr{S}_{\leqslant}(A, B)$
(a)	[0.06, 0.26]
(b) and w =0.5	[0.03, 0.14]
(c) and a = e	[0.04, 0.18]
(d) and p = 5	$[810^{-7}, 1.210^{-3}]$

and

$$M_{\Phi_{\mathscr{S}_\leqslant}(A',A)U}(y) = \mathscr{S}_\leqslant U(A',A).$$

Evidently $\Phi_{\mathscr{S}_\leqslant}(A',A)$ is an interval-valued fuzzy set on Y. We can point out the set $\Phi_{\mathscr{S}_\leqslant}(A',A)$ constructed in this way is such that $M_{\Phi_{\mathscr{S}_\leqslant}(A',A)}(y) = \text{const}$ for all $y \in Y$.

In these conditions, the algorithm we present for obtaining the conclusion of the GMP is expressed in the three following items:

(i) The interval-valued similarity measure A' in A is determined, that is, $\mathscr{S}_\leqslant(A',A)$.

(ii) Next we construct the set $\Phi_{\mathscr{S}_\leqslant}(A',A)$.

(iii) The conclusion $B'_{\mathscr{S}_\leqslant}$ is generated in the following way

$$B'_{\mathscr{S}_\leqslant}(A',A) = \wedge\big(\Phi_{\mathscr{S}_\leqslant}(A',A), B\big).$$

7.2. Characteristics of this algorithm

Proposition 6. *In the conditions of the algorithm above,*

- $(GMP\mathscr{S}_\leqslant 1)$ *if* $A' = A$, *then* $B'_{\mathscr{S}_\leqslant}(A',A) = B$;
- $(GMP\mathscr{S}_\leqslant 2)$ *if* $A' < A$, *then* $B'_{\mathscr{S}_\leqslant}(A',A) = \wedge(\Phi_{\Upsilon_\leqslant}(A,A'), B)$;
- $(GMP\mathscr{S}_\leqslant 3)$ *for all* $A', A \in IVFSs(X)$, $B'_{\mathscr{S}_\leqslant}(A',A) \leqslant B$ *holds*;
- $(GMP\mathscr{S}_\leqslant 4)$ *if* $A' = A_c$ *and* A' *or* A *are normal interval-valued fuzzy sets, then*

  $$B'_{\mathscr{S}_\leqslant}(A',A) = \{\langle y, M_{B'_{\mathscr{S}_\leqslant}(A',A)} = [0,0]\rangle | y \in Y\};$$
- $(GMP\mathscr{S}_\leqslant 5)$ *for all* $A', A \in IVFSs(X)$, *then* $\mathscr{S}_\leqslant\Big(B'_{\mathscr{S}_\leqslant}(A',A), B\Big) = \Upsilon_\leqslant\Big(B, B'_{\mathscr{S}_\leqslant}(A',A)\Big)$;
- $(GMP\mathscr{S}_\leqslant 6)$ *if* $\mathscr{S}_\leqslant(A',A) = [0,1]$, *then*

  $$B'_{\mathscr{S}_\leqslant}(A',A) = \{\langle y, M_{B'_{\mathscr{S}_\leqslant}(A',A)}(y) = [0, M_{BU}(y)]\rangle | y \in Y\}$$

holds.

Remark 3. From $(GMP\mathscr{S}_\leqslant 1)$ we can deduce that this algorithm satisfies (F1). However, from $(GMP\mathscr{S}_\leqslant 2)$ it happens that it does not always satisfy if $A' < A$, then $B'_{\mathscr{S}_\leqslant}(A',A) = B$. This being a property that differentiates this algorithm from the one presented in Section 5. We must point out that in $B'_{\mathscr{S}_\leqslant}(A',A) = \wedge(\Phi_{\Upsilon_\leqslant}(A,A'), B)$ the order of the sets A' and A changes.

The property $(GMP\mathscr{S}_\leqslant 3)$ is the same as $(GMP\Upsilon_\leqslant 3)$.

In the property $(GMP\mathscr{S}_\leqslant 4)$ we demand $A' = A_c$ and normal A' or A, whereas in $(GMP\Upsilon_\leqslant 5)$ we demanded that $A' = A_c$ be a normal set.

In $(GMP\mathscr{S}_\leqslant 5)$ we studied the interval-valued similarity measure between the conclusion of the GMP obtained by this algorithm and B.

Note that if $A' \sqsubseteq A$, then $B'_{\mathscr{S}_{\leqslant}}(A', A) = \wedge(\Phi_{\Upsilon_{\leqslant}}(A, A'), B)$, we only need to recall item (iii) of Theorem 6.

Example 5. In the same conditions as in Example 2 we have:

(a) We know that for the example (a) on Table 4, $\mathscr{S}_{\leqslant}(A', A) = [0.06, 0.26]$. By item (ii) we have $\Phi_{\mathscr{S}_{\leqslant}(A',A)} = \{\langle x_i, [0.06, 0.26]\rangle | x_i \in X, i = 1, \ldots, 14\}$, and by item (iii) the conclusion, represented as opposed to A' in Fig. 14, is

$$\begin{aligned} B'_{\Phi_{\mathscr{S}_{\leqslant}(A',A)}} = \{ & \langle x_1, [0,0]\rangle, \langle x_2, [0,0]\rangle, \langle x_3, [0.06, 0.26]\rangle, \langle x_4, [0.06, 0.26]\rangle, \\ & \langle x_5, [0.06, 0.26]\rangle, \langle x_6, [0.00, 0.26]\rangle, \langle x_7, [0,0]\rangle, \langle x_8, [0,0]\rangle, \\ & \langle x_9, [0,0]\rangle, \langle x_{10}, [0,0]\rangle, \langle x_{11}, [0,0]\rangle, \langle x_{12}, [0,0]\rangle, \langle x_{13}, [0,0]\rangle, \\ & \langle x_{14}, [0,0]\rangle\}. \end{aligned}$$

(b) The interval-valued similarity measure between A' and A for (b) in Table 4 (with $\omega = 0.5$) is $\mathscr{S}_{\leqslant}(A', A) = [0.03, 0.14]$, in these conditions the conclusion of the generalized modus ponens and the representation of its membership functions as opposed to those of A' (Fig. 15), are

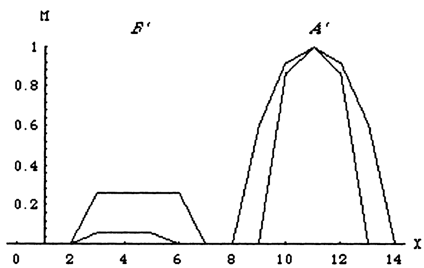


Fig. 14.

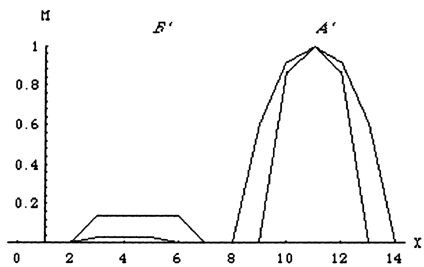


Fig. 15.

$$B'_{\Phi_{\mathscr{S}_{\leqslant}(A',A)}} = \{\langle x_1,[0,0]\rangle,\langle x_2,[0,0]\rangle,\langle x_3,[0.03,0.14]\rangle,\langle x_4,[0.03,0.14]\rangle,\\ \langle x_5,[0.03,0.14]\rangle,\langle x_6,[0,0.14]\rangle,\langle x_7,[0,0]\rangle,\langle x_8,[0,0]\rangle,\\ \langle x_9,[0,0]\rangle,\langle x_{10},[0,0]\rangle,\langle x_{11},[0,0]\rangle,\langle x_{12},[0,0]\rangle,\langle x_{13},[0,0]\rangle,\\ \langle x_{14},[0,0]\rangle\}.$$

(c) For the case (c) in Table 4 with $a = e$ we have $\mathscr{S}_{\leqslant}(A',A) = [0.04, 0.18]$, that is (Fig. 16),

$$B'_{\Phi_{\mathscr{S}_{\leqslant}(A',A)}} = \{\langle x_1,[0,0]\rangle,\langle x_2,[0,0]\rangle,\langle x_3,[0.04,0.18]\rangle,\langle x_4,[0.04,0.18]\rangle,\\ \langle x_5,[0.04,0.18]\rangle,\langle x_6,[0,0.18]\rangle,\langle x_7,[0,0]\rangle,\langle x_8,[0,0]\rangle,\\ \langle x_9,[0,0]\rangle,\langle x_{10},[0,0]\rangle,\langle x_{11},[0,0]\rangle,\langle x_{12},[0,0]\rangle,\langle x_{13},[0,0]\rangle,\\ \langle x_{14},[0,0]\rangle\}.$$

(d) For (d) with $p = 0$ we have: $\mathscr{S}_{\leqslant}(A',A) = [810^{-7}, 1.210^{-3}]$ and thus (Fig. 17)

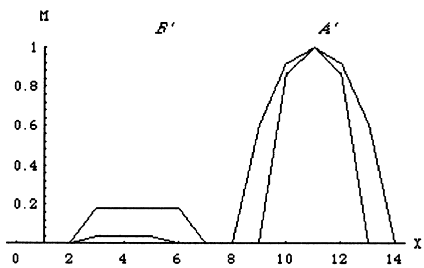


Fig. 16.

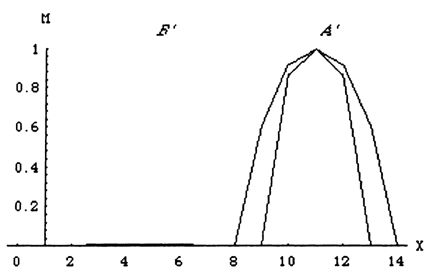


Fig. 17.

$$B'_{\Phi_{\mathscr{S}_{\leqslant}(A',A)}} = \{\langle x_1,[0,0]\rangle,\langle x_2,[0,0]\rangle,\langle x_3,[810^{-7},1.210^{-3}]\rangle,\langle x_4,[810^{-7},1.210^{-3}]\rangle, \langle x_5,[810^{-7},1.210^{-3}]\rangle,\langle x_6,[0.0,1.210^{-3}]\rangle,\langle x_7,[0,0]\rangle,\langle x_8,[0,0]\rangle, \langle x_9,[0,0]\rangle,\langle x_{10},[0,0]\rangle,\langle x_{11},[0,0]\rangle,\langle x_{12},[0,0]\rangle,\langle x_{13},[0,0]\rangle, \langle x_{14},[0,0]\rangle\}.$$

7.3. Method for the multiconditional approximate reasoning

We obtain conclusion B' from the three following steps:
(i) For each $i = \cdots, m$ we calculate the interval-valued similarity measure between A' and A_i, that is, $\mathscr{S}_{\leqslant}(A',A_i)$.
(ii) Next we construct the sets

$$\Phi^i_{\mathscr{S}_{\leqslant}}(A',A_i) = \{\langle y, M_{\Phi^i_{\mathscr{S}_{\leqslant}}(A',A_i)}(y) = [\mathscr{S}_{\leqslant}L(A',A_i), \mathscr{S}_{\leqslant}U(A',A_i)] > | y \in Y\},$$

with $i = 1,\ldots,m$.
(iii) We generate B' in the following way:

$$B'_{\mathscr{S}_{\leqslant}}(A',A_i) = \bigvee_{i=1}^{m}\Big(\wedge(\Phi^i_{\mathscr{S}_{\leqslant}}(A',A_i),B_i)\Big).$$

Note that the inclusion grade indicators of A' with each of sets A_i, $i = 1,\ldots,m$, are determined consecutively.

7.4. Characteristics of this algorithm

Proposition 7. *In the conditions of the algorithm above,*

- ($MAR\mathscr{S}_{\leqslant}1$) $B'_{\mathscr{S}_{\leqslant}}(A',A_i) \leqslant \bigvee_{i=1}^{n} B_i$;
- ($MAR\mathscr{S}_{\leqslant}2$) $\mathscr{S}_{\leqslant}\Big(B'_{\mathscr{S}_{\leqslant}}(A',A_i),\bigvee_{i=1}^{n} B_i\Big) = \Upsilon_{\leqslant}\Big(\bigvee_{i=1}^{n} B_i, B'_{\mathscr{S}_{\leqslant}}(A',A_i)\Big) \leqslant \bigwedge_{i=1}^{n} \Upsilon_{\leqslant}(B_i, B'_{\mathscr{S}_{\leqslant}}(A',A_i))$;
- ($MAR\mathscr{S}_{\leqslant}3$) *if* $A' = A_j$ *with* $1 \leqslant j \leqslant m$, *then* $B'_{\mathscr{S}_{\leqslant}}(A',A_i) = \bigvee_{(i=1),(i\neq j)}^{m} \Big(B_j, \wedge(\Phi^i_{\mathscr{S}_{\leqslant}}(A',A_i),B_i)\Big)$;
- ($MAR\mathscr{S}_{\leqslant}4$) *if* $A' = A_jc$, *and* A_j *or* A' *are normal interval-valued fuzzy sets, then* $B'_{\mathscr{S}_{\leqslant}}(A',A_i) = \bigvee_{(i=1),(i\neq j)}^{m}\Big(\wedge(\Phi^i_{\mathscr{S}_{\leqslant}}(A',A_i),B_i)\Big)$;
- ($MAR\mathscr{S}_{\leqslant}5$) *if for a value of* $i = 1,\ldots,m$, *for example for* j,

$$\{x|[M_{A'L}(x), M_{A'U}(x)] = [1,1] \textit{ and } [M_{A_jL}(x), M_{A_jU}(x)] = [0,0]\} \neq \emptyset$$

or

$$\{x|[M_{A'L}(x), M_{A'U}(x)] = [0,0] \textit{ and } [M_{A_jL}(x), M_{A_jU}(x)] = [1,1]\} \neq \emptyset,$$

then $B'_{\mathscr{S}_\leqslant}(A', A_i) = \bigvee_{(i=1),(i\neq j)}^{m} \left(\wedge(\Phi^i_{\mathscr{S}_\leqslant}(A', A_i), B_i)\right)$;

- $(MAR\mathscr{S}_\leqslant 6)$ *fixed j verifies*

$$\mathscr{S}_{\leqslant L}(B'_{\mathscr{S}_\leqslant}(A', A_i), B_j) \geqslant \wedge \left\{ \bigwedge_{i=1} \left(\vee (\Upsilon_{\leqslant L}(\Phi^i_{\mathscr{S}_\leqslant}(A', A_i), B_j), \Upsilon_{\leqslant L}(B_i, B_j)\right), \right.$$
$$\left. \bigwedge_{i=1} \left(\wedge (\Upsilon_{\leqslant L}(B_j, \Phi^i_{\mathscr{S}_\leqslant}(A', A_i)), \Upsilon_{\leqslant L}(B_j, B_i)\right) \right\};$$

- $(MAR\mathscr{S}_\leqslant 7)$ $B'_{\mathscr{S}_\leqslant}(A', A_i) \leqslant \bigvee_{i=1}^{m} \Phi^i_{\mathscr{S}_\leqslant}(A', A_i)$;
- $(MAR\mathscr{S}_\leqslant 8)$ *if* $\mathscr{S}_\leqslant(A', A_i) = [0,1]$ *for each* $i = 1, \ldots, m$, *then*

$$B'_{\mathscr{S}_\leqslant}(A', A_i) = \left\{ \left\langle y, M_{B_{\mathscr{S}_\leqslant}'(A',A_i)}(y) = \left[0, \bigvee_{i=1}^{m} M_{B_iU}(y)\right] \right\rangle \middle| y \in Y \right\}$$

holds.

Remark 4. From the properties $(MAR\mathscr{S}_\leqslant 4)$ and $(MAR\mathscr{S}_\leqslant 5)$ we can deduce that the respective set B_j is eliminated from the point (iii) of the algorithm, we only need to recall $\mathscr{S}_\leqslant(A_j, A_j) = [1,1]$.

The property $(MAR\mathscr{S}_\leqslant 6)$ expresses the lower extreme of the interval-valued similarity measure between the conclusion $B'_{\mathscr{S}_\leqslant}$ and any set B_j of the collection of sets B_i, $(i = 1, \ldots, m)$. Note that with regards to $\mathscr{S}_{\leqslant U}(B'_{\mathscr{S}_\leqslant}(A', A_i), B_j)$ we cannot assure anything.

Example 6. In the same conditions as in Example 3 we have that the similarity measures $\mathscr{S}_\leqslant(A', A_i)$ for $g(x) = x$ and $c(x) = 1 - x$ with $i = 1, \ldots, 5$ are given in Table 5.

According to (iii) of the algorithm above the conclusion, whose representation of the membership functions as opposed to those of A' can be found in Fig. 18, is given by

Table 5

$\mathscr{S}_\leqslant(A', A_1)$	[0.00, 0.00]
$\mathscr{S}_\leqslant(A', A_2)$	[0.00, 0.06]
$\mathscr{S}_\leqslant(A', A_3)$	[0.00, 0.00]
$\mathscr{S}_\leqslant(A', A_4)$	[0.00, 0.00]
$\mathscr{S}_\leqslant(A', A_5)$	[0.00, 0.00]

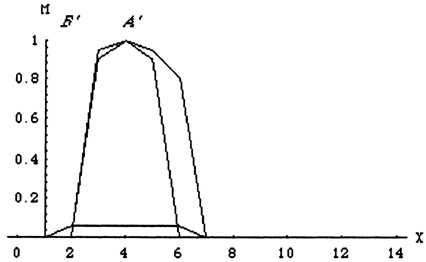


Fig. 18.

$$B'_{\Phi_{\mathscr{S}_\leqslant}(A',A)} = \{\langle x_1,[0,0]\rangle,\langle x_2,[0,0.06]\rangle,\langle x_3,[0,0.06]\rangle,\langle x_4,[0,0.06]\rangle,$$
$$\langle x_5,[0,0.06]\rangle,\langle x_6,[0,0.06]\rangle,\langle x_7,[0,0]\rangle,\langle x_8,[0,0]\rangle,\langle x_9,[0,0]\rangle,$$
$$\langle x_{10},[0,0]\rangle,\langle x_{11},[0,0]\rangle,\langle x_{12},[0,0]\rangle,\langle x_{13},[0,0]\rangle,\langle x_{14},[0,0]\rangle\}.$$

8. Comparison and generalization

In this section we present a study of the differences and similarities between the methods of the Sections 5 and 7. We also compare our results to those obtained by means of a classic method of inference in the literature of interval-valued fuzzy sets, Gorzalczany's [29]. We will represent as B' the conclusion when we talk indistinctly of any of the three methods.

The algorithms in Sections 5, 7, and Gorzalczany's have the same items (i), (ii) and (iii). Only item (i) changes, while in the first we use the interval-valued inclusion grade indicator and in the second the interval-valued similarity measure, Gorzalczany uses the grade of compatibility between *IVFSs*.

Taking into account the interval-valued similarity measure that we present in this paper is obtained from the interval-valued inclusion grade indicator, it easy to prove that the relation existing between the conclusions $B'_{\mathscr{S}_\leqslant}(A',A_i)$ and $B'_{\Upsilon_\leqslant}(A',A_i)$ is given by:

$$B'_{\mathscr{S}_\leqslant}(A',A_i) \leqslant \wedge\left(B'_{\Upsilon_\leqslant}(A',A_i),B'_{\Upsilon_\leqslant}(A_i,A')\right) \leqslant B'_{\Upsilon_\leqslant}(A',A_i),$$

being $B'_{\Upsilon_\leqslant}(A_i,A') = \bigvee_{i=1}^{m}\left(\wedge(\Phi^i_{\Upsilon_\leqslant}(A_i,A),B_i)\right)$.

This inequality makes the conclusion $B'_{\mathscr{S}_\leqslant}(A',A_i)$ be less restrictive than $B'_{\Upsilon_\leqslant}(A',A_i)$ since:

1. If $A' < A$, only when we work with Υ can be assure that the conclusion for the generalized modus ponens is equal to B, (see $(GMP\Upsilon_\leqslant 2)$), fulfilling (F1).
2. If $A' < A_j$, only when we using Υ the corresponding B_j participates completely in the $\bigvee_{i=1}^{m}$ of item (iii).

These two considerations are the ones that characterize fundamentally the algorithm presented with respect to the interval-valued inclusion grade indicator as opposed to, not only the algorithm developed with respect to the interval-valued similarity measure, but Gorzalczany's algorithm as well. It is clear that the choice of one algorithm or another will be imposed by the interests of the *particular* application to be developed in each situation. If we want the conditional: if $A' < A$, then $B' = B$, to be fulfilled, we will choose the first algorithm and otherwise the second.

Other important differences between the tree algorithms are the following:

(a) In the algorithm in Section 5: If $A'_1 < A'_2$, then $B'_{\Upsilon_\leqslant(A'_1,A)} \geqslant B'_{\Upsilon_\leqslant(A'_2,A)}$ holds. However, in the algorithm in Section 7 we cannot say anything about the conclusions if $A'_1 < A'_2$.

(b) If $A' = A_ic$ and it happens that set A_i or set A' is normal (one of the two or both), then by the algorithm developed in accordance with the interval-valued similarity measure the corresponding B_i, is eliminated from the expression presented in item (iii) of said algorithm, (see $(MAR\mathscr{S}_\leqslant 4)$ and Remark 4).

(c) If $A' = A_ic$ and A' is normal, then by means of the algorithm described in accordance with the interval-valued inclusion grade indicator the corresponding B_i is eliminated from the expression presented in item (iii) of said algorithm, (see $(MAR\Upsilon_\leqslant 3)$ and Remark 2).

(d) In Gorzalczany's algorithm we manage to eliminate the corresponding B_i, when $A' \cap A_i = \emptyset$.

From our point of view, a quality of the proposed algorithms as opposed to Gorzalczany's is that since we do not use the compatibility measure defined in [29], our algorithms do not demand at any time that there be at least one element $x \in X$ such that $M_{AL}(x) \neq 0$. That is, Gorzalczany's algorithm is not applicable when A' is the null set.

Among the main similarities between the three algorithms, we will point out the following:

1. In the three algorithms, if in the generalized modus ponens $A' = A$, then $B' = B$. Therefore in the three cases (F1) stands.
2. $B' \leqslant B$ for the GMP and $B' \leqslant \bigvee_{i=1}^m B_i$ for a system of rules with the three algorithms.
3. If A' is identical to some A_i, the respective set B_i enters in its full form into la expression described in item (iii) of the algorithms, (see $(GMP\Upsilon_\leqslant 1)$, $(MAR\Upsilon_\leqslant 3)$, $(GMP\mathscr{S}_\leqslant 1)$ and $(MAR\mathscr{S}_\leqslant 3)$).
4. If $\Upsilon_\leqslant(A', A_i) = [0, 1] = \mathscr{S}_\leqslant(A', A_i)$ for each $i = 1, \ldots, m$, with both algorithms represented in this paper we obtain the same solution, $(MAR\mathscr{S}_\leqslant 8)$ and $(MAR\Upsilon_\leqslant 9)$.
5. For the generalized modus ponens none of the three algorithms maintains the fuzzy linguistic hedge operators as defined by Zadeh [76,77] (small, very small, etc), that is, if for example $A' = A^2$ $(M_{A'L}(x) = M^2_{AL}(x)$ and

$M_{A'U}(x) = M^2_{AU}(x)$), generally we cannot say that $B' = B^2$. This is a great inconvenience of the methodology developed in this paper. Nevertheless, as we said before, with the algorithm in Section 5, (F2)(i) holds.
6. For the generalized modus ponens, none of the three algorithms satisfies (F3).
7. The computational complexity, both in time and memory , of the three algorithms is similar. A precise study of this complexity can be found in [29]. We should point out that the efficiency in time of the algorithm in Section 5 is slightly higher than the efficiency in time of the algorithm presented in Section 7.

It is possible to make a generalization of Gorzalczany's algorithm and of those presented in this paper using any t-norm β and its corresponding t-conorm dual α, instead of 'min' and 'max', so that the conclusion is obtained by the expression:

$B'_{\Upsilon_\leqslant}(A', A_i) = \underset{i=1}{\overset{m}{\alpha}}\left(\beta(\Phi^i_{\Upsilon_\leqslant}(A', A_i), B_i)\right)$ when we use $\Upsilon_\leqslant$, or

$B'_{\mathscr{S}_\leqslant}(A', A_i) = \underset{i=1}{\overset{m}{\alpha}}\left(\beta(\Phi^i_{\mathscr{S}_\leqslant}(A', A_i), B_i)\right)$ when we use $\mathscr{S}_\leqslant$.

However, we have used $\wedge$ and $\vee$ due to the fact that some of the axioms demanded of the interval-valued inclusion grade indicators have been given in accordance with 'min' and 'max'. Nevertheless, a first study of the algorithm in accordance with any B and a can be found in [3,12].

9. Indicators as selectors

In the previous sections we have presented an algorithm that allows us to obtain, from the interval-valued inclusion grade indicator for interval-valued fuzzy sets, the conclusions of the generalized modus ponens as well as a system of rules. This algorithm has been developed as an example of a possible application of said indicator. In this section we propose another application of the interval-valued indicator, which is that of being the element that *selects* one among the methods of obtaining the conclusion of the generalized modus ponens existing in the literature.

9.1. Classic conditional outline for the choice of a method for the GMP

With respect to the algorithms in Sections 5 and 7, we have said they have advantages in the generalized modus ponens, for example:
- if $A' = A$, then $B'_{\Upsilon_\leqslant(A',A)} = B$ and $B'_{\mathscr{S}_\leqslant(A',A)} = B$ (Axiom (F1));
- if $A' < A$, then $B'_{\Upsilon_\leqslant(A',A)} = B$ (Axiom (F1)(i) for the algorithm in Section 5);

and its disadvantages, for example:
- if $\Upsilon_\leqslant(A', A) = [0, 0]$, then $B'_{\Upsilon_\leqslant(A',A)} = \{\langle y, [0, 0]\rangle | y \in Y\}$;

- if $\Upsilon_{\leqslant}(A', A) = [0, 0]$ or $\Upsilon_{\leqslant}(A, A') = [0, 0]$, then $B'_{\mathscr{S}_{\leqslant}(A',A)} = \{\langle y, [0, 0]\rangle | y \in Y\}$.

Evidently, a problem with the algorithms in Sections 5 and 7 is the possibility of obtaining conclusions like:

$$B' = \{\langle y, [0, 0]\rangle \,|\, y \in Y\}.$$

If $B \neq \{\langle y, [0, 0]\rangle \,|\, y \in Y\}$, this type of results are obtained when $\Upsilon_{\leqslant}(A', A) = [0, 0]$, that is, if

$$\{x | M_{A'}(x) = [1, 1] \text{ and } M_A(x) = [0, 0]\} \neq \emptyset$$

for the first algorithm or when $\Upsilon_{\leqslant}(A', A) = [0, 0]$ or $\Upsilon_{\leqslant}(A, A') = [0, 0]$, that is,

$$\{x | (M_{A'}(x) = [1, 1] \text{ and } M_A(x) = [0, 0]) \text{ or } (M_{A'}(x) = [0, 0] \text{ and } M_A(x) = [1, 1])\} \neq \emptyset;$$

for the second algorithm.

Therefore, in the generalized modus ponens we can have $B' = \{\langle y, [0, 0]\rangle | y \in Y\}$ in situations as different as the represented in (a) and (b) of Fig. 19.

Basically, the difference between these two figures is that in (a) $A' \wedge A \neq \{\langle x, [0, 0]\rangle | x \in X\}$, whereas in (b) $A' \wedge A = \{\langle x, [0, 0]\rangle | x \in X\}$.

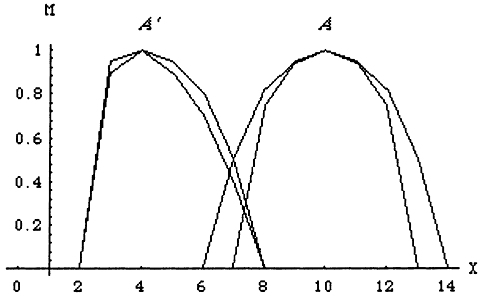
We have said in Section 5 that in this paper we are going to study the generalized modus ponens using *IVFSs* and the ideas of Nafarie [42], Gorzalczany [29], Baldwin [4] etc., which can be summarized in the two following steps:

(1) first relate A to A',

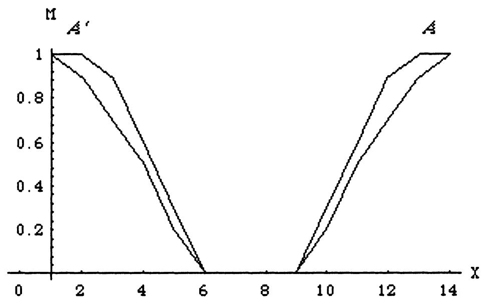
(2) build the consequence B' using the result of the comparison above and B.

We will carry out step (1) using the inclusion grade indicator of the interval-valued fuzzy set A' in A.

Certainly, the solution of GMP: $B' = \{\langle y, [0, 0]\rangle | y \in Y\}$ for (b) in Fig. 19 is a consequence of the methodology developed up to now based on first relating A' and A (step (1)). Evidently this relation when $A' \wedge A = \{\langle x, [0, 0]\rangle | x \in X\}$, should not be given by the interval-valued inclusion grade indicator. Such indicator in situations like (b), gives little information with regards to the relation existing between A and A', moreover, the grade of compatibility defined by Gorzalczany, does not give valid information in these cases, ($\Gamma(A', A) = [0, 0]$). Thus, if $A' \wedge A = \{\langle x, [0, 0]\rangle | x \in X\}$, it is not advisable to use steps (1) and (2). In this situation it is more reasonable to use Zadeh's compositional rule for the GMP.



(a)



(b)

Fig. 19.

For (a) in Fig. 19, $\Upsilon_{\leqslant}(A', A) = [0, 0]$ also holds and therefore, the conclusion of the generalized modus ponens given by the algorithm in Section 5 is $B'_{\Upsilon_{\leqslant}(A',A)} = \{\langle y, [0, 0]\rangle | y \in Y\}$. However, in these conditions $A' \wedge A \neq \{\langle x, [0, 0]\rangle | x \in X\}$, from which we deduce the grade of compatibility defined by Gorzalczany in [29] different from [0,0], ($\Gamma(A', A) \neq [0, 0]$), fact that allows us to assure that the conclusion by his algorithm, (also based on the idea of steps (1) and (2)), is not null.

These considerations, along with the desire to obtain solutions for the generalized modus ponens that maintain the linguistic hedge operators as defined by Zadeh in [76,77], have led us to present the following classic conditional outline, by which we can choose the most advisable method to use in the generalized modus ponens. In this outline the interval-valued inclusion grade indicator is the one that selects between the methods to choose from. To simplify, we employ the following notation: $\Omega_0 \equiv (\Gamma(A', A) = [0, 0])$, $\Omega_1 \equiv (\Upsilon(A', A) = [1, 1])$, $\Omega_2 \equiv (\Upsilon(A, A') = [1, 1])$, $\Omega_3 \equiv (\Upsilon(A', A) = [0, 0])$ and $\Omega_4 \equiv (\Upsilon(A, A') = [0, 0])$,

If Ω_0 **then** $C.M.$

else **if** Ω_1 **then** $(M1)$

else **if** Ω_2 **then** $(M2)$

else if Ω_3 **then** $M.Gorz.$

$$\textbf{else if } \Omega_4 \textbf{ then } \begin{cases} B'_{\Upsilon_{\leq}(A',A)} \\ \text{or} \\ \text{M. Gorz.} \end{cases}$$

$$\textbf{else } \begin{cases} B'_{\Upsilon_{\leq}(A',A)} \\ \text{or} \\ B'_{\mathcal{S}_{\leq}(A',A)} \\ \text{or} \\ \text{M. Gorz.} \end{cases}$$

where C.M. is any method based on Zadeh's compositional rule (adapted to *IVFSs*), (M1) and (M2) are no matter which methods that maintain the linguistic hedges, (it may happen that both methods are the same) and Gorz is Gorzalczany's method.

In the conditional outline above, the interval-valued inclusion grade indicator is in the condition '**If** Condition **then** Action', for this reason we say that it acts as 'selector' between the methods to employ.

We have made the choice of the conditional outline above for the following reasons:

(A) As we said before, if $A' \wedge A = \{\langle x, [0,0]\rangle | x \in X\}$, then it is not reasonable to relate A' and A by means of the interval-valued inclusion grade indicator nor of Gorzalczany's compatibility measure. In these cases, the use of the classical methods based on Zadeh's compositional rule is imposed. Evidently, any method we use must be adapted to *IVFSs* [3,11–13]. The use of $\Gamma(A',A)$ in this step of the conditional above is due to, as we said in the preliminaries, $\Gamma(A',A) = [0,0]$ if and only if $A' \wedge A = \{\langle x, [0,0]\rangle | x \in X\}$.

(B) If $\Upsilon(A',A) = [1,1]$, then $A' \leqslant A$. For example: if $A' = A^n$, with $n = 1,2,3,\ldots$, (evidently $\Upsilon(A',A) = [1,1]$), then the conditional above selects a method (M1) that will allow us to conclude $B' = B^n$. That is, if $n = 2$, then B' is *very B*, if $n = 4$, then B' is *very very B*, etc. In these conditions the linguistic hedge operators are maintained and therefore, axioms (F1) and (F2)(ii) are met. These two axioms were presented by Fukami et al. for fuzzy sets, in this paper we are working *IVFSs*, so it is necessary to make clear that when we say that axiom (F1) is met: If

$A' = A$, then $B' = B$, we are saying that said axiom is met independently by the lower extremes and the upper extremes, that is, if $A' = A$, then $M_{A'L}(x) = M_{AL}(x)$ and $M_{A'U}(x) = M_{AU}(x)$ for all $x \in X$, then the fulfillment of axiom $(F1)$ for *IVFSs* means that $M_{B'L}(y) = M_{BL}(y)$ and $M_{B'U}(y) = M_{BU}(y)$ for all $y \in Y$. As for $F2(ii)$ such axiom means that if $M_{A'L}(x) = M^2_{AL}(x)$ and $M_{A'U}(x) = M^2_{AU}(x)$ for all $x \in X$, then $M_{B'L}(y) = M^2_{BL}(y)$ and $M_{B'U}(y) = M^2_{BU}(y)$ for all $y \in Y$. (Note that according to the conditional outline above, it may happen that $A' \leqslant A$ and $A' \neq A^n$ and also the same method (M1) be used).

(C) If $\Upsilon(A', A) \neq [1,1]$ and $\Upsilon(A, A') = [1,1]$, then $A \leqslant A'$. For example: if $A' = A^n$, with $n \in (0,1)$, then the outline above selects a method (M2) so that $B' = B^n$. Let us point out the case $n = \frac{1}{2}$, that is, when A' is *more or less A*, then by method (M2) we have that B' is *more or less B*, satisfying axiom (F3). Like in the item above the fulfillment of (F3) for *IVFSs* means that if $M_{A'L}(x) = M^{1/2}_{AL}(x)$ and $M_{A'U}(x) = M^{1/2}_{AU}(x)$ for all $x \in X$, then $M_{B'L}(y) = M^{1/2}_{BL}(y)$ and $M_{B'U}(y) = M^{1/2}_{BU}(y)$ for all $y \in Y$.

(D) If $\Upsilon(A', A) \neq [1,1]$ and $\Upsilon(A, A') \neq [1,1]$, then $A \nleqslant A'$ and $A' \nleqslant A$, and if it happens that $\Upsilon(A', A) = [0,0]$, in these conditions we cannot calculate the conclusion by the algorithm in Sections 5 and 7, for in both cases the result is $\{\langle y, [0,0]\rangle | y \in Y\}$. However, if we use Gorzalczany's algorithm based on the same methodology as the one developed in Sections 5 and 7, that is, relating first A' and A, by means of the grade of compatibility between A' and A, then we obtain $B' \neq \{\langle y, [0,0]\rangle | y \in Y\}$. Note that if in these conditions $\Upsilon(A, A') = [0,0]$ holds, then there exist at least $x_1, x_2 \in X$ such that $M_{A'}(x_1) = [1,1], M_A(x_1) = [0,0], M_A(x_2) = [1,1]$ and $M_{A'}(x_2) = [0,0]$ and therefore we find ourselves in situations like the ones represented in (a) in Fig. 19.

(E) If $\Upsilon(A', A) \neq [1,1], \Upsilon(A, A') \neq [1,1], \Upsilon(A', A) \neq [0,0]$ and $\Upsilon(A, A') = [0,0]$, we take the algorithm developed in Section 5 or Gorzalczany's. In these conditions the choice of one over the other will depend on the particular problem we are dealing with.

(F) If $\Upsilon(A', A) \neq [1,1], \Upsilon(A, A') \neq [1,1], \Upsilon(A', A) \neq [0,0]$ and $\Upsilon(A, A') \neq [0,0]$, we take the algorithm developed in V and VII or Gorzalczany's. As before, in these conditions the choice of one over the other will depend our on interests at that particular time.

(G) We must point out that in the classical conditional outline above $B' \leqslant B$ always stands, except perhaps when $A' \wedge A = \{\langle x, [0,0]\rangle | x \in X\}$.

(H) With the conditional outline above, it results that the method chosen for obtaining the conclusion of the generalized modus ponens when working with *IVFSs* is such that on the one hand the lower extremes of the membership intervals and on the other the upper extremes satisfy axioms (F1), (F2) (ii) and F(3). However, we can say nothing about axiom (F4):

$$\text{Either}\begin{cases} \text{(i) If } A' = A_c, \text{ then } B' = Y \text{ (from not } A \text{ ignorance follows)} \\ \text{or} \\ \text{(ii) If } A' = A_c, \text{ then } B' = B_c. \end{cases}$$

From our point of view, for interval-valued fuzzy sets, the conclusion for which we can say that we have 'total lack of information' is that which for each element, its membership interval is [0,1]. In these conditions we do not have knowledge relative to the degree of membership of each element to the set. For this reason we consider that with *IVFSs*, axiom (F4) should not be demanded. Nevertheless we must keep in mind that for fuzzy sets this axiom has caused great controversies, beginning with the choice of (F4)(i) or (F4)(ii), (see Dubois and Prade [21]). On the other hand if sets A and A' are crisp sets and $A' = A_c$, then $A' \wedge A = \{\langle x, [0,0]\rangle | x \in X\}$ and therefore, we obtain the conclusion by C.M.

Fig. 20 represents the conditional outline above, where the interval-valued inclusion grade indicator appears as *selector* of the method to be used.

9.2. A possible method (M1) or (M2)

With regards to methods (M1) and (M2) we have to point out that we can use any of the ones existing in the literature of fuzzy sets that maintain the linguistic terms (small, very small, etc.), for example those presented by Nafarieh and Keller [42] or those studied in [14,45].

Next we present in six items a possible (M1) for interval-valued fuzzy sets (this method can be taken in the conditional outline above as method (M2)):

(i) By means of the method of least squares, approximate the lower extremes of the membership intervals to set A to functions of type $a_1 x^{b_1}$, so that we can write $M_{\tilde{A}L}(x) = a_1 x^{b_1}$.

(ii) To do the same as the item above with the lower extremes of the membership intervals to set A', that is, $M_{\widetilde{A'L}}(x) = a'_1 x^{b'_1}$.

(iii) Take x of $M_{\tilde{A}L} = a_1 x^{b_1}$ and substitute in $M_{\widetilde{A'L}} = a'_1 x^{b'_1}$, that is,

$$M'_{\tilde{A}}L = \frac{a'_1}{a_1^{(b'_1)/(b_1)}} M_{\tilde{A}L}^{(b'_1)/(b_1)}$$

(with $b_1 \neq 0$).

(iv) Build fuzzy set

$$A_1 = \left\{ \left\langle y, M_{A_1}(y) = \min\left(1, \frac{a'_1}{a_1^{(b_1\prime)/(b_1)}} M_{BL}^{(b_1\prime)/(b_1)}(y)\right)\right\rangle \middle| y \in Y \right\}.$$

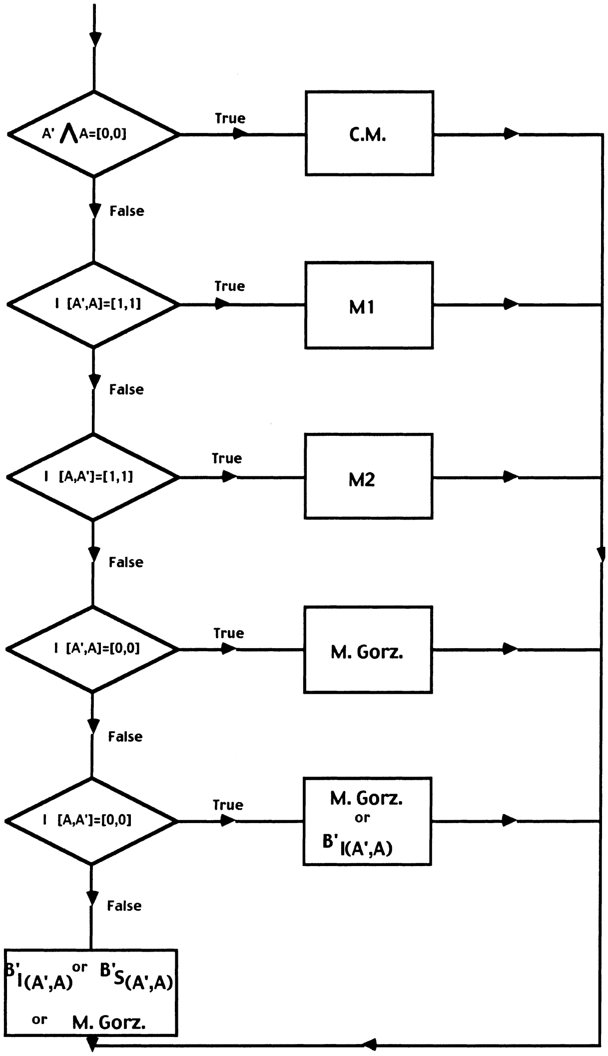


Fig. 20.

(v) Repeat the four steps above for the upper extreme so that:

$$A_2 = \left\{ \left\langle y, M_{A_2}(y) = \min\left(1, \frac{a_2'}{a_2^{(b_2')/(b_2)}} M_{BU}^{(b_2')/(b_2)}(y)\right)\right\rangle \middle| y \in Y \right\}.$$

(vi) Build conclusion B' as follows

$$B' = \{\langle y, M_{B'}(y) = [\min(M_{A_1}(y), M_{A_2}(y)), \max(M_{A_1}(y), M_{A_2}(y))]\rangle | y \in Y\}.$$

Now, for convenience we take the universes of discourse as follows:

(1) $X = \{x_1, \ldots, x_n\}, Y = \{y_1, \ldots, y_m\}$,

(2) $x_i, y_j, \in (0, 1]$ for all $i = 1, \ldots, n, j = 1, \ldots, m$,

(3) $x_i < x_{i+1}$ and $y_j < y_{j+1}$ for all i, j.

We carry out the approximation indicated in the item (i) resolving the following linear algebraic equations:

$$\begin{aligned} &n\mathrm{Ln}\, a_1 + \left(\sum_{i=1}^{n} \ln \quad x_i\right) b_1 = \sum_{i=1}^{n} \mathrm{Ln}\, M_{AL}(x_i), \\ &\left(\sum_{i=1}^{n} \mathrm{Ln}\, x_i\right) \mathrm{Ln}\, a_1 + \left(\sum_{i=1}^{n} \mathrm{Ln}^2\, x_i\right) b_1 = \sum_{i=1}^{n} \mathrm{Ln}\, x_i \mathrm{Ln}\, M_{AL}(x_i) \end{aligned} \tag{7}$$

obtained from taking logarithms in the expression $M_{AL}(x_i) = a_1 x_i^{b_1}$ and applying the method of least squares. *In the construction of Eq.* (7) *we do not take into account elements like* $M_{AL}(x_i) = 0$, *that is, these elements are ignored in order to obtain the systems of linear algebraic Eq.* (7).

Theorem 7. *Let* $p \in \mathbb{R}^+ \cup \{0\}$ *and let* $A \in IVFSs(X)$. *In the conditions above,* (*of the six item method*), *if* $A' = A^p$, *then* $B' = B^p$.

In the conditions of Theorem 7, like $p \geqslant 0$ functions are increasing, then for all $y \in Y$, since $M_{BL}(y) \leqslant M_{BU}(y)$, we have $M_{B'L}(y) = M^p_{BL}(y) \leqslant M^p_{BU}(y) = M_{B'U}(y)$, therefore

$$\begin{aligned} M_{B'L}(y) &= \min(M_{A_1}(y), M_{A_2}(y)) = M^p_{BL}(y), \\ M_{B'U}(y) &= \max(M_{A_1}(y), M_{A_2}(y)) = M^p_{BU}(y). \end{aligned}$$

Example 7. Let us consider the referential sets $X = Y = \{x_1 = 0.1, x_2 = 0.2, \ldots, x_9 = 0.9, x_{10} = 1.0\}$, and let A and B be as follows:

$$\begin{aligned} A = \{ &\langle x_1, [1.0, 1.0]\rangle, \langle x_2, [0.17, 0.8]\rangle, \langle x_3, [0.06, 0.6]\rangle, \langle x_4, [0.0, 0.4]\rangle, \\ &\langle x_5, [0.0, 0.2]\rangle, \langle x_6, [0.0, 0.0]\rangle, \langle x_7, [0.0, 0.0]\rangle, \langle x_8, [0.0, 0.0]\rangle, \\ &\langle x_9, [0.0, 0.0]\rangle, \langle x_{10}, [0.0, 0.0]\rangle\}, \end{aligned}$$

$$\begin{aligned} B = \{ &\langle x_1, [0.0, 0.0]\rangle, \langle x_2, [0.5, 0.55]\rangle, \langle x_3, [0.86, 0.87]\rangle, \langle x_4, [0.90, 1.0]\rangle, \\ &\langle x_5, [0.68, 0.70]\rangle, \langle x_6, [0.50, 0.50]\rangle, \langle x_7, [0.34, 0.40]\rangle, \langle x_8, [0.25, 0.30]\rangle, \\ &\langle x_9, [0.15, 0.20]\rangle, \langle x_{10}, [0.0, 0.0]\rangle\}. \end{aligned}$$

If A' is given by (Fig. 21)

$$\begin{aligned} A' = A^2 = \{ &\langle x_1, [1.0, 1.0]\rangle, \langle x_2, [0.03, 0.64]\rangle, \langle x_3, [0.003, 0.36]\rangle, \langle x_4, [0.0, 0.16]\rangle, \\ &\langle x_5, [0.0, 0.04]\rangle, \langle x_6, [0.0, 0.0]\rangle, \langle x_7, [0.0, 0.0]\rangle, \langle x_8, [0.0, 0.0]\rangle, \\ &\langle x_9, [0.0, 0.0]\rangle, \langle x_{10}, [0.0, 0.0]\rangle\}, \end{aligned}$$

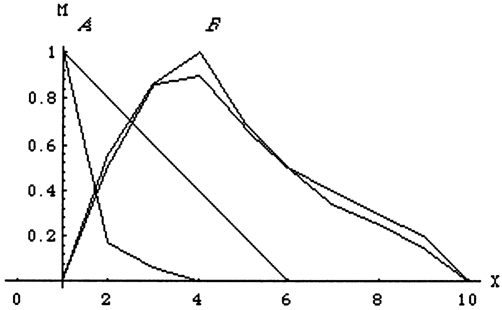


Fig. 21.

it seems evident that the interval-valued inclusion grade indicator of $A' = A^2$ in A is $\Upsilon(A', A) = [1, 1]$, then by the conditional outline above we apply a method (M1) that will maintain the linguistic hedge operators as defined by Zadeh [76,77], for example the six item method developed above.

We carry out the approximation indicated in item (i) resolving the following linear algebraic Eq. (7). For the lower extremes of the intervals of set A above we have $a_1 = 0.003$ and $b_1 = -2.51$.

By item (ii), that is, resolving (7) for the lower extremes of the membership intervals of set A', we have $a_1' = (0.003)^2$ and $b_1' = -5.02$. From item (iii) and item (iv) we build

$$\begin{aligned} A_1 &= \{\langle y, M_{A_1}(y) = 1 \cdot M_{BL}^2(y)\rangle | y \in Y\} \\ &= \{\langle y_1, 0.0\rangle, \langle y_2, 0.25\rangle, \langle y_3, 0.74\rangle, \langle y_4, 0.81\rangle, \langle y_5, 0.46\rangle, \langle y_6, 0.25\rangle, \langle y_7, 0.11\rangle, \\ &\qquad \langle y_8, 0.065\rangle, \langle y_9, 0.02\rangle, \langle y_{10}, 0.0\rangle\}. \end{aligned}$$

We apply item (v) so that we obtain: $a_2 = 0.155111$, $b_2 = -0.900814$, $a_2' = 0.02$ and $b_2' = -1.82$, and therefore

$$\begin{aligned} A_2 &= \{\langle y, M_{A_2}(y) = 1 \cdot M_{BU}^2(y)\rangle | y \in Y\} \\ &= \{\langle y_1, 0.0\rangle, \langle y_2, 0.30\rangle, \langle y_3, 0.76\rangle, \langle y_4, 1.0\rangle, \langle y_5, 0.49\rangle, \langle y_6, 0.25\rangle, \langle y_7, 0.16\rangle, \\ &\qquad \langle y_8, 0.09\rangle, \langle y_9, 0.04\rangle, \langle y_{10}, 0.0\rangle\}, \end{aligned}$$

by item (vi) we have the following conclusion: (Fig. 22)

$$\begin{aligned} B' = B^2 = \{&\langle x = y_1, [0.0, 0.0]\rangle, \langle y_2, [0.25, 0.30]\rangle, \langle y_3, [0.74, 0.76]\rangle, \\ &\langle y_4, [0.81, 1.0]\rangle, \langle y_5, [0.46, 0.49]\rangle, \langle y_6, [0.25, 0.25]\rangle, \langle y_7, [0.11, 0.16]\rangle, \\ &\langle y_8, [0.065, 0.09]\rangle, \langle y_9, [0.02, 0.04]\rangle, \langle y_{10}, [0.0, 0.0]\rangle\}. \end{aligned}$$

The advantages and disadvantages as well as the conditions to be satisfied in order to apply the six item method, (i.e., $a_1 \neq 0$, $b_1 \neq 0, \ldots$), are found studied for fuzzy sets in [14]. It is important to say that the primary and fundamental *disadvantage* of the six item method is the following: *to approximate the*

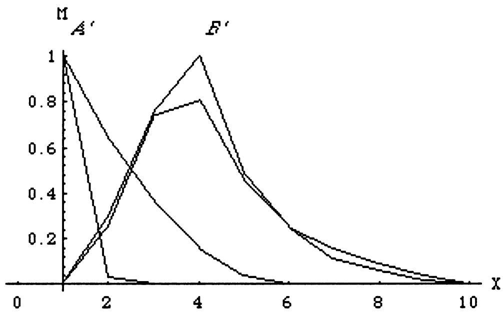


Fig. 22.

membership functions M_L and M_U to ax^b type functions, it should be noticed that ax^b type functions are monotone. If M_L or M_U are not monotone, the approximate between M_L and M_U and ax^b using the method of least squares is unreasonable. In this case, the six item method must be improved, and it will be necessary to use another method (M1), as for example Nafarie and Keller's method [42].

Lastly, we have to indicate that the six item method is one of the many that exist in the literature relative to maintaining the linguistic terms. As always, the use of one method or another will be imposed by the needs in each particular case. For this reason, in the conditional outline above we have pointed out the importance of the interval-valued inclusion grade indicator for *IVFSs* as *selector* of different methods, without specifying in each case which we are to choose, ((M1) and (M2) are no matter which, not necessarily the six item method above). Similar considerations are valid when $\Upsilon(A',A) \neq [1,1], \Upsilon(A',A) \neq [1,1], \Upsilon(A',A) \neq [0,0]$ and $\Upsilon(A,A') \neq [0,0]$ leads us to take conclusions $B'_{\Upsilon_{\leqslant}(A',A)}, B'_{\mathscr{S}_{\leqslant}(A',A)}$ or Gorzalczany's. Evidently, in these conditions these are the only possible methods to use [15]. Therefore, the conclusion can be obtained by methods different from the ones in Sections 5 and 7.

10. Conclusions

The main objectives of this paper have been:

(a) To study and justify the axiom that we think must verify the inclusion grade indicators for interval-valued fuzzy sets.

(b) To present an expression of said indicator.

(c) To show some applications of the indicator.

The desire to justify axioms like the first one and to show an application of the interval-valued inclusion grade indicators has led us to develop a method of inference in approximate reasoning based on *IVFSs* and on the interval-valued inclusion grade indicator for these sets.

Basing ourselves on the works of Sinha and Dougherty for fuzzy sets we have presented a set of nine axioms for the inclusion grade indicator for interval-valued fuzzy sets. The justification of each of them has been made, sometimes while exposing the axioms and at times (like for example demanding that the indicator be an interval), by means of the study of a method of inference in approximate reasoning based on *IVFSs*.

On the other hand, like Sinha, and Dougherty and De Baets and Kerre, we have presented an expression for the interval-valued inclusion grade indicator. The main advantage of this expression is that we can generate interval-valued inclusion grade indicators from the same continuous and strictly increasing functions (that is, continuous and strictly increasing functions $g : [0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$) such that generate involutive fuzzy complementations (see Trillas [61]).

Besides, from inclusion grade indicators we have obtained an expression for the similarity measure for *IVFSs*. Therefore, from the same functions that generate fuzzy complementations we can also build similarity measures for interval-valued fuzzy sets.

As an example of a possible application of the indicators we have presented two algorithms for obtaining the conclusion of the generalized modus ponens and a system of rules. These algorithms have their advantages and disadvantages, it is the later that have led us to use these indicators as elements that allow us *to select* the method to carry out at each time.

We conclude saying that the fact of working with interval-valued fuzzy sets and not fuzzy sets does not lessen generality from the developments made in the paper, but just the opposite, for everything exposed in this paper is valid for fuzzy sets as well by just taking intervals with amplitude 0. Having this objective in mind has lead us to impose Axiom 4.

Acknowledgements

I would like to thank P. Bonissone for his suggestions and the effort made in reading this paper.

Appendix A. Proof of key displayed results

Proof of Proposition 1.

(i) $\Rightarrow$) $\Upsilon(A, B) = 1$, then for all $x \in X$, $1 \leqslant 1 - g(\mu_A(x)) + g(\mu_B(x))$, then $g(\mu_A(x)) \leqslant g(\mu_B(x))$, since g is strictly increasing $\mu_A(x) \leqslant \mu_B(x)$ for all $x \in X$.

$\Leftarrow$) If $A \leqslant B$, then $\mu_A(x) \leqslant \mu_B(x)$ for all $x \in X$, since g is strictly increasing we have $g(\mu_A(x)) \leqslant g(\mu_B(x))$ therefore $1 \leqslant 1 - g(\mu_A(x)) + g(\mu_B(x))$ for all $x \in X$, then $\Upsilon(A, B) = 1$.

(ii) $\Rightarrow$) If $\Upsilon(A,B)=0$, then there is at least one $x\in X$ such that $1-g(\mu_A(x))+g(\mu_B(x))=0$, therefore $g(\mu_A(x))=1+g(\mu_B(x))$, then $g(\mu_B(x))=0$ and $g(\mu_A(x))=1$, therefore $\mu_A(x)=1$ and $\mu_B(x)=0$.

$\Leftarrow$) Since there is at least one $x\in X$ such that $\mu_A(x)=1$ and $\mu_B(x)=0$, we have $g(\mu_A(x))=1$ and $g(\mu_B(x))=0$, therefore for that x we have $0=1-g(\mu_A\ (x))+g(\mu_B(x))$, then $\Upsilon(A,B)=0$.

(iii) If $B\leqslant C$, then $\mu_B(x)\leqslant\mu_C(x)$ for all $x\in X$, therefore $g(\mu_B(x))\leqslant g(\mu_C(x))$, then $g(c(\mu_A(x)))+g(\mu_B(x))\leqslant g(c(\mu_A(x)))+g(\mu_C(x))$ for all $x\in X$, then $\Upsilon(A,B)\leqslant\Upsilon(A,C)$.

(iv) If $B\leqslant C$, then $\mu_B(x)\leqslant\mu_C(x)$, therefore $c(\mu_C(x))\leqslant c(\mu_B(x))$, since g is strictly increasing we have $g(c(\mu_C(x)))\leqslant g(c(\mu_B(x)))$, then $g(c(\mu_C(x)))+g(\mu_A\ (x))\leqslant g(c(\mu_B(x)))+g(\mu_A(x))$, for all $x\in X$, therefore $\Upsilon(C,A)\leqslant\Upsilon(B,A)$.

(v)

$$\begin{aligned}\Upsilon(B_c,A_c)&=\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(c(\mu_B(x))))+g(c(\mu_A(x))))\}\\&=\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(\mu_A(x)))+g(\mu_B(x)))\}=\Upsilon(A,B).\end{aligned}$$

Hereinafter in the demonstrations of the following items we will use $\langle\vee,\wedge,c\rangle$ is dual with respect to any fuzzy complement, besides since $g$is strictly increasing, then $g(\wedge(a,b))=\wedge(g(a),g(b))$ and $g(\vee(a,b))=\vee(g(a),g(b))$ being $a,b\in[0,1]$.

(vi) $\Upsilon(B\vee C,A)$

$$\begin{aligned}&=\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(\vee(\mu_B(x),\mu_C(x))))+g(\mu_A(x)))\}\\
&=\operatorname*{Inf}_{x\in X}\{\wedge(1,g(\wedge(c(\mu_B(x)),c(\mu_C(x))))+g(\mu_A(x)))\}\\
&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(\mu_B(x))),g(c(\mu_C(x))))+g(\mu_A(x)))\}\\
&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(\mu_B(x)))+g(\mu_A(x)),g(c(\mu_C(x)))+g(\mu_A(x))))\}\\
&=\operatorname*{Inf}_{x\in X}\{\wedge(\wedge(1,g(c(\mu_B(x)))+g(\mu_A(x))),\wedge(1,g(c(\mu_C(x)))\\
&\qquad+g(\mu_A(x))))\},\ \text{by Theorem 0}\\
&=\wedge\Big\{\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(\mu_B(x)))+g(\mu_A(x)))\},\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(\mu_C(x)))\\
&\qquad+g(\mu_A(x)))\}\Big\}\\
&=\wedge(\Upsilon(B,A),\Upsilon(C,A)).\end{aligned}$$

(vii) Similar to the one above.

(viii) $\Upsilon(A, B \vee C)$

$$= \operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(\mu_A(x))) + g(\vee(\mu_B(x), \mu_C(x))))\}$$

$$= \operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(\mu_A(x))) + \vee(g(\mu_B(x)), g(\mu_C(x))))\}$$

$$= \operatorname*{Inf}_{x\in X}\{\wedge(1, \vee(g(c(\mu_A(x))) + g(\mu_B(x)), g(c(\mu_A(x))) + g(\mu_C(x)))))\}$$

$$= \operatorname*{Inf}_{x\in X}\{\vee(\wedge(1, g(c(\mu_A(x))) + g(\mu_B(x))), \wedge(1, g(c(\mu_A(x))) + g(\mu_C(x)))))\},$$

by (1)

$$\geqslant \vee\left\{\operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(\mu_A(x))) + g(\mu_B(x)))\}, \operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(\mu_A(x))) + g(\mu_C(x)))\}\right\}$$

$$= \vee(\Upsilon(A, B), \Upsilon(A, C)).$$

(ix) Similar to the one above. $\square$

Proof of Proposition 2.

(i)
$$\Upsilon\left(\bigvee_{i=1}^{n} A_i, B\right) = \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1, g\left(c\left(\bigvee_{i=1}^{n}\mu_{A_i}(x)\right)\right) + g(\mu_B(x))\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1, g\left(\bigwedge_{i=1}^{n} c(\mu_{A_i}(x))\right) + g(\mu_B(x))\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1, \bigwedge_{i=1}^{n}\big(g(c(\mu_{A_i}(x)))\big) + g(\mu_B(x))\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1, \bigwedge_{i=1}^{n}\big(g(c(\mu_{A_i}(x))) + g(\mu_B(x))\big)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\bigwedge_{i=1}^{n}\big(\wedge\big(1, g(c(\mu_{A_i}(x))) + g(\mu_B(x))\big)\big)\right\}$$

$$= \bigwedge_{i=1}^{n}\left(\operatorname*{Inf}_{x\in X}\{\wedge\big(1, g(c(\mu_{A_i}(x))) + g(\mu_B(x))\big)\}\right)$$

$$= \bigwedge_{i=1}^{n}(\Upsilon(A_i, B)).$$

(ii) Similar to the one carried out in the item above.

$$\text{(iii) } \Upsilon\left(A, \bigvee_{i=1}^{n} B_i\right) = \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1, g(c(\mu_A(x))) + g\left(\bigvee_{i=1}^{n}\mu_{B_i}(x)\right)\right)\right\}$$
$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1, g(c(\mu_A(x))) + \bigvee_{i=1}^{n}(g(\mu_{B_i}(x)))\right)\right\}$$
$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1, \bigvee_{i=1}^{n}(g(c(\mu_A(x))) + g(\mu_{B_i}(x)))\right)\right\}$$
$$= \operatorname*{Inf}_{x\in X}\left\{\bigvee_{i=1}^{n}(\wedge(1, g(c(\mu_A(x))) + g(\mu_{B_i}(x))))\right\}$$
$$\geqslant \bigvee_{i=1}^{n}\left(\operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(\mu_A(x))) + g(\mu_{B_i}(x)))\}\right)$$
$$= \bigvee_{i=1}^{n}(\Upsilon(A, B_i)).$$

(iv) Similar to the one carried out in the item above. □

Proof of Theorem 2.

Axiom 1. Evidently $\Upsilon_{\leqslant_L}(A,B) \leqslant \Upsilon_{\leqslant_U}(A,B)$ for all $A,B \in IVFSs(X)$.

Axiom 2. ⇒) $\Upsilon_{\leqslant}(A,B) = [1,1]$, then $\Upsilon_{\leqslant_L}(A,B) = 1$ and obviously $\Upsilon_{\leqslant_U}(A,B) = 1$, therefore we only need to keep in mind that $\Upsilon_{\leqslant_L}(A,B) = 1$. Then

$$1 \leqslant \wedge(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x))) + g(M_{BU}(x))),$$

therefore $1 \leqslant 1 - g(M_{AL}(x)) + g(M_{BL}(x))$ and $1 \leqslant 1 - g(M_{AU}(x)) + g(M_{BU}(x))$, since g is strictly increasing we have $M_{AL}(x) \leqslant M_{BL}(x)$ and $M_{AU}(x) \leqslant M_{BU}(x)$ for all $x \in X$.

⇐) If $A \leqslant B$, then $M_{AL}(x) \leqslant M_{BL}(x)$ and $M_{AU}(x) \leqslant M_{BU}(x)$, therefore since g is strictly increasing we have $g(M_{AL}(x)) \leqslant g(M_{BL}(x))$ and $g(M_{AU}(x)) \leqslant g(M_{BU}(x))$, that is, $1 \leqslant 1 - g(M_{AL}(x)) + g(M_{BL}(x))$ and $1 \leqslant 1 - g(M_{AU}(x)) + g(M_{BU}(x))$, then $1 \leqslant g(c(M_{AL}(x))) + g(M_{BL}(x))$ and $1 \leqslant g(c(M_{AU}(x))) + g(M_{BU}(x))$, therefore $\Upsilon_{\leqslant_L}(A,B) = 1$ and obviously $\Upsilon_{\leqslant_U}(A,B) = 1$, that is, $\Upsilon(A,B) = [1,1]$.

Axiom 3. ⇒) $\Upsilon_{\leqslant}(A,B) = [0,0]$, then $\Upsilon_{\leqslant_U}(A,B) = 0$, that is, there is at least one $x \in X$ such that

$$\vee(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x))) + g(M_{BU}(x))) = 0,$$

therefore $g(c(M_{AL}(x))) + g(M_{BL}(x)) = 0$ and $g(c(M_{AU}(x))) + g(M_{BU}(x)) = 0$, keeping in mind the conditions imposed on g we have: $g(c(M_{AL}(x))) = 0$, $g(M_{BL}(x)) = 0$, $g(c(M_{AU}(x))) = 0$ and $g(M_{BU}(x)) = 0$, therefore we have that $M_{AL}(x) = 1, M_{BL}(x) = 0, M_{AU}(x) = 1, M_{BU}(x) = 0$. Therefore $[M_{AL}(x), M_{AU}(x)] = [1,1]$ and $[M_{BL}(x), M_{BU}(x)] = [0,0]$.

$\Leftarrow$) Now $\{x|[M_{AL}(x), M_{AU}(x)] = [1,1]$, and $[M_{BL}(x), M_{BU}(x)] = [0,0]\} \neq \emptyset$, then

$$g(c(M_{AL}(x))) + g(M_{BL}(x)) = g(c(1)) + g(0) = g(0) + g(0) = 0,$$

$$g(c(M_{AU}(x))) + g(M_{BU}(x)) = g(c(1)) + g(0) = g(0) + g(0) = 0,$$

therefore $\Upsilon_{\leqslant} L(A,B) = 0$ and $\Upsilon_{\leqslant} U(A,B) = 0$, that is, $\Upsilon(A,B) = [0,0]$.

Axiom 4. If A and B are fuzzy sets, then $M_{AL}(x) = M_{AU}(x)$ and $M_{BL}(x) = M_{BU}(x)$ for all $x \in X$, then $g(c(M_{AL}(x))) + g(M_{BL}(x)) = g(c(M_{AU}(x))) + g(M_{BU}(x))$, therefore

$$\wedge\,(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x))) + g(M_{BU}(x)))$$
$$= \vee(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x))) + g(M_{BU}(x))),$$

then $\Upsilon_{\leqslant_L}(A,B) = \Upsilon_{\leqslant_U}(A,B)$.

Axiom 5. Since $B_c = \{\langle x, [c(M_{BU}(x)), c(M_{BL}(x))]\rangle | x \in X\}$, $A_c = \{\langle x, [c(M_{AU}(x)), c(M_{AL}(x))]\rangle | x \in X\}$ and c is involutive fuzzy complement, we have that

$$\begin{aligned}\Upsilon_{\leqslant_L}(B_c, A_c) &= \operatorname*{Inf}_{x\in X}\{\wedge\,(1, \wedge(g(c(c(M_{BU}(x)))) + g(c(M_{AU}(x))),\\ &\qquad g(c(c(M_{BL}(x)))) + g(c(M_{AL}(x)))))\}\\ &= \operatorname*{Inf}_{x\in X}\{\wedge\,(1, \wedge(g(M_{BU}(x)) + g(c(M_{AU}(x))),\\ &\qquad g(M_{BL}(x)) + g(c(M_{AL}(x)))))\}\\ &= \Upsilon_{\leqslant_L}(A,B),\end{aligned}$$

$$\begin{aligned}\Upsilon_{\leqslant_U}(B_c, A_c) &= \operatorname*{Inf}_{x\in X}\{\wedge\,(1, \vee(g(c(c(M_{BU}(x)))) + g(c(M_{AU}(x))),\\ &\qquad g(c(c(M_{BL}(x)))) + g(c(M_{AL}(x)))))\}\\ &= \operatorname*{Inf}_{x\in X}\{\wedge\,(1, \vee(g(M_{BU}(x)) + g(c(M_{AU}(x))),\\ &\qquad g(M_{BL}(x)) + g(c(M_{AL}(x)))))\}\\ &= \Upsilon_{\leqslant_U}(A,B).\end{aligned}$$

Axiom 6. If $B \leqslant C$, then $M_{BL}(x) \leqslant M_{CL}(x)$ and $M_{BU}(x) \leqslant M_{CU}(x)$ for all $x \in X$, since g is strictly increasing we have $g(M_{BL}(x)) \leqslant g(M_{CL}(x))$ and $g(M_{BU}(x)) \leqslant g(M_{CU}(x))$, therefore

$$\begin{aligned}\Upsilon_{\leqslant_L}(A,B) = \operatorname*{Inf}_{x\in X}\{\wedge\,(1, \wedge(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x)))\\ + g(M_{BU}(x)))))\}\end{aligned}$$

$$\leqslant \operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{AL}(x)))+g(M_{CL}(x)),$$
$$g(c(M_{AU}(x)))+g(M_{CU}(x))))\}$$
$$=\Upsilon_{\leqslant}L(A,C).$$

In a similar way we prove that $\Upsilon_{\leqslant}U(A,B)\leqslant\Upsilon_{\leqslant}U(A,C)$.

Axiom 7. Since $B\leqslant C$, then $M_{BL}(x)\leqslant M_{CL}(x)$ and $M_{BU}(x)\leqslant M_{CU}(x)$, besides $c(M_{BL}(x))\geqslant c(M_{CL}(x))$ and $c(M_{BU}(x))\geqslant c(M_{CU}(x))$ for all $x\in X$. Since g is strictly increasing we have $g(c(M_{BL}(x)))\geqslant g(c(M_{CL}(x)))$ and $g(c(M_{BU}(x)))\geqslant g(c(M_{CU}(x)))$, therefore

$$\begin{aligned}\Upsilon_{\leqslant_L}(C,A)&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{CL}(x)))+g(M_{AL}(x)),g(c(M_{CU}(x)))\\&\qquad+g(M_{AU}(x))))\}\\&\leqslant\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{BL}(x)))g(M_{AL}(x)),\\&\qquad g(c(M_{BU}(x)))+g(M_{AU}(x))))\}\\&=\Upsilon_{\leqslant}L(B,A),\end{aligned}$$

$$\begin{aligned}\Upsilon_{\leqslant_U}(C,A)&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\vee(g(c(M_{CL}(x)))+g(M_{AL}(x)),g(c(M_{CU}(x)))\\&\qquad+g(M_{AU}(x))))\}\\&\leqslant\operatorname*{Inf}_{x\in X}\{\wedge(1,\vee(g(c(M_{BL}(x)))+g(M_{AL}(x)),\\&\qquad g(c(M_{BU}(x)))+g(M_{AU}(x))))\}\\&=\Upsilon_{\leqslant_U}(B,A).\end{aligned}$$

Axiom 8.

$$\begin{aligned}\Upsilon_{\leqslant_L}(A,B\vee C)&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{AL}(x)))+g(\vee(M_{BL}(x),M_{CL}(x))),\\&\qquad g(c(M_{AU}(x)))+g(\vee(M_{BU}(x),M_{CU}(x)))))\}\\&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{AL}(x)))+\vee(g(M_{BL}(x)),g(M_{CL}(x))),\\&\qquad g(c(M_{AU}(x)))+\vee(g(M_{BU}(x)),g(M_{CU}(x)))))\}\\&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(\vee(g(c(M_{AL}(x)))+g(M_{BL}(x)),g(c(M_{AL}(x)))\\&\qquad+g(M_{CL}(x)))),\vee g(c(M_{AU}(x)))+(M_{BU}(x)),(c(M_{AU}(x)))\\&\qquad+g(M_{CU}(x))))))\}.\end{aligned}$$

By (1) we have that

$$\geqslant\operatorname*{Inf}_{x\in X}\{\wedge(1,\vee(\wedge(g(c(M_{AL}(x)))+g(M_{BL}(x)),g(c(M_{AU}(x)))+g(M_{BU}(x)))),$$

$$\wedge(g(c(M_{AL}(x))) + g(M_{CL}(x)), g(c(M_{AU}(x))) + g(M_{CU}(x))))))\}$$

$$= \operatorname*{Inf}_{x\in X}\{ \vee \{ \wedge (1, \wedge(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x)))$$

$$\geqslant \vee \Big(\operatorname*{Inf}_{x\in X}\{ \wedge (1, \wedge(g(c(M_{AL}(x))) + g(M_{BL}(x)),$$

$$g(c(M_{AU}(x))) + g(M_{BU}(x))))\}, \operatorname*{Inf}_{x\in X}\{ \wedge (1, \wedge(g(c(M_{AL}(x)))$$

$$+g(M_{CL}(x)), g(c(M_{AU}(x))) + g(M_{CU}(x))))\}\Big)$$

$$= \vee(\Upsilon_{\leqslant}L(A,B), \Upsilon_{\leqslant}L(A,C)).$$

$$\Upsilon_{\leqslant U}(A, B\vee C) = \operatorname*{Inf}_{x\in X}\{ \wedge (1, \vee(g(c(M_{AL}(x))) + g(\vee(M_{BL}(x), M_{CL}(x))),$$

$$g(c(M_{AU}(x))) + g(\vee(M_{BU}(x), M_{CU}(x)))))\}$$

$$= \operatorname*{Inf}_{x\in X}\{ \wedge (1, \vee(g(c(M_{AL}(x))) + \vee(g(M_{BL}(x)), g(M_{CL}(x))),$$

$$g(c(M_{AU}(x))) + \vee(g(M_{BU}(x)), g(M_{CU}(x)))))\}$$

$$= \operatorname*{Inf}_{x\in X}\{ \wedge (1, \vee(\vee (g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AL}(x)))$$

$$+g(M_{CL}(x)))), \vee(g(c(M_{AU}(x))) + g(M_{BU}(x)),$$

$$g(c(M_{AU}(x))) + g(M_{CU}(x))))))\}.$$

By Theorem 0 we have that

$$= \operatorname*{Inf}_{x\in X}\{ \wedge (1, \vee(\vee (g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x)))$$

$$+ g(M_{BU}(x)))), \vee(g(c(M_{AL}(x))) + g(M_{CL}(x)), g(c(M_{AU}(x)))$$

$$+ g(M_{CU}(x))))))\}$$

$$= \operatorname*{Inf}_{x\in X}\{ \vee \{ \wedge (1, \vee(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x)))$$

$$+ g(M_{BU}(x)))) \wedge (1, \vee(g(c(M_{AL}(x))) + g(M_{CL}(x)), g(c(M_{AU}(x)))$$

$$+ g(M_{CU}(x))))\}\}$$

$$\geqslant \vee \Big(\operatorname*{Inf}_{x\in X}\{ \wedge (1, \vee(g(c(M_{AL}(x))) + g(M_{BL}(x)), g(c(M_{AU}(x))) + g(M_{BU}(x))))\},$$

$$\operatorname*{Inf}_{x\in X}\{ \wedge (1, \vee(g(c(M_{AL}(x))) + g(M_{CL}(x)), g(c(M_{AU}(x)))$$

$$+ g(M_{CU}(x))))\}\Big)$$

$$= \vee(\Upsilon_{\leqslant U}(A,B), \Upsilon_{\leqslant U}(A,C)).$$

Axiom 9. Similar to the one carried out in Axiom 8. □

Proof of Corollary 1.

$$\text{(i)}\quad \Upsilon_{\leqslant L}\left(\bigvee_{i=1}^{n} A_i, B\right)$$

$$= \operatorname*{Inf}_{x\in X}\left\{ \wedge\left(1, \wedge\left(g\left(c\left(\bigvee_{i=1}^{n} M_{A_iL}(x)\right)\right) + g(M_{BL}(x)), g\left(c\left(\bigvee_{i=1}^{n} M_{A_iU}(x)\right)\right)\right.\right.\right.$$

$$\left.\left.\left. + g(M_{BU}(x))\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{ \wedge\left(1, \wedge\left(g\left(\bigwedge_{i=1}^{n} c(M_{A_iL}(x))\right) + g(M_{BL}(x)), g\left(\bigwedge_{i=1}^{n} c(M_{A_iU}(x))\right)\right.\right.\right.$$

$$\left.\left.\left. + g(M_{BU}(x))\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{ \wedge\left(1, \wedge\left(\bigwedge_{i=1}^{n}(g(c(M_{A_iL}(x)))) + g(M_{BL}(x)), \bigwedge_{i=1}^{n}(g(c(M_{A_iU}(x))))\right.\right.\right.$$

$$\left.\left.\left. + g(M_{BU}(x))\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{ \wedge\left(1, \wedge\left(\bigwedge_{i=1}^{n}(g(c(M_{A_iL}(x))) + g(M_{BL}(x))), \bigwedge_{i=1}^{n}(g(c(M_{A_iU}(x)))\right.\right.\right.$$

$$\left.\left.\left. + g(M_{BU}(x)))\right)\right)\right\}.$$

By Theorem 0 we have

$$= \operatorname*{Inf}_{x\in X}\left\{ \wedge\left(1, \bigwedge_{i=1}^{n}\left(\wedge\left(g(c(M_{A_iL}(x))) + g(M_{BL}(x)), g(c(M_{A_iU}(x)))\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + g(M_{BU}(x))\right)\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{ \bigwedge_{i=1}^{n}\left(\wedge\left(1, \wedge\left(g(c(M_{A_iL}(x))) + g(M_{BL}(x)), g(c(M_{A_iU}(x)))\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + g(M_{BU}(x))\right)\right)\right)\right\}$$

$$= \bigwedge_{i=1}^{n}\left(\operatorname*{Inf}_{x\in X}\left\{\wedge\left(1,\wedge\left(g(c(M_{A_iL}(x)))+g(M_{BL}(x)),g(c(M_{A_iU}(x)))\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\,g(M_{BU}(x))\right)\right)\right\}\right) = \bigwedge_{i=1}^{n}(\Upsilon_{\leqslant L}(A_i,B)).$$

$$\Upsilon_{\leqslant U}\left(\bigvee_{i=1}^{n}A_i,B\right)$$

$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1,\vee\left(g\left(c\left(\bigvee_{i=1}^{n}M_{A_iL}(x)\right)\right)+g(M_{BL}(x)),g\left(c\left(\bigvee_{i=1}^{n}M_{A_iU}(x)\right)\right)\right.\right.\right.$$

$$\left.\left.\left.+\,g(M_{BU}(x))\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1,\vee\left(g\left(\bigwedge_{i=1}^{n}c(M_{A_iL}(x))\right)+g(M_{BL}(x)),g\left(\bigwedge_{i=1}^{n}c(M_{A_iU}(x))\right)\right.\right.\right.$$

$$\left.\left.\left.+\,g(M_{BU}(x))\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1,\vee\left(\bigwedge_{i=1}^{n}(g(c(M_{A_iL}(x))))+g(M_{BL}(x)),\bigwedge_{i=1}^{n}(g(c(M_{A_iU}(x))))\right.\right.\right.$$

$$\left.\left.\left.+\,g(M_{BU}(x))\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1,\vee\left(\bigwedge_{i=1}^{n}(g(c(M_{A_iL}(x)))+g(M_{BL}(x))),\bigwedge_{i=1}^{n}(g(c(M_{A_iU}(x)))\right.\right.\right.$$

$$\left.\left.\left.+\,g(M_{BU}(x)))\right)\right)\right\}.$$

By Theorem 0 we have

$$\leqslant \operatorname*{Inf}_{x\in X}\left\{\wedge\left(1,\bigwedge_{i=1}^{n}\left(\vee\left(g(c(M_{A_iL}(x)))+g(M_{BL}(x)),g(c(M_{A_iU}(x)))\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\,g(M_{BU}(x))\right)\right)\right)\right\}$$

$$= \operatorname*{Inf}_{x\in X}\left\{\bigwedge_{i=1}^{n}\left(\wedge\left(1,\vee\left(g(c(M_{A_iL}(x)))+g(M_{BL}(x)),g(c(M_{A_iU}(x)))\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\,g(M_{BU}(x))\right)\right)\right)\right\}$$

$$= \bigwedge_{i=1}^{n}\left(\operatorname*{Inf}_{x\in X}\left\{\wedge\left(1,\vee\left(g(c(M_{A_iL}(x)))+g(M_{BL}(x)),g(c(M_{A_iU}(x)))\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\,g(M_{BU}(x))\right)\right)\right\}\right)$$

$$= \bigwedge_{i=1}^{n}(\Upsilon_{\leqslant U}(A_i,B)).$$

In a similar way we can prove (ii), (iii) and (iv). □

Proof of Theorem 3. (i) We know that if $A \preceq B$, then $M_{AL}(x) \leqslant M_{BL}(x) \leqslant M_{BU}(x) \leqslant M_{AU}(x)$ for all $x \in X$, besides $g(M_{AL}(x)) \leqslant g(M_{BL}(x)) \leqslant g(M_{BU}(x)) \leqslant g(M_{AU}(x))$, then $g(c(M_{AL}(x))) + g(M_{BL}(x)) = 1 -\ g(M_{AL}(x)) + g(M_{BL}(x)) \geqslant 1$ and $g(c(M_{AU}(x))) + g(M_{BU}(x)) = 1 - g(M_{AU}(x))\ + g(M_{BU}(x)) \leqslant 1$, in these conditions:

$$\begin{aligned}\Upsilon_{\leqslant_L}(A,B) &= \operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{AL}(x)))+g(M_{BL}(x)),g(c(M_{AU}(x)))\\&\qquad +g(M_{BU}(x))))\}\\&= \operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(1-g(M_{AL}(x))+g(M_{BL}(x)),1-g(M_{AU}(x))\\&\qquad +g(M_{BU}(x))))\}\\&= \operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(M_{AU}(x)))+g(M_{BU}(x)))\}.\end{aligned}$$

In a similar way we prove that $\Upsilon_{\leqslant_U}(A,B) = 1$.

(ii) We only need to keep in mind the definitions of $\Upsilon_{\leqslant_L}(A,B)$ and $\Upsilon_{\leqslant_U}(A,B)$, that $M_{AL}(x) \leqslant M_{AU}(x) \leqslant M_{BL}(x) \leqslant M_{BU}(x)$ for all $x \in X$ and that g is strictly increasing. □

Proof of Theorem 4. Analogous to the proof of Theorem 2. □

Proof of Corollary 2. (i) Since $\Upsilon_{\leqslant U}(A,B) = 1$ we have that for all $x \in X$ is verified that

$$\vee(g(c(M_{AL}(x)))+g(M_{BL}(x)),g(c(M_{AU}(x)))+g(M_{BU}(x))) \geqslant 1.$$

Besides $\Upsilon_{\leqslant L}(A,B)=0$, therefore there is at least one $x\in X$ such that

$$\wedge(g(c(M_{AL}(x)))+g(M_{BL}(x)),g(c(M_{AU}(x)))+g(M_{BU}(x)))=0,$$

for that $x\in X$ two things can happen

(1) $g(c(M_{AL}(x)))+g(M_{BL}(x))=0=1-g(M_{AL}(x))+g(M_{BL}(x))$, then $M_{AL}(x)=1$ (therefore $M_{AU}(x)=1$) and $M_{BL}(x)=0$. Since also $\vee(g(c(M_{AL}(x)))+g(M_{BL}(x)),g(c(M_{AU}(x)))+g(M_{BU}(x)))\geqslant 1$, then we have $g(c(M_{AU}(x)))+g(M_{BU}(x))=1-g(M_{AU}(x))+g(M_{BU}(x))=1-g(1)+g(M_{BU}(x))=g(M_{BU}(x))\geqslant 1$, then $M_{BU}(x)=1$, therefore in this case $M_A(x)=[1,1]$ and $M_B(x)=[0,1]$.

(2) $g(c(M_{AU}(x)))+g(M_{BU}(x))=0=1-g(M_{AU}(x))+g(M_{BU}(x))$, then $M_{AU}(x)=1$ and $M_{BU}(x)=0$ (therefore $M_{BL}(x)=0$). Since also

$$\vee(g(c(M_{AL}(x)))+g(M_{BL}(x)),g(c(M_{AU}(x)))+g(M_{BU}(x)))\geqslant 1,$$

then we have $g(c(M_{AL}(x)))+g(M_{BL}(x))=1-g(M_{AL}(x))+g(M_{BL}(x))=1-g(M_{AL}(x))+g(0)\geqslant 1$, then $M_{AL}(x)=0$, therefore in this case $M_A(x)=[0,1]$ and $M_B(x)=[0,0]$.

(ii) Similar to the one above.

(iii)

$$\begin{aligned}\Upsilon_{\leqslant L}(A,A_c)&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{AL}(x)))+g(c(M_{AU}(x))),g(c(M_{AU}(x)))\\&\qquad+g(c(M_{AL}(x)))))\}\\&=\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(M_{AL}(x)))+g(c(M_{AU}(x))))\},\end{aligned}$$

$$\begin{aligned}\Upsilon_{\leqslant U}(A,A_c)&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\vee(g(c(M_{AL}(x)))+g(c(M_{AU}(x))),g(c(M_{AU}(x)))\\&\qquad+g(c(M_{AL}(x)))))\}\\&=\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(M_{AL}(x)))+g(c(M_{AU}(x))))\}=\Upsilon_{\leqslant L}(A,A_c).\end{aligned}$$

(iv) We only need to recall Axiom 3 and the item above.

(v) Similar to the demonstration in (ii).

(vi) We only need to recall Axiom 3 and the item above.

(vii) If $B=\{\langle x,M_B(x)=[0,0]\rangle | x\in X\}$, then for each $A\in IVFSs(X)$ we have

$$\begin{aligned}\Upsilon_{\leqslant L}(A,B)&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{AL}(x)))+g(0),g(c(M_{AU}(x)))+g(0)))\}\\&=\operatorname*{Inf}_{x\in X}\{\wedge(1,\wedge(g(c(M_{AL}(x))),g(c(M_{AU}(x)))))\},\end{aligned}$$

since $M_{AL}(x)\leqslant M_{AU}(x)$ for all $x\in X$ and $c(M_{AL}(x))\geqslant c(M_{AU}(x))$, then

$$\Upsilon_{\leqslant L}(A,B)=\operatorname*{Inf}_{x\in X}\{\wedge(1,g(c(M_{AL}(x))))\}=\operatorname*{Inf}_{x\in X}(g(c(M_{AU}(x)))).$$

In a similar way $\Upsilon_{\leqslant U}(A,B)=\operatorname{Inf}_{x\in X}(g(c(M_{AL}(x))))$ is proven. $\square$

Proof of Proposition 3.

- ($GMP\Upsilon_{\leqslant}1$) If $A' = A$, then $\Upsilon_{\leqslant}(A', A) = [1, 1]$, therefore by item (iii) of the algorithm $B'_{\Upsilon_{\leqslant}(A',A)} = B$.
- ($GMP\Upsilon_{\leqslant}2$) If $A' < A$, then $\Upsilon_{\leqslant}(A', A) = [1, 1]$, therefore $\Phi_{\Upsilon_{\leqslant}(A',A)} = \{\langle y, M_{\Phi_{\Upsilon_{\leqslant}(A',A)}}(y) = [1, 1]\rangle | y \in Y\}$, then $B'_{\Upsilon_{\leqslant}(A',A)} = \wedge(\Phi_{\Upsilon_{\leqslant}(A',A)}, B) = B$.
- ($GMP\Upsilon_{\leqslant}3$) Since $B'_{\Upsilon_{\leqslant}(A',A)} = \wedge(\Phi_{\Upsilon_{\leqslant}(A',A)}, B)$, we have that: $B'_{\Upsilon_{\leqslant}(A',A)} \leqslant B$.
- ($GMP\Upsilon_{\leqslant}4$) Since $A'_1 \leqslant A'_2$ by Axiom 7, we have $\Upsilon_{\leqslant}(A'_1, A) \geqslant \Upsilon_{\leqslant}(A'_2, A)$. By items (ii) and (iii) from algorithm we have $B'_{\Upsilon_{\leqslant}(A_1{}',A)} \geqslant B'_{\Upsilon_{\leqslant}(A_2{}',A)}$.
- ($GMP\Upsilon_{\leqslant}5$) If A' is a normal interval-valued fuzzy set, then there is at least one $x \in X$ such that $M_{A'}(x) = [1, 1]$, since $A' = A_c$, then $M_A(x) = [0, 0]$, by Axiom 3 we have $\Upsilon(A', A) = [0, 0]$, therefore by items (ii) and (iii) we have $B'_{\Upsilon_{\leqslant}(A_1{}',A)} = \{\langle y, [0, 0]\rangle | y \in Y\}$.
- ($GMP\Upsilon_{\leqslant}6$) Evident.
- ($GMP\Upsilon_{\leqslant}7$) We only need to keep in mind item (v) of Corollary 2.
- ($GMP\Upsilon_{\leqslant}8$) Consequence of item (i) of Theorem 3. □

Proof of Proposition 4.

- ($MAR\Upsilon_{\leqslant}1$) We only need to recall $\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i) \leqslant B_i$ with $i = 1, \ldots, m$.
- ($MAR\Upsilon_{\leqslant}2$) Consequence of Axiom 2.
- ($MAR\Upsilon_{\leqslant}3$) If $A' = A_jc$ is normal, then there is at least one $x \in X$ such that $M_{A'}(x) = [1, 1]$, then $M_{A_j}(x) = [0, 0]$, then by Axiom 3 we have $\Upsilon_{\leqslant}(A' = A_jc, A_j) = [0, 0]$.
- ($MAR\Upsilon_{\leqslant}4$) Consequence of Axiom 3, due to $\Upsilon_{\leqslant}(A', A_j) = [0, 0]$.
- ($MAR\Upsilon_{\leqslant}5$)

$$
\begin{aligned}
B'_{\Upsilon_{\leqslant}(A',A_i)} &= \bigvee_{i=1}^{m}\Big(\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i)\Big) \leqslant \bigvee_{i=1}^{m}\Big(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}\Big) \\
&= \bigvee_{i=1}^{m}\left(\left\{\left\langle y, M_{\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}}(y) = \Upsilon_{\leqslant}(A', A_i)\right\rangle \middle| y \in Y\right\}\right) \\
&= \left\{\left\langle y, \bigvee_{i=1}^{m} M_{\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}}(y) = \bigvee_{i=1}^{m}\big(\Upsilon_{\leqslant}(A', A_i)\big)\right\rangle \middle| y \in Y\right\} \text{ (by Corollary 1)} \\
&\leqslant \left\{\left\langle y, M_{\Phi^{(\vee_i A_i)}_{\Upsilon_{\leqslant}(A',A_i)}}(y) = \Upsilon_{\leqslant}\left(A', \bigvee_{i=1}^{m} A_i\right)\right\rangle \middle| y \in Y\right\} = \Phi^{(\vee_i A_i)}_{\Upsilon_{\leqslant}(A',A_i)}.
\end{aligned}
$$

- ($MAR\Upsilon_{\leqslant}6$) We only need to recall item (i) of Corollary 1 and item (ii) of the algorithm.

- $(MAR\Upsilon_{\leqslant}7)$

$$B'_{\Upsilon_{\leqslant}(A',A_i)} = \bigvee_{i=1}^{m}\Big(\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i)\Big) \leqslant \wedge\left(\bigvee_{i=1}^{m}\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, \bigvee_{i=1}^{m}B_i\right) \leqslant \bigvee_{i=1}^{m}\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}.$$

- $(MAR\Upsilon_{\leqslant}8)$ In accordance with Axiom 6, if $A_m \leqslant A_{m-1} \leqslant \cdots \leqslant A_2 \leqslant A_1$, then

$$\Upsilon_{\leqslant}(A',A_m) \leqslant \Upsilon_{\leqslant}(A',A_{m-1}) \leqslant \cdots \leqslant \Upsilon_{\leqslant}(A',A_2) \leqslant \Upsilon_{\leqslant}(A',A_1),$$

therefore

$$\Phi^m_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \Phi^{m-1}_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \cdots \leqslant \Phi^2_{\Upsilon_{\leqslant}(A',A_i)} \leqslant \Phi^1_{\Upsilon_{\leqslant}(A',A_i)},$$

then $\bigvee_{i=1}^{m}\Phi^i_{\Upsilon_{\leqslant}(A',A_i)} = \Phi^1_{\Upsilon_{\leqslant}(A',A_i)}$, besides

$$\begin{aligned} B'_{\Upsilon_{\leqslant}(A',A_i)} &= \bigvee_{i=1}^{m}\Big(\wedge(\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, B_i)\Big) \leqslant \wedge\left(\bigvee_{i=1}^{m}\Phi^i_{\Upsilon_{\leqslant}(A',A_i)}, \bigvee_{i=1}^{m}B_i\right) \\ &= \wedge\left(\Phi^1_{\Upsilon_{\leqslant}(A',A_i)}, \bigvee_{i=1}^{m}B_i\right). \end{aligned}$$

- $(MAR\Upsilon_{\leqslant}9)$ Evident. □

Proof of Theorem 5. It is easy to see that this expression satisfies properties (i), (ii) and (iii). But we have to prove (iv) here.

Since $A \leqslant B \leqslant C$, then

$$\begin{aligned} \mathscr{S}(A,B) &= \beta(\Upsilon(A,B),\Upsilon(B,A)) = \beta([1,1],\Upsilon(B,A)) \\ &= [\beta(1,\Upsilon_L(B,A)),\beta(1,\Upsilon_U(B,A))] = [\Upsilon_L(B,A)),\Upsilon_U(B,A))] \\ &= \Upsilon(B,A), \end{aligned}$$

$$\begin{aligned} \mathscr{S}(A,C) &= \beta(\Upsilon(A,C),\Upsilon(C,A)) = \beta([1,1],\Upsilon(C,A)) \\ &= [\beta(1,\Upsilon_L(C,A)),\beta(1,\Upsilon_U(C,A))] = [\Upsilon_L(C,A)),\Upsilon_U(C,A))] \\ &= \Upsilon(C,A) \end{aligned}$$

by Axiom 7 we have $\mathscr{S}(A,C) = \Upsilon(C,A) \leqslant \Upsilon(B,A) = \mathscr{S}(A,B)$.

$$\begin{aligned} \mathscr{S}(B,C) &= \beta(\Upsilon(B,C),\Upsilon(C,B)) = \beta([1,1],\Upsilon(C,B)) \\ &= [\beta(1,\Upsilon_L(C,B)),\beta(1,\Upsilon_U(C,B))] = [\Upsilon_L(C,B)),\Upsilon_U(C,B))] \\ &= \Upsilon(C,B), \end{aligned}$$

$$\begin{aligned} \mathscr{S}(A,C) &= \beta(\Upsilon(A,C),\Upsilon(C,A)) = \beta([1,1],\Upsilon(C,A)) \\ &= [\beta(1,\Upsilon_L(C,A)),\beta(1,\Upsilon_U(C,A))] = [\Upsilon_L(C,A)),\Upsilon_U(C,A))] \\ &= \Upsilon(C,A) \end{aligned}$$

by Axiom 6 we have $\mathscr{S}(B,C) = \Upsilon(C,B) \geqslant \Upsilon(C,A) = \mathscr{S}(A,C)$.

(v) By Axiom 4 it is evident. □

Proof of Proposition 5. We only need to keep in mind Axioms 2 and 3. □

Proof of Theorem 6.

(i) $\Rightarrow$)

$$\begin{aligned}\mathscr{S}_{\leqslant}(A,A_c) &= \wedge(\Upsilon_{\leqslant}(A,A_c),\Upsilon_{\leqslant}(A_c,A))\\ &= [\wedge(\Upsilon_{\leqslant L}(A,A_c),\Upsilon_{\leqslant L}(A_c,A)),\wedge(\Upsilon_{\leqslant U}(A,A_c),\Upsilon_{\leqslant U}(A_c,A))]\\ &= [0,0],\end{aligned}$$

therefore

$$\wedge(\Upsilon_{\leqslant L}(A,A_c),\Upsilon_{\leqslant L}(A_c,A)) = 0$$

and

$$\wedge(\Upsilon_{\leqslant U}(A,A_c),\Upsilon_{\leqslant U}(A_c,A)) = 0,$$

by Corollary 2 we know that

$$\Upsilon_{\leqslant L}(A,A_c) = \Upsilon_{\leqslant U}(A,A_c) = \operatorname*{Inf}_{x\in X}\{\wedge(1, g(c(M_{AL}(x))) + g(c(M_{AU}(x))))\}$$

and

$$\Upsilon_{\leqslant L}(A_c,A) = \Upsilon_{\leqslant U}(A_c,A) = \operatorname*{Inf}_{x\in X}\{\wedge(1, g(M_{AL}(x)) + g(M_{AU}(x)))\},$$

then

$$\wedge(\Upsilon_{\leqslant L}(A,A_c),\Upsilon_{\leqslant L}(A_c,A)) = \wedge(\Upsilon_{\leqslant U}(A,A_c),\Upsilon_{\leqslant U}(A_c,A)) = 0$$

in these conditions two things can happen:

(1) $\Upsilon_{\leqslant U}(A,A_c) = 0 = \mathrm{Inf}_{x\in X}\{\wedge(1, g(c(M_{AL}(x))) + g(c(M_{AU}(x))))\}$, then there is at least one $x \in X$ such that $M_{AL}(x) = 1 = M_{AU}(x)$, therefore A is normal.

(2) $\Upsilon_{\leqslant U}(A_c,A) = 0 = \mathrm{Inf}_{x\in X}\{\wedge(1, g(M_{AL}(x)) + g(M_{AU}(x)))\}$, then there is at least one $x \in X$ such that $M_{AL}(x) = 0 = M_{AU}(x)$, therefore A_c is normal.

$\Leftarrow$) Evident keeping in mind Corollary 2 and that $g : [0,1] \to [0,1]$ is a continuous and strictly increasing function such that $g(0) = 0$ and $g(1) = 1$.

(ii) $\Rightarrow$) $\mathscr{S}_{\leqslant}(A,B) = [\wedge(\Upsilon_{\leqslant L}(A,B),\Upsilon_{\leqslant L}(B,A)),\wedge(\Upsilon_{\leqslant U}(A,B),\Upsilon_{\leqslant U}(B,A))] = [1,1]$, therefore

$$\wedge(\Upsilon_{\leqslant L}(A,B),\Upsilon_{\leqslant L}(B,A)) = 1$$

and

$$\wedge(\Upsilon_{\leqslant U}(A,B),\Upsilon_{\leqslant U}(B,A)) = 1$$

since the maximum value $\Upsilon_{\leqslant L}$ and $\Upsilon_{\leqslant U}$ can take is one, then we have $\Upsilon_{\leqslant L}(A,B) = \Upsilon_{\leqslant L}(B,A) = 1$ and $\Upsilon_{\leqslant U}(A,B) = \Upsilon_{\leqslant U}(B,A) = 1$, therefore $\Upsilon(A,B) = [1,1]$, then by Axiom 2 we have that $A \leqslant B$ and since $\Upsilon(B,A) = [1,1]$, then by Axiom 2 $B \leqslant A$ holds, therefore $A = B$.

$\Leftarrow$) If $A = B$, then by Axiom 2, $\Upsilon_{\leqslant}(A,B) = [1,1]$ and $\Upsilon_{\leqslant}(B,A) = [1,1]$, therefore $\mathscr{S}_{\leqslant}(A,B) = \wedge(\Upsilon(A,B), \Upsilon(B,A)) = [\wedge(1,1), \wedge(1,1)] = [1,1]$.

(iii) We only need to recall item (ii) of Theorem 3.

(iv) Similar to the one carried out in item (i) of Corollary 2. □

Proof of Proposition 6.

Similar to the one carried out in Proposition 3 keeping in mind the definition and properties of the interval-valued similarity measure. □

Proof of Proposition 7. We only need to recall Axiom 1, Corollary 1 and Axiom 9. □

Proof of Theorem 7. Let $A', A \in IVFSs(X)$ we will represent as $M_{\tilde{A}L}(x_i) = a_1 x_i^{b_1}$ and $M_{\tilde{A'}L}(x_i) = a_1' x_i^{b_1'}$ the approximations obtained of the lower extremes by the method of least squares when we approximate to functions of the type ax^b. From (7) we deduce that:

$$\begin{aligned} n\mathrm{Ln}\, a_1' + \left(\sum_{i=1}^{n} \mathrm{Ln}\, x_i\right) b_1' &= \sum_{i=1}^{n} \mathrm{Ln}\, M_{A'L}(x_i) = \sum_{i=1}^{n} \mathrm{Ln}\, M_{AL}^{p}(x_i) \\ &= p \sum_{i=1}^{n} \mathrm{Ln}\, M_{AL}(x_i) \\ &= pn\mathrm{Ln}\, a_1 + p\left(\sum_{i=1}^{n} \mathrm{Ln}\, x_i\right) b_1, \end{aligned} \tag{8}$$

$$\begin{aligned} &\left(\sum_{i=1}^{n} \mathrm{Ln}\, x_i\right) \mathrm{Ln}\, a_1' + \left(\sum_{i=1}^{n} \mathrm{Ln}^2\, x_i\right) b_1' = \sum_{i=1}^{n} \mathrm{Ln}\, x_i \mathrm{Ln}\, M_{A'L}(x_i) \\ &= \sum_{i=1}^{n} \mathrm{Ln}\, x_i \mathrm{Ln}\, M_{AL}^{p}(x_i) = p \sum_{i=1}^{n} \mathrm{Ln}\, x_i \mathrm{Ln}\, M_{AL}(x_i) \\ &= p\left(\sum_{i=1}^{n} \mathrm{Ln}\, x_i\right) \mathrm{Ln}\, a_1 + p\left(\sum_{i=1}^{n} \mathrm{Ln}^2\, x_i\right) b_1. \end{aligned} \tag{9}$$

Solving in (8) we have

$$\mathrm{Ln}\, \frac{a_1'}{a_1^p} = \frac{(\sum_{i=1}^{n} \mathrm{Ln}\, x_i)(p \cdot b_1 - b_1')}{n},$$

from (9) we have

$$\left(\sum_{i=1}^{n} \operatorname{Ln} x_i\right) \operatorname{Ln} \frac{a_1'}{a_1^p} = \left(\sum_{i=1}^{n} \operatorname{Ln}^2 x_i\right)(p \cdot b_1 - b_1'),$$

substituting Ln (a'/a^p) in this expression and taking into account that

$$\left(\sum_{i=1}^{n} \operatorname{Ln} x_i\right)^2 - n\left(\sum_{i=1}^{n} \operatorname{Ln}^2 x_i\right) \neq 0,$$

we have $b_1' = p \cdot b_1$ and $a_1' = a_1^p$, therefore

$$\frac{b_1'}{b_1} = p \quad \text{and} \quad \frac{a_1'}{a_1^{(b_1')/(b_1)}} = \frac{a_1'}{a_1^p} = \frac{a_1^p}{a_1^p} = 1.$$

The upper extremes are proven in a similar way. □

References

[1] C. Alsina, E. Trillas, L. Valverde, On some logical connectives for fuzzy sets theory, J. Math. Anal. Appl. 93 (1983) 15–26.

[2] T. Arnould, S. Tano, Interval valued fuzzy backward reasoning, IEEE Trans. Fuzzy Syst. 3 (4) (1995) 425–437.

[3] K. Atanassov, H. Bustince, P. Burillo, V. Mohedano, A method for inference in approximate reasoning for the one-dimensional case based on normal intuitionistic fuzzy sets, in: IFSA'95, vol. I, Sao Paulo, Brazil, 1995, pp. 149–152.

[4] J.F. Baldwin, N.C.F. Guild, Feasible algorithms for approximate reasoning using fuzzy logic, Fuzzy Sets Syst. 3 (1980) 225–251.

[5] J.F. Baldwin, B.W. Pilsworth, Axiomatic approach to implication for approximate reasoning with fuzzy logic, Fuzzy Sets Syst. 3 (1980) 193–219.

[6] W. Bandler, L. Kohout, Fuzzy power sets and fuzzy implication operators, Fuzzy Sets Syst. 4 (1980) 13–30.

[7] R. Bellman, M. Giertz, On the analytic formalism of the theory of fuzzy sets, Information Sci. 5 (1973) 149–156.

[8] Z. Bien, M.G. Chun, An inference network for bidirectional approximate reasoning based on a equality measure, IEEE Trans. Fuzzy Syst. 2 (1995) 177–180.

[9] G. Birkhoff, Lattice Theory, Third ed., vol XXV, American Mathematical Society Colloquium Publications, Providence, RI, 1967.

[10] P. Burillo, H. Bustince, Entropy on intuitionistic fuzzy sets and on interval valued fuzzy sets, Fuzzy Sets Syst. 78 (1996) 305–316.

[11] P. Burillo, H. Bustince, Intuitionistic fuzzy relations Part I, Mathware Soft Comput. 2 (1995) 5–38.

[12] H. Bustince, Conjuntos intuicionistas e intervalo valorados difusos propiedades y construcción, Relaciones Intuicionistas Fuzzy, Thesis, Universidad Pública de Navarra, 1994.

[13] H. Bustince, P. Burillo, Interval valued fuzzy relations in a set structures, J. Fuzzy Math. 4 (1996) 765–785.

[14] H. Bustince, M. Calderon, V. Mohedano, About a least squares model for fuzzy rules of inference (I), in: SC'96, International Workshop on Soft Computing, Kazan, Russia, 1996, pp. 59–65.

[15] S.M. Chen, W.H. Hsiao, W.T. Jong, Bidirectional approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets Syst. 91 (1997) 339–353.
[16] B. De Baets, Ordering alternatives in MCDM problems, in: D. Ruan, P. D'hondt, P. Govaerts, E. Kerre, (Eds.), Fuzzy Logic and Intelligent Technologies in Nuclear Science, Proceedings of the First International Workshop on Fuzzy Logic and Intelligent Technologies in Nuclear Science, World Scientific, Singapore, 1994, pp. 48–53.
[17] B. De Baets, E. Kerre, Fuzzy inclusion and the inverse problem, in H.-J. Zimmermann (Ed.), Proceedings of the EUFIT'94, and European Congress on Intelligent Techniques and Soft Computing, vol. 2, Aachen, Germany, ELITE, Aachen, 1994, pp. 940–945.
[18] D. Dubois, H. Prade, A class of fuzzy measures based on triangular norms, Int. J. Gen. Syst. 8 (1982) 43–61.
[19] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[20] D. Dubois, H. Prade, Fuzzy logics and generalized modus ponens revised, Int. J. Cybernet. Syst. 15 (1984) 293–331.
[21] D. Dubois, H. Prade, Fuzzy sets in approximate reasoning. Part 1 Inference with possibility distributions, Fuzzy Sets Syst. 40 (1991) 143–202.
[22] F. Esteva, E. Trillas, X. Domingo, Weak and strong negation function for fuzzy set theory, in: Proceedings of the 11th IEEE International Symposium on Multivalued Logic, Norman, Oklahoma, 1981, pp. 23–27.
[23] Y. Ezawa, A. Kandel, Robust fuzzy inference, Int. J. Intelligent Syst. 6 (1991) 185–197.
[24] J. Fodor, Contrapositive summetry of fuzzy implications, Fuzzy Sets Syst. 69 (1995) 141–156.
[25] S. Fukami, M. Mizumoto, K. Tanaka, Some considerations on fuzzy conditional inference, Fuzzy Sets Syst. 4 (1980) 243–273.
[26] L.W. Fung y K.S. Fu, An axiomatic approach to rational decision-making in a fuzzy environment, in: L.A. Zadeh, K.S. Fu, K. Tanaka, M. Shimura (Eds.), Fuzzy Seats and Decision Processes, Academic Press, New York, 1975, pp. 227–256.
[27] J.A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.
[28] J.A. Goguen, The logic of inexact reasoning, Synthese 19 (1969) 325–373.
[29] M.B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued, Fuzzy Sets Fuzzy Sets Syst. 21 (1987) 1–17.
[30] J.A. Goguen, The logic of inexact concepts, Synthese 19 (1969) 325–373.
[31] M.M. Gupta, J. Qi, Theory of T-norms and fuzzy inference methods, Fuzzy Sets Syst. 40 (1991) 431–450.
[32] M. Higashi, G.J. Klir, On measure of fuzzyness and fuzzy complements, Int. J. Gen. Syst. 8 (3) (1982) 169–180.
[33] S. Jenei, A more efficient method for defining fuzzy connectives, Fuzzy Sets Syst. 90 (1997) 25–35.
[34] A. Kaufmann, Introduction a la Théorie des Sous-Ensembles Flous, vols. I–IV, Masson, 1977.
[35] E.P. Klement, Operations on fuzzy sets and fuzzy numbers related to triangular norms.
[36] G. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications, Prentice-Hall, Upper Saddle River, NJ, 1995.
[37] L.J. Kohout, W. Bandler, Fuzzy interval inference utilizing the checklist paradigm and BK-relational products, in; R.B. Kearfort et al. (Eds.), Applications of Interval Computations, Kluwer Academic Publisheres, Dordrecht, 1996, pp. 291–335.
[38] R. Lowen, On fuzzy complements, Information Sci. 14 (2) (1978) 107–113.
[39] K. Menger, Statistical metrics, in: Proceedings of the N. A. S, vol. 28, 1942.
[40] M. Mizumoto, Extended fuzzy reasoning, in: Gupta, Kandel, Blander and Kiszka, (Eds.), Reasoning in Expert Systems, North-Holland, Amsterdam, 1985, pp. 71–85.
[41] M. Mizumoto, H.-J. Zimmermann, Comparison of fuzzy reasoning methods, Fuzzy Sets Syst. 8 (1982) 253–283.

[42] A. Nafarieh, J.M. Keller, A new approach to inference in approximate reasoning, Fuzzy Sets Syst. 41 (1991) 17–37.
[43] S.V. Ovchinnikov, Structure of fuzzy binary relations, Fuzzy Sets Syst. 6 (2) (1981) 169–195.
[44] S.V. Ovchinnikov, General negations in fuzzy set theory, J. Math. Anal. Appl. 92 (1) (1983) 234–239.
[45] T.D. Pham, S. Valliappan, A least squares model for fuzzy rules of inference, Fuzzy Sets Syst. 64 (1994) 207–212.
[46] C.P. Papis, N.I. Karacapilidis, A comparative assesment of measures of similarity of fuzzy values, Fuzzy Sets Syst. 56 (1993) 171–174.
[47] C. Ponsard, Hiérarchie des places centrales et graphes Φ-Flous, Environ. Planning A 9 (1977) 1233–1252.
[48] Qiang Zuo, I.B. Turksen, Hung T. Nguyen, et al., In expert systems, even if we fix AND/OR operations, a natural answer to a composite query is the interval of possible degrees of belief, Reliable Computing, 1995, Supplement (Extended Abstracts of APIC'95: International Workshop on Applications of Interval Computations, El Paso, TX, Febr. 23–25, 1995) pp. 236–240.
[49] M.K. Roy, R. Biswas, I-v fuzzy relations and Sanchez's approach for medical diagnosis, Fuzzy Sets Syst. 47 (1992) 35–38.
[50] S. Roychowdhury, New triangular operator generators for fuzzy systems, IEEE Trans. Fuzzy Syst. 5 (2) (1997) 189–198.
[51] D. Ruan, E.E. Kerre, Fuzzy implication operators and generalized fuzzy method of cases, Fuzzy Sets Syst. 54 (1993) 23–37.
[52] R. Sambuc, Functions Φ-Flous Application à l'aide au Diagnostic en Pathologie Thyroidienne. Thèse de Doctorat en Médecine, Marseille, 1975.
[53] D. Sinha, E.R. Dougherty, A general axiomatic theory of intrinsically fuzzy mathematical morfologies, IEEE Trans. Fuzzy Syst. 4 (3) (1995) 389–403.
[54] D. Sinha, E.R. Dougherty, Fuzzification of set inclusion: Theory and applications, Fuzzy Sets Syst. 55 (1993) 15–42.
[55] D. Sinha, E.R. Dougherty, Fuzzy mathematical morphology, J. Visual Commun. Image Representation 3 (3) (1992) 286–302.
[56] P. Smets, P. Magrez, Implication in fuzzy logic, Int. J. Approximate Reasoning 1 (1987) 327–347.
[57] M.H. Smith et al., Optimal strategy of switching reasoning methods in fuzzy control, in: H.T. Nguyen, M. Sugeno, R. Tong, R. Yager (Eds.), Chapter 6.
[58] M. Sugeno, Fuzzy measures and fuzzy integrals: A survey, in: M.M. Gupta, G.N. Saridis, B.R. Gaines (Eds.), Fuzzy Automata and Decision Processes, North-Holland, Amsterdam, 1977, pp. 89–102 (Institute of Technology, 1974).
[59] M. Sugeno, T. Terano, A model of learning based on fuzzy information, Kybernetes 6 (3) (1977) 157–166.
[60] E. Trillas, Sobre funciones de negacion en la teoria de conjuntos difusos, Stochastica III-1 (1979) 47–59.
[61] E. Trillas, C. Alsina, J.M. Terricabras, Introducción a la lógica borrosa, Ariel Matemática, 1995.
[62] E. Trillas, L. Valverse, On mode and implication in approximate reasoning, in: M.M. Gupta, A. Kandel, W. Bandler, J.B. Kiszka (Eds.), Approximate Reasoning in Expert Systems, North-Holland, Amsterdam, 1985, pp. 157–166.
[63] I.B. Turksen, Interval-valued fuzzy sets based on normal forms, Fuzzy Sets Syst. 20 (1986) 191–210.
[64] I.B. Turksen, Z. Zhong, An approximate analogical reasoning schema based on similarity measures and interval-valued fuzzy sets, Fuzzy Sets Syst. 34 (1990) 323–346.
[65] I.B. Turksen, Interval-valued fuzzy sets and compensatory and, Fuzzy Sets Syst. 51 (1992) 295–307.

[66] I.B. Turksen, Interval-valued strict preference with Zadeh triples, Fuzzy Sets Syst. 78 (1996) 183–195.
[67] E. Tsiporkova-Hristoskova, B. De Baets, E. Kerre, A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mapping, Fuzzy Sets Syst. 85 (1997) 93–108.
[68] P.Z. Wang , Theory of Fuzzy Sets and Their Applications, Shanghai Science and Technology Publishing House, 1982.
[69] S. Weber, A general concept of fuzzy connectives, negations and implications based on t-norms and t-conorms, Fuzzy Sets Syst. 11 (1983) 115–134.
[70] W.-J. Wang, New similarity on fuzzy sets and on elements, Fuzzy Sets Syst. 85 (1997) 305–309.
[71] L. Xuecheng, Entropy, distance measure and similarity measure of fuzzy sets and their relations, Fuzzy Sets Syst. 52 (1992) 305–318.
[72] R.R. Yager, On the measure of fuzziness and negation Part I: Membership in the unit interval, Int. J. Gen. Syst. 5 (1979) 221–229.
[73] R.R. Yager, On the measure of fuzziness and negation Part II lattices, Information Control 44 (3) (1979) 236–260.
[74] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes interval-valued fuzzy sets, IEEE Trans. Syst. Man Cybernet. 3 (1973) 28–44.
[75] L.A. Zadeh, Theory of approximate reasoning, in: J.E. Hayes, D. Michie, L.I. Mikulich (Eds.), Machine Intelligence, Ellis Horwood, Chichester, 1970, pp. 149–194.
[76] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Part II, Information Sci. 8 (1975) 301–357.
[77] L.A. Zadeh, A fuzzy-set theoretic interpretation of linguistic hedges, J. Cybernet. 2 (1972) 4–34.