Maximum a Posteriori Tree Augmented Naive Bayes Classifiers

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Abstract

Bayesian classifiers such as Naive Bayes or Tree Augmented Naive Bayes (TAN) have shown excellent performance given their simplicity and heavy underlying independence assumptions. In this paper we prove that under suitable conditions it is possible to calculate efficiently the maximum a posterior TAN model. Furthermore, we prove that it is also possible to calculate a weighted set with the k maximum a posteriori TAN models. This allows efficient TAN ensemble learning and accounting for model uncertainty. These results can be used to construct two classifiers. Both classifiers have the advantage of allowing the introduction of prior knowledge about structure or parameters into the learning process. Empirical results show that both classifiers lead to an improvement in error rate and accuracy of the predicted class probabilities over established TAN based classifiers with equivalent complexity.

Keywords: Bayesian networks, Bayesian network classifiers, Naive Bayes, decomposable distributions, Bayesian model averaging.

1 Introduction

Bayesian classifiers as *Naive Bayes* [11] or *Tree Augmented Naive Bayes* (TAN) [7] have shown excellent performance in spite of their simplicity and heavy underlying independence assumptions.

In our opinion, the TAN classifier, as presented in [7], has two weak points: not taking into account model uncertainty and lacking a theoretically well founded explanation for the use of softening of the induced model parameters (see section 2.2).

In [3] an alternative classifier based on empirical local Bayesian model averaging was proposed as a possible improvement for the first weak point. Furthermore, in [4] the fact that decomposable distributions over TANs allow the tractable calculation of the model averaging integral was used to construct SSTB-MATAN, a classifier that takes into account model uncertainty in a theoretically well founded way and that provides improved classification accuracy.

In [3] an alternative softening is proposed with a theoretically more appealing derivation based on multinomial sampling.

In this paper both weak points are addressed. A computationally more efficient alternative to the first weak point is introduced and a well founded softening alternative is proposed that solves the second weak point. More concretely, we show that under the assumption of decomposable distributions over TANs, we can efficiently compute the TAN model with a maximum a posteriori (MAP) probability. This result allows the construction of MAPTAN, a classifier that provides a well founded alternative to the softening proposed in [7] and improves its error rate and the accuracy of the predicted class probabilities. Furthermore, we will also prove that under this assumption we can efficiently compute the k most probable TAN models and their relative probabilities. This result allows the construction of MAPTAN+BMA, a classifier that takes into consideration model uncertainty to some extent and improves in time complexity and accuracy over its equivalent presented in [3]. Furthermore, established TAN classifiers do not easily allow the introduction of prior knowledge into the learning process. Being able to compute MAP TAN structures means that we can easily do that, whenever our prior knowledge can be represented as a decomposable distribution over TANs.

These results point out the relevance of decomposable distribution over TANs, which are conjugate to TAN models, for the construction of classifiers based on the TAN model.

The paper is structured as follows. In section 2 Tree Augmented Naive Bayes is presented and the notation to be used in the rest of the paper is introduced. In section 3 we present decomposable distributions over TANs. In section 4 we give the main results for finding MAP TAN structures. In section 5 we construct MAPTAN and MAPTAN+BMA, using the previously stated results. In section 6 we provide the empirical results showing that our classifiers improve over established TAN classifiers. We end up with some conclusions and future work in section 7.

2 Tree Augmented Naive Bayes

Tree Augmented Naive Bayes (TAN) appears as a natural extension to the *Naive Bayes* classifier [10, 11, 6]. TAN models are a restricted family of Bayesian networks in which the class variable has no parents and each attribute has as parents the class variable and at most one other attribute. An example of TAN model can be seen in Figure 1(c).

In this section we start introducing the notation to be used in the rest of the paper. After that we discuss the TAN induction algorithm presented in [7].

2.1 Formalization and Notation

The notation used in the paper is an effort to put together the different notations used in [3, 8, 7, 13] and some conventions in the machine learning literature.

2.1.1 The Discrete Classification Problem

A *discrete attribute* is a finite set, for example we can define attribute $Pressure$ as $Pressure = \{Low, Medium, High\}$. A *discrete domain* is a finite set of discrete attributes. We will note $\Omega = \{X_1, \ldots, X_m\}$ for a discrete domain, where $X_1, \ldots, X_m$ are the attributes in the domain. A *classified discrete domain* is a discrete domain where one of the attributes is distinguished as "class". We will use $\Omega_C = \{A_1, \ldots, A_n, C\}$ for a classified discrete domain. In the rest of the paper we will refer to an attribute either as X_i (when it is considered part of a discrete domain), A_i (when it is considered part of a classified discrete domain and it is not the class) and C (when it is the class of a classified discrete domain). We will note as $V = \{A_1, \ldots, A_n\}$ the set of attributes in a classified discrete domain that are not the class.

Given an attribute A, we will note $\#A$ as the number of different values of A. We define $\#\Omega = \prod_{i=1}^{m} \#X_i$ and $\#\Omega_C = \#C \prod_{i=1}^{n} \#A_i$.

An *observation* x in a classified discrete domain Ω_C is an ordered tuple $x = (x_1, \ldots, x_n, x_C) \in A_1 \times \ldots \times A_n \times C$. An *unclassified observation* S in Ω_C is an ordered tuple $S = (s_1, \ldots, s_n) \in A_1 \times \ldots \times A_n$. To be homogeneous we will abuse this notation a bit noting s_C for a possible value of the class for S. A *dataset* $\mathcal{D}$ in Ω_C is a multiset of classified observations in Ω_C.

We will note N for the number of observations in the dataset. We will also note $N_i(x_i)$ for the number of observations in $\mathcal{D}$ where the value for A_i is x_i, $N_{i,j}(x_i, x_j)$ the number of observations in $\mathcal{D}$ where the value for A_i is x_i and the value for A_j is x_j and similarly for $N_{i,j,k}(x_i, x_j, x_k)$ and so on. We note similarly $f_i(x_i), f_{i,j}(x_i, x_j), \ldots$ the frequencies in $\mathcal{D}$. It is worth noticing that f defines a probability distribution over $A_1 \times \ldots \times A_n \times C$.

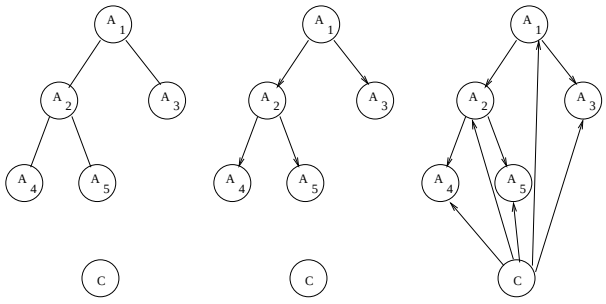
A *classifier* in a classified discrete domain Ω_C is a procedure that given a dataset $\mathcal{D}$ in Ω_C and an unclassified observation S in Ω_C assigns a class to S.

2.1.2 Bayesian Networks for Discrete Classification

Bayesian networks offer a solution for the discrete classification problem. The approach is to define a random variable for each attribute in Ω (the class is included but not distinguished at this time). We will note $\mathbf{U} = \{\mathcal{X}_1, \ldots, \mathcal{X}_m\}$ where each $\mathcal{X}_i$ is a random variable over its corresponding attribute X_i. We extend the meaning of this notation to $\mathcal{A}_i$, $\mathcal{C}$ and $\mathcal{V}$. A *Bayesian network* over $\mathbf{U}$ is a pair $B = \langle G, \Theta \rangle$. The first component, G, is a directed acyclic graph whose vertices correspond to the random variables $\mathcal{X}_1, \ldots, \mathcal{X}_m$ and whose edges represent direct dependencies between the variables. The graph G encodes independence assumptions: each variable $\mathcal{X}_i$ is independent of its non-descendants given its parents in G. The second component of the pair, namely Θ, represents the set of parameters that quantifies the network. It contains a parameter $\theta_{i|\Pi_i}(x_i, \Pi_{x_i}) = P_B(x_i|\Pi_{x_i})$ for each $x_i \in X_i$ and $\Pi_{x_i} \in \Pi_{X_i}$, where Π_{X_i} denotes the Cartesian product of every X_j such that $\mathcal{X}_j$ is a parent of $\mathcal{X}_i$ in G. Π_i is the list of parents of $\mathcal{X}_i$ in G. We will note $\overline{\Pi_i} = \mathbf{U} - \{\mathcal{X}_i\} - \Pi_i$. A Bayesian network defines a unique joint probability distribution over $\mathbf{U}$ given by

$$P_B(x_1, \ldots, x_m) = \prod_{i=1}^{m} P_B(x_i|\Pi_{x_i}) = \prod_{i=1}^{m} \theta_{i|\Pi_i}(x_i|\Pi_{x_i}) \tag{1}$$

The application of Bayesian networks for classification can be very simple. For example suppose we have an algorithm that given a classified discrete domain Ω_C and a dataset $\mathcal{D}$ over Ω_C returns a Bayesian network B over $\mathbf{U} = \{\mathcal{A}_1, \ldots, \mathcal{A}_n, \mathcal{C}\}$ where each $\mathcal{A}_i$ (resp. $\mathcal{C}$) is a random variable over A_i (resp. C). Then if we are given a new unclassified observation S we can easily classify S into class $\underset{s_C \in C}{argmax}(P_B(s_1, \ldots, s_n, s_C))$. This simple mechanism allows us to see any Bayesian network learning algorithm as a classifier.



(a) E (b) $\overline{E}$ (c) $\overline{E}^*$

Figure 1: Notation for learning with trees

2.1.3 Learning with Trees

Given a classified domain Ω_C we will note $\mathcal{E}$ the set of undirected graphs E over $\{\mathcal{A}_1, \ldots, \mathcal{A}_n\}$ such that E is a tree (has no cycles). We will use $u, v \in E$ instead of $(\mathcal{A}_u, \mathcal{A}_v) \in E$ for simplicity. We will note as $\overline{E}$ a directed tree for E. Every $\overline{E}$ uniquely determines the structure of a Tree Augmented Naive Bayes classifier, because from $\overline{E}$ we can construct $\overline{E}^* = \overline{E} \cup \{(\mathcal{C}, \mathcal{A}_i) | 1 \leq i \leq n\}$ as can be seen in an example in Figure 1. We note the root of a directed tree $\overline{E}$ as $\rho_{\overline{E}}$ (i.e. in Figure 1(b) we have that $\rho_{\overline{E}} = A_1$).

We will note as $\Theta_{\overline{E}^*}$ the set of parameters that quantify the Bayesian network $M = \langle \overline{E}^*, \Theta_{\overline{E}^*} \rangle$. More concretely:

$\Theta_{\overline{E}^*} = (\boldsymbol{\theta}_C, \boldsymbol{\theta}_{\rho_{\overline{E}}|C}, \{\boldsymbol{\theta}_{v|u,C} | u, v \in \overline{E}\})$

$\boldsymbol{\theta}_C = \{\theta_C(c) | c \in C\}$ where $\theta_C(c) = P(\mathcal{C} = c|M)$

$\boldsymbol{\theta}_{\rho_{\overline{E}}|C} = \{\theta_{\rho_{\overline{E}}|C}(i, c) | i \in A_{\rho_{\overline{E}}}, c \in C\}$ where
$\theta_{\rho_{\overline{E}}|C}(i, c) = P(\mathcal{A}_{\rho_{\overline{E}}} = i | \mathcal{C} = c, M)$

For each $u, v \in \overline{E}$:
$\boldsymbol{\theta}_{v|u,C} = \{\theta_{v|u,C}(j, i, c) | j \in A_v, i \in A_u, c \in C\}$ where
$\theta_{v|u,C}(j, i, c) = P(\mathcal{A}_v = j | \mathcal{A}_u = i, \mathcal{C} = c, M)$.

2.2 Learning Maximum Likelihood TAN

One of the measures used to learn Bayesian networks is the *log likelihood*:

$$LL(B|\mathcal{D}) = \sum_{x \in \mathcal{D}} \log(P_B(x)) \tag{2}$$

An interesting property of the TAN family is that we have an efficient procedure [7] for identifying the structure of the network which maximizes likelihood. To learn the maximum likelihood TAN we should use the following equation to compute the parameters.

$$\theta_{i|\Pi_i}(x_i, \Pi_{x_i}) = \frac{N_{i,\Pi_i}(x_i, \Pi_{x_i})}{N_{\Pi_i}(\Pi_{x_i})} \tag{3}$$

where $N_{i,\Pi_i}(x_i, \Pi_{x_i})$ stands for the number of times in the dataset that attribute i has value x_i and its parents have values Π_{x_i}. Equivalently, $N_{\Pi_i}(\Pi_{x_i}$ is the number of times in the dataset that the parents of attribute i have values Π_{x_i}.

It has been shown [7] that equation 3 leads to "overfitting" the model. Also in [7] Friedman et al. propose to use the parameters as given by

$$\theta_{i|\Pi_i}(x_i, \Pi_{x_i}) = \frac{N_{i,\Pi_i}(x_i, \Pi_{x_i})}{N_{\Pi_i}(\Pi_{x_i}) + N^0_{i|\Pi_i}} + \\ + \frac{N^0_{i|\Pi_i}}{N_{\Pi_i}(\Pi_{x_i}) + N^0_{i|\Pi_i}} \frac{N_i(x_i)}{N} \tag{4}$$

and suggest setting $N^0_{i|\Pi_i} = 5$ based on empirical results. Using equation 4 to fix the parameters improves the accuracy of the classifier. In our opinion, no well founded justification is given for the improvement. In the next section we introduce decomposable distribution over TANs, a family of probability distributions over the space of TAN models that allow to derive a well founded softening alternative.

3 Decomposable Distributions over TANs

Decomposable priors were introduced by Meila and Jaakola in [13] where it was demonstrated for tree belief networks that if we assume a decomposable prior, the posterior probability is also decomposable and can be completely determined analytically in polynomial time.

In this section we introduce decomposable distributions over TANs, which are probability distributions in the space $\mathcal{M}$ of TAN models and an adaptation of decomposable priors, as they appear in [13], to the task of learning TAN.

Decomposable distributions are constructed in two steps. In the first step, a distribution over the set of different undirected tree structures is defined. Every directed tree structure is defined to have the same probability as its undirected equivalent. In the second step, a distribution over the set of parameters is defined so that it is also independent on the structure. In the rest of the paper we will assume ξ implies a decomposable distribution over $\mathcal{M}$ with hyperparameters $\boldsymbol{\beta}, \mathbf{N}'$ (these hyperparameters will be explained along the development). Under this assumption, the probability for a model $M = \langle \overline{E}^*, \Theta_{\overline{E}^*} \rangle$ (a TAN with fixed tree structure $\overline{E}^*$ and fixed parameters $\Theta_{\overline{E}^*}$) is determined by:

$$P(M|\xi) = P(\overline{E}^*, \Theta_{\overline{E}^*}|\xi) = P(\overline{E}^*|\xi) P(\Theta_{\overline{E}^*}|\overline{E}^*, \xi) \tag{5}$$

In the following sections we specify the value of $P(\overline{E}^*|\xi)$ (decomposable distribution over structures) and $P(\Theta_{\overline{E}^*}|\overline{E}^*, \xi)$ (decomposable distribution over parameters).

3.1 Decomposable Distribution over TAN Structures

One of the hyperparameters of a decomposable distribution is an $n \times n$ matrix $\boldsymbol{\beta} = (\beta_{u,v})$ such that $\forall u, v : 1 \leq u, v \leq n \ : \beta_{u,v} = \beta_{v,u} \geq 0 \ ; \ \beta_{v,v} = 0$. We can interpret $\beta_{u,v}$ as a measure of how possible is under ξ that the edge $(\mathcal{A}_u, \mathcal{A}_v)$ is contained in the TAN model underlying the data.

Given ξ, the probability of a TAN structure $\overline{E}^*$ is defined as:

$$P(\overline{E}^*|\xi) = \frac{1}{Z_\beta} \prod_{u,v \in E} \beta_{u,v} \tag{6}$$

where Z_β is a normalization constant with value:

$$Z_\beta = \sum_{E \in \mathcal{E}} \prod_{u,v \in E} \beta_{u,v} \tag{7}$$

It is worth noting that $P(\overline{E}^*|\xi)$ depends only on the underlying undirected tree structure E.

3.2 Decomposable Distribution over TAN Parameters

Applying equation 1 to the case of TAN we have that

$$P(\Theta_{\overline{E}^*}|\overline{E}^*,\xi) = P(\boldsymbol{\theta}_C|\overline{E}^*,\xi)P(\boldsymbol{\theta}_{\rho_{\overline{E}}|C}|\overline{E}^*,\xi)\times \\ \times \prod_{u,v\in\overline{E}} P(\boldsymbol{\theta}_{v|u,C}|\overline{E}^*,\xi) \quad (8)$$

A decomposable distribution has a hyperparameter set $\mathbf{N}' = \{N'_{v,u,C}(j,i,c)|1 \leq u \neq v \leq n \ ; \ j \in A_v \ ; \ i \in A_u \ ; \ c \in C\}$ with the constraint that exist $N'_{u,C}(i,c)$, $N'_C(c)$, N' such that for every u,v:

$$N'_{u,C}(i,c) = \sum_{j\in A_v} N'_{v,u,C}(j,i,c) \quad (9)$$

$$N'_C(c) = \sum_{i\in A_u} N'_{u,C}(i,c) \quad (10)$$

$$N' = \sum_{c\in C} N'_C(c) \quad (11)$$

Given ξ, a decomposable probability distribution over parameters with hyperparameter $\mathbf{N}'$ is defined by equation 8 and the following set of Dirichlet distributions:

$$P(\boldsymbol{\theta}_C|\overline{E},\xi) = D(\theta_C(.); N'_C(.)) \quad (12)$$

$$P(\boldsymbol{\theta}_{\rho_{\overline{E}}|C}|\overline{E},\xi) = \prod_{c\in C} D(\theta_{\rho_{\overline{E}}|C}(.,c); N'_{\rho_{\overline{E}},C}(.,c)) \quad (13)$$

$$P(\boldsymbol{\theta}_{v|u,C}|\overline{E},\xi) = \prod_{c\in C}\prod_{i\in A_u} D(\theta_{v|u,C}(.,i,c); N'_{v,u,C}(.,i,c)) \quad (14)$$

If the conditions in equations 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14 hold, we will say that $P(M|\xi)$ follows a decomposable distribution with hyperparameters $\boldsymbol{\beta}, \mathbf{N}'$.

3.3 Learning with Decomposable Distributions

Assume that the data is generated by a TAN model and that $P(M|\xi)$ follows a decomposable distribution with hyperparameters $\boldsymbol{\beta}$, $\mathbf{N}'$. Then, $P(M|\mathcal{D},\xi)$, the posterior probability distribution after observing a dataset $\mathcal{D}$, is a decomposable distribution with parameters $\boldsymbol{\beta^*}$, $\mathbf{N'}^*$ given by:

$$\beta^*_{u,v} = \beta_{u,v} W_{u,v} \quad (15)$$

$$N'^*_{u,v,C}(j,i,c) = N'_{u,v,C}(j,i,c) + N_{u,v,C}(j,i,c) \quad (16)$$

where

$$W_{u,v} = \prod_{c \in C} \prod_{i \in A_u} \frac{\Gamma(N'_{u,C}(i,c))}{\Gamma(N'_{u,C}(i,c) + N_{u,C}(i,c))} \prod_{c \in C} \prod_{j \in A_v} \frac{\Gamma(N'_{v,C}(j,c))}{\Gamma(N'_{v,C}(j,c) + N_{v,C}(j,c))} \prod_{c \in C} \prod_{i \in A_u} \prod_{j \in A_v} \frac{\Gamma(N'_{v,u,C}(j,i,c) + N_{v,u,C}(j,i,c))}{\Gamma(N'_{v,u,C}(j,i,c))} \tag{17}$$

The proof appears in [5].

3.4 Classifying with Decomposable Distributions given an undirected structure

Assume that the data is generated by a TAN model and that $P(M|\xi)$ follows a decomposable distribution with hyperparameters $\boldsymbol{\beta}$, $\mathbf{N}'$. Then, $P(\mathcal{C} = s_C|\mathcal{V} = S, E, \xi)$, the probability of a class s_C given an unclassified instance S and an undirected TAN structure E, fulfills

$$P(\mathcal{C} = s_C|\mathcal{V} = S, E, \xi) \propto h_0^{S,s_C} \prod_{u,v \in E} h_{u,v}^{S,s_C} \tag{18}$$

where

$$h_0^{S,s_C} = \frac{1}{Z_\beta} \frac{1}{N'} \prod_{A_u \in V} N'_{u,C}(s_u, s_C) \tag{19}$$

$$h_{u,v}^{S,s_C} = \frac{N'_{v,u,C}(s_v, s_u, s_C)}{N'_{u,C}(s_u, s_C) N'_{v,C}(s_v, s_C)} \tag{20}$$

The proof appears in [2].

4 Maximum a Posteriori results for decomposable distributions over TANs

In this section we show that if we assume a decomposable distribution over TANs as prior over the set of models, the undirected tree structure underlying the MAP TAN can be found in $\mathcal{O}((N + r^3) \cdot n^2)$ time where $r = \max(\max_{i \in V} \#A_i, \#C)$. Furthermore, we also show that we can find the k undirected tree structures underlying the k MAP TAN models and their relative weights in $\mathcal{O}((N + r^3 + \log(\beta(n^2, n)) + k) \cdot n^2)$, where $\beta(m, n)$ is defined to be $\min\{i | \log^{(i)} n \leq m/n\}$ and $\log^{(i)} x$ denotes the log function iterated i times. For almost all practical considerations the time complexity of finding the k MAP TAN models is equivalent to $\mathcal{O}((N + r^3 + k) \cdot n^2))$.

Both results are supported by the next result, that shows that computing the most probable undirected tree structure under a decomposable distribution over TANs with hyperparameters $\boldsymbol{\beta}, \mathbf{N}'$ can be reduced to calculating the maximum weighted spanning tree (MWST) for the graph with adjacency matrix $\log(\boldsymbol{\beta})$.

4.1 Calculating the most probable undirected tree structure under a decomposable distribution over TANs

From the definition of decomposable distribution, concretely from equation 6, it is easy to see that the most probable undirected tree given a decomposable distribution over TANs with hyperparameters $\boldsymbol{\beta}, \mathbf{N}'$ is given by

$$MPT(\boldsymbol{\beta}, \mathbf{N}') = \underset{E \in \mathcal{E}}{argmax} \prod_{u,v \in E} \beta_{u,v} \tag{21}$$

We can see that $MPT(\boldsymbol{\beta}, \mathbf{N}')$ does not depend on $\mathbf{N}'$. Furthermore, assuming that $\forall u, v \; u \neq v; \; \beta_{u,v} > 0$, we can take the logarithm of the r.h.s. having

$$MPT(\boldsymbol{\beta}, \mathbf{N}') = \underset{E \in \mathcal{E}}{argmax} \sum_{u,v \in E} \log(\beta_{u,v}) \tag{22}$$

Considering the matrix $\log(\boldsymbol{\beta})$ as an adjacency matrix, $MPT(\boldsymbol{\beta}, \mathbf{N}')$ is the MWST for the graph represented by that adjacency matrix. Hence, if we are given a decomposable distribution over TANs with hyperparameter $\boldsymbol{\beta}$, we can find the most probable undirected tree by calculating the logarithm of every element in the matrix and then running any algorithm for finding the MWST. The complexity of the MWST algorithm for a complete graph is $\mathcal{O}(n^2)$ [15].

4.2 Calculating the MAP TAN structure given a prior decomposable distribution over TANs

In section 3.3 we enunciated that if we assume a decomposable prior distribution over TANs with hyperparameters $\boldsymbol{\beta}, \mathbf{N}'$ the posterior distribution after a dataset $\mathcal{D}$ follows a decomposable distribution over TANs with hyperparameters given by equations 15, 16 and 17. Since the posterior is a decomposable distribution over trees, we can apply the former result for finding the most probable undirected tree over it and we get the MAP tree. We can translate this result into algorithm 1, that calculates the MAP undirected tree given a dataset $\mathcal{D}$ and prior hyperparameters $\boldsymbol{\beta}$,$\mathbf{N}'$. Since the computation of MWST is $\mathcal{O}(n^2)$, the time complexity of `MAPTreeStructure` is bounded by `CalcN'Posterior`, which has complexity $\mathcal{O}((N + r^3) \cdot n^2)$.

4.3 Calculating the k MAP TAN structures and their relative probability weights given a prior decomposable distribution over TANs

The problem of computing the k MWST in order is well known and can be solved in $\mathcal{O}((\log(\beta(n^2, n)) + k) \cdot n^2)$ for a complete graph [9]. It is easy to see

```
procedure MAPTANStructure (Dataset 𝒟,Matrix β,CountingSet N′)
    var
        CountingSet N′;
        Matrix lβ*;
    begin
        N′* = CalcN'PosteriorTAN(𝒟,N′);
        lβ* = CalcLogBetaPosteriorTAN(β,N′,N′*);
        return MWST(lβ*);

procedure CalcN'PosteriorTAN (Dataset 𝒟,CountingSet N′)
    var
        CountingSet N′*;
    begin
        foreach attribute u
            foreach attribute v < u
                foreach value x_u ∈ A_u
                    foreach value x_v ∈ A_v
                        foreach value c ∈ C
                            N′*_{u,v,C}(x_u, x_v, c) = N′_{u,v}(x_u, x_v, c);
        foreach attribute x ∈ 𝒟
            foreach attribute u
                foreach attribute v < u
                    N′*_{u,v,C}(x_u, x_v, x_C) = N′*_{u,v,C}(x_u, x_v, x_C) + 1;
        return N′*;

procedure CalcLogBetaPosteriorTAN (Matrix β,CountingSet N′, N′*)
    var
        Matrix lβ*;
    begin
        foreach attribute u
            foreach attribute v < u
                lβ*_{u,v} = log β_{u,v} + CalcLogWTAN(N′,N′*,u,v);
        return lβ*;

procedure CalcLogWTAN (CountingSet N′, N′*, int u, v)
    begin
        w = 0;
        foreach value c ∈ C
            foreach value x_u ∈ A_u
                w = w + logΓ(N′_{u,C}(x_u, c)) - logΓ(N′*_{u,C}(x_u, c));
            foreach value x_v ∈ A_v
                w = w + logΓ(N′_{v,C}(x_v, c)) - logΓ(N′*_{v,C}(x_v, c));
            foreach value x_u ∈ A_u
                foreach value x_v ∈ A_v
                    w = w + logΓ(N′*_{u,v,C}(x_u, x_v, c))
                          - logΓ(N′_{u,v,C}(x_u, x_v, c));
        return w;
```

Algorithm 1: Computation of the MAP TAN

that if in the last step of `MAPTreeStructure` instead of calculating the MWST we calculate the k MWST and their relative weights as shown in algorithm 2, the algorithm will return the k MAP TANs and their relative probabilities. The time complexity of the new algorithm is simply the addition of the complexity of `CalcN'Posterior` with that of computing the k MAP trees and that of computing the weights, giving $\mathcal{O}((N + r^3 + \log(\beta(n^2, n)) + k) \cdot n^2)$ which, as previously mentioned, can be understood as $\mathcal{O}((N + r^3 + k) \cdot n^2)$ for all practical purposes.

5 Constructing the MAPTAN and MAPTAN+BMA classifiers

The previously introduced results allow us to efficiently compute MAP TAN structures. We know from equation 18 that

$$P(\mathcal{C} = s_C|\mathcal{V} = S, E, \xi) \propto h_0^{S,S_C} \prod_{u,v \in \overline{E}} h_{u,v}^{S,S_C} \tag{23}$$

In fact, given an undirected TAN structure E, it is easy to see that the probability distribution $P(\mathcal{C} = s_C|\mathcal{V} = S, E, \xi)$ can be represented as a TAN model with structure $\overline{E}^*$, such that its undirected version coincides with E and its parameter set is given by

$$\begin{aligned} \theta_{u|v,C}(s_u, s_v, s_C) &= \frac{N'_{u,v,C}(s_u,s_v,s_C)}{N'_{v,C}(s_v,s_C)} \\ \theta_{u|C}(s_u, s_C) &= \frac{N'_{u,C}(s_u,s_C)}{N'_C(s_C)} \\ \theta_C(s_C) &= \frac{N'_C(s_C)}{N'} \end{aligned} \tag{24}$$

A similar result in the case of decomposable distribution over trees can also be found in [14]. Given a decomposable prior we can calculate the decomposable posterior using the result in section 3.3 and then apply the result we have just enunciated to the posterior. The posterior probability distribution $P(\mathcal{C} = s_C|\mathcal{V} = S, E, \mathcal{D}, \xi)$ can be represented as a TAN model with structure $\overline{E}^*$, such that its undirected version coincides with E and its parameter set is given by

$$\begin{aligned} \theta_{u|v,C}(s_u, s_v, s_C) &= \frac{N'_{u,v,C}(s_u,s_v,s_C)+N_{u,v,C}(s_u,s_v,s_C)}{N'_{v,C}(s_v,s_C)+N_{v,C}(s_v,s_C)} \\ \theta_{u|C}(s_u, s_C) &= \frac{N'_{u,C}(s_u,s_C)+N_{u,C}(s_u,s_C)}{N'_C(s_C)+N_C(s_C)} \\ \theta_C(s_C) &= \frac{N'_C(s_C)+N_C(s_C)}{N'+N} \end{aligned} \tag{25}$$

Given this result and our previous results for determining the MAP TAN structure and the k MAP TAN structures and their relative probability weights, it is very easy to construct two new classifiers by simple composition. First of all we have to fix a set of prior hyperparameters. In [4] we argued that

$$\forall u, v \ ; \ 1 \leq u \neq v \ \leq n \ ; \ \beta_{u,v} = 1 \tag{26}$$

$$\begin{aligned} &\forall u, v; 1 \leq u \neq v \leq n; \forall j \in A_v; \forall i \in A_u; \forall c \in C \\ &N'_{v,u,C}(j, i, c) = \frac{\lambda}{\#C\#A_u\#A_v} \end{aligned} \tag{27}$$

where λ is an equivalent sample size, provide a reasonable choice of the hyperparameters if no information from the domain is available.

5.1 MAPTAN classifier

After fixing the prior hyperparameters, the learning step for MAPTAN classifier consists in:

1. Applying algorithm 1 to find the undirected tree E underlying the MAP TAN structure given a dataset $\mathcal{D}$.
2. Randomly choose a root, create a directed tree $\overline{E}$ and from it a directed TAN structure $\overline{E}^*$.
3. Use equation 25 to fix the TAN parameters.

For classifying an unclassified observation, we have to apply the TAN that has been learned for each of the $\#C$ classes to construct a probability distribution over the values of the class C and then choose the most probable class.

This classification algorithm runs in $\mathcal{O}((N+r^3)\cdot n^2)$ learning time and $\mathcal{O}(nr)$ classification time.

```
procedure k-MAPTANs (Dataset 𝒟,Matrix β,CountingSet N', int k)
    var
        CountingSet N';
        WeightedTreeSet WTS;
        Matrix lβ*;
    begin
        N'* = CalcN'PosteriorTAN(𝒟,N');
        lβ* = CalcLogBetaPosteriorTAN(β,N',N'*);
        WTS = k-MWST(lβ*,k);
        CalcTreeWeights(WTS,lβ*);
        return WTS;
```

Algorithm 2: Computation of the k MAP TANs

5.2 MAPTAN+BMA classifier

After fixing the prior hyperparameters, the learning stage for MAPTAN+BMA classifier consists in:

1. Applying algorithm 2 to find the k undirected trees underlying the k MAP TAN structures and their relative probability weights given a dataset $\mathcal{D}$.
2. Generate a TAN model for each of the undirected tree structures as we did in MAPTAN.
3. Assign to each TAN model the weight of its corresponding undirected tree.

The resulting probabilistic model will be a mixture of TANs. For classifying an unclassified observation, we have to apply the k TAN models for the $\#C$ classes and calculate the weighted average to construct a probability distribution over the values of the class C and then choose the most probable class.

This classification algorithm runs in $\mathcal{O}((N + r^3 + \log(\beta(n^2, n)) + k) \cdot n^2)$ learning time and $\mathcal{O}(nrk)$ classification time.

5.3 Relevant characteristics of MAPTAN and MAPTAN+BMA

We have shown that decomposable distributions over TANs can be used to construct two well founded classifiers: MAPTAN and MAPTAN+BMA. In the introduction we highlighted two possible ways in which the TAN classifier, as presented in [7], could be improved: by taking into account model uncertainty and by providing a theoretically well founded explanation for the use of softening.

We have seen that MAPTAN+BMA provides a theoretically well founded way of dealing with model uncertainty. Its learning time complexity regarding N almost equivalent to that of STAN, and it grows polynomially on k. This is much more efficient than the algorithm for learning k TAN models proposed in [3]. MAPTAN+BMA has a classification time complexity, $\mathcal{O}(nrk)$ reasonably higher than that of STAN. Furthermore, we can use k as an *effort knob*, in the sense of [16], hence providing a useful feature for data mining users that allows them to decide how much computational power they want to spend in the task. In our opinion, MAPTAN+BMA provides a good complexity tradeoff to deal with model uncertainty when learning TAN.

Both MAPTAN and MAPTAN+BMA can be interpreted as using softening in both the structure search and the parameter fixing. This softening appears, in a natural way, as the result of assuming a decomposable distribution over TANs as the prior over the set of models. In our opinion MAPTAN is theoretically more appealing than STAN.

Both MAPTAN and MAPTAN+BMA, share with SSTBMATAN$^{\text{o}}$ the relevant characteristic of allowing the use of some form of prior information if such is available, specially structure related information. For example, if we have expert knowledge that tell us that one of the edges of the tree is much more (equiv. much less) likely than the others it is very easy to incorporate this knowledge when fixing the prior hyperparameter matrix $\boldsymbol{\beta}$. Evidently, as was pointed out in [12], decomposable distributions do not allow the expression of some types of prior information such as "if edge (u, v) exists then edge (w, z) is very likely to exist".

6 Empirical results

We tested four algorithms over 17 datasets from the Irvine repository [1]. To discretize continuous attributes we used equal frequency discretization with 5 intervals. For each dataset and algorithm we tested both error rate and *LogScore*. *LogScore* is calculated by adding the minus logarithm of the probability assigned by the classifier to the correct class and gives an idea of how well the classifier is estimating probabilities (the smaller the score the better the result).

If we name the test set $\mathcal{D}'$ we have

$$LogScore(M, \mathcal{D}') = \\ = \sum_{(S,s_C)\in\mathcal{D}'} -\log(P(\mathcal{C} = s_C|\mathcal{V} = S, M)) \quad (28)$$

For the evaluation of both error rate and *LogScore* we used 10 fold cross validation. We tested the algorithm with the 10%, 50% and 100% of the learning data for each fold, in order to get an idea of the influence of the amount of data in the behaviors of both error rate and *LogScore* for the algorithm.

The error rates appear in Tables 1, 2 and 3 with the best method for each dataset boldfaced. *LogScore*'s appear in Tables 4, 5 and 6. The columns of the tables are the induction methods and the rows are the datasets. The meaning of the column headers are:

- STAN is the softened TAN induction algorithm as presented in [7].
- STAN+BMA is the classifier resulting from applying local Bayesian model averaging (see [3]) to STAN.
- MAPTAN, is the classifier based on the MAP TAN model described in section 5.
- MAPTAN+BMA is the classifier based on the weighted average of the k MAP TAN models described also in section 5.

6.1 Interpretation of the results

Summarizing the empirical results in the tables, we can conclude that:

- MAPTAN improves STAN error rate for most datasets and has a similar *LogScore*.
- MAPTAN+BMA improves MAPTAN's *LogScore* for most datasets. When little data is available, it also improves its error rate.
- MAPTAN+BMA improves STAN+BMA error rate and *LogScore* for many datasets.

In the rest of the section we discuss and justify these assertions into more detail.

Dataset	MAPTAN	MAPTAN+BMA	sTAN	sTAN+BMA
ADULT	**17.18 ± 0.68**	17.19 ± 0.71	17.60 ± 0.82	17.60 ± 0.80
AUSTRALIAN	19.91 ± 1.14	**19.62 ± 1.13**	25.39 ± 1.18	24.96 ± 1.13
BREAST	17.23 ± 1.21	16.89 ± 1.28	8.73 ± 0.87	**7.73 ± 0.93**
CAR	17.19 ± 1.04	**16.50 ± 0.84**	19.38 ± 0.95	17.60 ± 0.77
CHESS	9.55 ± 0.80	**9.48 ± 0.86**	10.89 ± 0.56	10.91 ± 0.53
CLEVE	**28.12 ± 1.68**	28.14 ± 1.59	32.37 ± 1.00	31.89 ± 1.27
CRX	19.77 ± 0.91	**19.16 ± 1.00**	25.14 ± 0.87	24.18 ± 0.98
FLARE	23.50 ± 1.09	23.16 ± 1.09	19.94 ± 0.85	**19.92 ± 0.88**
GLASS	47.02 ± 1.66	**45.72 ± 1.59**	59.19 ± 1.78	58.54 ± 1.83
GLASS2	33.69 ± 1.74	**32.87 ± 1.82**	37.75 ± 1.39	36.63 ± 1.37
IRIS	28.67 ± 2.33	26.27 ± 2.30	25.87 ± 3.07	**24.80 ± 2.96**
LETTER	30.22 ± 0.96	**30.19 ± 0.97**	36.11 ± 1.39	34.68 ± 1.37
LIVER	45.52 ± 1.26	44.96 ± 1.06	42.39 ± 0.94	**41.24 ± 1.37**
NURSERY	7.87 ± 1.03	**7.57 ± 1.04**	8.88 ± 1.12	8.50 ± 1.12
PRIMARY-TUMOR	74.52 ± 1.73	74.28 ± 1.66	**71.67 ± 1.54**	71.73 ± 1.44
SOYBEAN	26.53 ± 1.30	**26.51 ± 1.33**	30.79 ± 1.28	30.82 ± 1.33
VOTES	**9.61 ± 0.94**	9.67 ± 0.99	14.14 ± 0.93	14.13 ± 0.71

Table 1: Averages and standard deviations of error rate using 10% of the learning data

Dataset	MAPTAN	MAPTAN+BMA	sTAN	sTAN+BMA
ADULT	**16.26 ± 0.75**	16.28 ± 0.77	16.46 ± 0.78	16.45 ± 0.83
AUSTRALIAN	15.36 ± 0.94	**15.13 ± 1.09**	18.14 ± 0.91	17.74 ± 0.80
BREAST	5.92 ± 0.74	5.84 ± 0.78	5.26 ± 0.84	**4.75 ± 0.72**
CAR	7.62 ± 0.75	**7.55 ± 0.76**	8.68 ± 0.68	8.09 ± 0.58
CHESS	**7.87 ± 0.44**	7.90 ± 0.44	8.25 ± 0.49	8.15 ± 0.49
CLEVE	**19.82 ± 1.30**	20.27 ± 1.27	24.01 ± 1.31	23.57 ± 1.28
CRX	15.47 ± 1.01	**15.30 ± 1.02**	18.12 ± 0.92	17.68 ± 0.85
FLARE	19.83 ± 0.72	19.81 ± 0.65	18.55 ± 0.62	**18.54 ± 0.72**
GLASS	24.02 ± 1.22	**23.31 ± 1.48**	33.79 ± 1.14	33.86 ± 0.97
GLASS2	23.69 ± 1.62	22.81 ± 1.61	**22.38 ± 1.53**	23.40 ± 1.54
IRIS	11.60 ± 1.22	11.07 ± 1.08	8.40 ± 1.00	**8.27 ± 0.82**
LETTER	**14.79 ± 0.78**	**14.79 ± 0.78**	15.62 ± 0.91	15.31 ± 0.83
LIVER	37.33 ± 1.16	36.90 ± 1.15	36.73 ± 1.60	**35.17 ± 1.34**
NURSERY	6.39 ± 0.92	6.37 ± 0.89	7.09 ± 0.80	**6.03 ± 0.97**
PRIMARY-TUMOR	59.15 ± 1.67	**59.09 ± 1.63**	60.23 ± 1.17	59.87 ± 1.33
SOYBEAN	6.64 ± 0.77	**6.50 ± 0.85**	7.88 ± 0.71	7.80 ± 0.82
VOTES	**6.22 ± 0.83**	6.25 ± 0.84	7.63 ± 0.93	7.76 ± 0.93

Table 2: Averages and standard deviations of error rate using 50% of the learning data

Dataset	MAPTAN	MAPTAN+BMA	sTAN	sTAN+BMA
ADULT	**16.35 ± 0.73**	16.35 ± 0.73	16.46 ± 0.68	16.42 ± 0.72
AUSTRALIAN	13.68 ± 0.75	**13.65 ± 0.74**	16.49 ± 0.65	16.43 ± 0.72
BREAST	4.75 ± 0.53	4.63 ± 0.48	4.29 ± 0.66	**3.72 ± 0.45**
CAR	**5.76 ± 0.52**	5.78 ± 0.45	6.23 ± 0.55	6.16 ± 0.53
CHESS	7.71 ± 0.25	**7.67 ± 0.21**	7.89 ± 0.38	7.68 ± 0.44
CLEVE	18.74 ± 1.15	**18.53 ± 1.18**	19.99 ± 1.26	19.73 ± 1.18
CRX	13.67 ± 0.53	**13.53 ± 0.58**	15.71 ± 0.66	15.79 ± 0.74
FLARE	19.71 ± 0.49	19.71 ± 0.55	18.46 ± 0.30	**18.31 ± 0.24**
GLASS	**18.46 ± 1.20**	18.74 ± 1.29	26.58 ± 1.22	25.99 ± 1.28
GLASS2	19.81 ± 0.85	20.25 ± 1.39	19.61 ± 1.42	**18.06 ± 1.43**
IRIS	7.73 ± 1.70	7.47 ± 1.66	8.13 ± 1.44	**7.20 ± 1.43**
LETTER	**11.49 ± 0.74**	**11.49 ± 0.74**	12.69 ± 0.77	12.48 ± 0.83
LIVER	34.35 ± 0.86	33.99 ± 0.77	33.36 ± 0.98	**33.19 ± 1.10**
NURSERY	6.33 ± 0.89	6.26 ± 0.91	6.62 ± 0.75	**4.81 ± 0.76**
PRIMARY-TUMOR	55.09 ± 1.24	**54.68 ± 1.02**	56.74 ± 1.09	56.32 ± 0.93
SOYBEAN	5.47 ± 0.62	**5.27 ± 0.62**	5.97 ± 0.50	5.94 ± 0.49
VOTES	**5.89 ± 0.74**	5.89 ± 0.72	6.26 ± 0.81	6.34 ± 0.56

Table 3: Averages and standard deviations of error rate using 100% of the learning data

Dataset	MAPTAN	MAPTAN+BMA	sTAN	sTAN+BMA
ADULT	562.25 ± 3.75	**561.39 ± 3.71**	567.09 ± 3.92	567.64 ± 4.00
AUSTRALIAN	18.54 ± 0.95	17.68 ± 0.96	17.85 ± 0.64	**17.06 ± 0.60**
BREAST	23.59 ± 1.67	18.24 ± 1.56	8.12 ± 0.69	**7.56 ± 0.65**
CAR	34.89 ± 1.02	**32.79 ± 0.98**	38.55 ± 0.91	36.52 ± 0.86
CHESS	32.50 ± 0.89	**32.25 ± 0.91**	35.39 ± 0.58	35.40 ± 0.59
CLEVE	11.15 ± 1.06	10.09 ± 0.96	8.49 ± 0.74	**8.23 ± 0.76**
CRX	19.44 ± 1.06	18.30 ± 1.00	17.84 ± 1.05	**16.89 ± 1.00**
FLARE	51.12 ± 1.17	**49.48 ± 1.15**	24332.38 ± 56.59	24332.03 ± 56.59
GLASS	20.49 ± 1.45	**17.14 ± 1.40**	11713.24 ± 72.91	11713.00 ± 72.91
GLASS2	6.45 ± 0.79	5.49 ± 0.64	4.68 ± 0.57	**4.57 ± 0.54**
IRIS	4.58 ± 0.68	4.06 ± 0.69	4.04 ± 0.67	**3.96 ± 0.70**
LETTER	3535.93 ± 12.92	3495.14 ± 13.52	1385.73 ± 8.95	**1300.23 ± 8.38**
LIVER	18.71 ± 0.95	15.87 ± 0.92	12.62 ± 0.79	**11.71 ± 0.65**
NURSERY	112.72 ± 2.47	**111.95 ± 2.47**	3126.39 ± 77.45	3123.62 ± 77.45
PRIMARY-TUMOR	71.74 ± 2.08	**69.08 ± 2.05**	75927.03 ± 123.39	75926.94 ± 123.39
SOYBEAN	68.52 ± 1.77	**65.29 ± 1.55**	41125.59 ± 108.25	41125.46 ± 108.25
VOTES	5.66 ± 0.66	**5.17 ± 0.60**	6.09 ± 0.50	6.03 ± 0.48

Table 4: Averages and standard deviations of *LogScore* using 10% of the learning data

Dataset	MAPTAN	MAPTAN+BMA	sTAN	sTAN+BMA
ADULT	507.82 ± 3.82	**507.52 ± 3.81**	520.03 ± 3.93	518.82 ± 3.91
AUSTRALIAN	12.86 ± 0.82	**12.57 ± 0.84**	14.79 ± 0.76	14.41 ± 0.59
BREAST	10.95 ± 0.67	9.20 ± 0.69	5.17 ± 0.64	**4.40 ± 0.62**
CAR	15.96 ± 0.44	**15.90 ± 0.40**	20.44 ± 0.51	19.73 ± 0.48
CHESS	26.70 ± 0.66	**26.66 ± 0.68**	27.32 ± 0.73	27.12 ± 0.79
CLEVE	6.83 ± 0.69	**6.70 ± 0.66**	7.38 ± 0.66	7.15 ± 0.63
CRX	13.28 ± 0.89	**12.93 ± 0.85**	15.62 ± 1.11	15.21 ± 1.07
FLARE	39.81 ± 1.16	**39.45 ± 1.15**	4233.42 ± 41.82	4233.31 ± 41.82
GLASS	11.05 ± 0.73	**9.37 ± 0.84**	309.52 ± 24.49	309.25 ± 24.49
GLASS2	5.06 ± 0.73	4.64 ± 0.64	3.86 ± 0.53	**3.68 ± 0.51**
IRIS	1.87 ± 0.35	1.77 ± 0.34	1.52 ± 0.37	**1.48 ± 0.34**
LETTER	1030.65 ± 9.50	1030.65 ± 9.50	574.47 ± 6.13	**559.56 ± 6.17**
LIVER	13.03 ± 0.89	12.21 ± 0.77	10.78 ± 0.74	**10.39 ± 0.71**
NURSERY	96.60 ± 2.40	**96.52 ± 2.42**	1596.96 ± 67.06	1594.32 ± 67.06
PRIMARY-TUMOR	44.24 ± 1.25	**43.00 ± 1.23**	12028.93 ± 51.79	12028.74 ± 51.79
SOYBEAN	6.79 ± 0.83	**6.47 ± 0.74**	907.34 ± 42.43	907.27 ± 42.43
VOTES	3.66 ± 0.58	**3.54 ± 0.52**	5.04 ± 0.80	4.50 ± 0.69

Table 5: Averages and standard deviations of *LogScore* using 50% of the learning data

6.1.1 MAPTAN **vs** STAN

MAPTAN improves STAN error rate in a statistically significant way for most datasets and has a similar *LogScore*. After performing a 5% statistical significance t-test, we have that MAPTAN error rate is significantly better than STAN for 12, 11 and 7 datasets with 10%, 50% and 100% of the learning data respectively, whilst STAN error rate is better than MAPTAN in a statistically significant way only for 4, 2 and 2 datasets. *LogScore* results favor MAPTAN slightly. MAPTAN *LogScore* is significantly better than STAN for 6, 9 and 9 datasets with 10%, 50% and 100% of the learning data respectively, whilst STAN *LogScore* is better than MAPTAN in a statistically significant way for 7, 5 and 6 datasets respectively.

Dataset	MAPTAN	MAPTAN+BMA	sTAN	sTAN+BMA
ADULT	495.88 ± 3.68	**495.70 ± 3.67**	508.10 ± 3.07	508.01 ± 3.07
AUSTRALIAN	10.65 ± 0.46	**10.47 ± 0.44**	12.90 ± 0.65	12.66 ± 0.61
BREAST	8.96 ± 0.87	7.89 ± 0.61	4.85 ± 0.50	**4.28 ± 0.54**
CAR	**14.11 ± 0.40**	14.12 ± 0.40	16.29 ± 0.39	16.31 ± 0.41
CHESS	26.12 ± 0.40	**26.09 ± 0.32**	26.46 ± 0.46	26.22 ± 0.36
CLEVE	6.10 ± 0.43	**6.05 ± 0.38**	6.51 ± 0.44	6.29 ± 0.51
CRX	11.34 ± 0.62	**11.05 ± 0.60**	13.97 ± 0.68	13.76 ± 0.58
FLARE	35.82 ± 0.92	**35.61 ± 0.90**	1532.39 ± 0.62	1532.22 ± 0.65
GLASS	8.53 ± 1.05	7.50 ± 1.03	7.40 ± 0.59	**7.12 ± 0.52**
GLASS2	4.20 ± 0.56	3.91 ± 0.54	3.20 ± 0.39	**3.08 ± 0.37**
IRIS	1.29 ± 0.53	1.22 ± 0.53	1.18 ± 0.44	**1.16 ± 0.44**
LETTER	612.99 ± 7.71	612.99 ± 7.71	441.94 ± 5.61	**433.37 ± 5.84**
LIVER	10.79 ± 0.63	10.61 ± 0.66	**9.59 ± 0.44**	9.72 ± 0.60
NURSERY	94.59 ± 2.45	94.57 ± 2.43	91.52 ± 2.41	**89.41 ± 2.30**
PRIMARY-TUMOR	35.28 ± 0.99	**34.64 ± 0.94**	6327.87 ± 38.33	6327.64 ± 38.33
SOYBEAN	3.45 ± 0.50	**3.38 ± 0.49**	4.49 ± 0.51	4.45 ± 0.48
VOTES	3.74 ± 0.59	**3.57 ± 0.58**	3.96 ± 0.55	3.76 ± 0.46

Table 6: Averages and standard deviations of *LogScore* using 100% of the learning data

6.1.2 MAPTAN+BMA **vs** MAPTAN

MAPTAN+BMA improves MAPTAN *LogScore* in a statistically significant way for most datasets. When little data is available, this improvement translates in an improvement in error rate. After performing a 5% statistical significance t-test, we have that MAPTAN+BMA error rate is significantly better than MAPTAN's for 6, 0 and 2 datasets with 10%, 50% and 100% of the learning data respectively, whilst MAPTAN error rate is never better than MAPTAN+BMA's in a statistically significant way. *LogScore* results favor MAPTAN+BMA more clearly. MAPTAN+BMA *LogScore* is significantly better than MAPTAN for 16, 13 and 12 datasets with 10%, 50% and 100% of the learning data respectively, whilst MAPTAN *LogScore* never improves SSTBMATAN's in a statistically significant way.

6.1.3 MAPTAN+BMA **vs** STAN+BMA

MAPTAN+BMA improves STAN+BMA error rate and *LogScore* in a statistically significant way for many datasets. After performing a 5% statistical significance t-test, we have that MAPTAN+BMA error rate is significantly better than STAN+BMA for 10, 10 and 8 datasets with 10%, 50% and 100% of the learning data respectively, whilst STAN+BMA error rate is only better than MAPTAN+BMA in a statistically significant way for 5, 4 and 4 datasets . *LogScore* results favor MAPTAN+BMA slightly. MAPTAN+BMA *LogScore* is significantly better than STAN+BMA for 7, 9 and 7 datasets with 10%, 50% and 100% of the learning data respectively, whilst STAN+BMA *LogScore* is only better MAPTAN+BMA in a statistically significant way for 5 datasets independently of the amount of data.

7 Conclusions and future work

We have seen that under a decomposable distribution over TANs it is possible to efficiently determine the MAP undirected TAN structure and the set of k MAP TAN structures and their relative probability weights. We used these results to construct two new classifiers: MAPTAN and MAPTAN+BMA. We have provided empirical results showing that both classifiers improve over established TAN based classifiers with equivalent complexity. From a practical point of view, selecting when to use SSTBMATAN,MAPTAN+BMA,MAPTAN depends mainly on two factors: the amount of uncertainty a posteriori in the models we expect to have and the ratio between the value of accuracy and the value of efficiency for the user. We can see a qualitative sketch of when to choose each classifier in figure 2. For any value of the ratio, we will choose MAPTAN when uncertainty a posteriori in models is low, SSTBMATAN when it is high and MAPTAN+BMAinbetween. If learning takes place in an environment where accuracy is much more important than efficiency, then our threshold in uncertainty to use MAPTAN+BMA and SSTBMATAN will be lower. If learning takes place in an environment where efficiency is much more important than accuracy, then MAPTAN will be our choice most of the times unless uncertainty in models is very high.

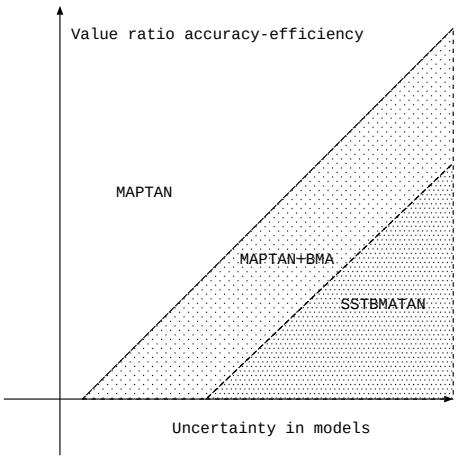


Figure 2: Selecting between SSTBMATAN, MAPTAN+BMA and MAPTAN

7.1 Future work

Since the amount of uncertainty a posteriori in the models can be measured after the learning step has been done, and the computational overload for finding the k MAP TAN models is low, it is possible to construct a metaclassifier that is able to make the selection between the three classifiers in figure 2 automatically. This metaclassifier could receive a threshold in the amount of uncertainty in model

selection. From the k MAP weights the classifier can easily calculate a lower bound on the amount of uncertainty in model selection using the k MAP models, and the single MAP model and select the classifier to be used according this bound. Furthermore, this evaluation can be performed periodically (each 1000 instances for example). Assuming the dataset is i.i.d., once the uncertainty in a single MAP model is under the threshold, the learning algorithm can assume that the structure has been learnt and from there on its learning time will be $\mathcal{O}(n)$, instead of on the current $\mathcal{O}(n^2)$. This means that we can use the results in this paper to construct an almost linear TAN learning algorithm. Developing these ideas remains as future work.

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