Corrigendum

Logics preserving degrees of truth from varieties of residuated lattices

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Abstract

A wrong argument in the proof of one of the main results in the paper is corrected. The result itself remains true. The right proof incorporates the basic ideas in the originally alleged proof, but in a more restricted construction.

Keywords: substructural logic, many-valued logic, degrees of truth, residuated lattices, protoalgebraic logic.
The proof of the last implication in Theorem 4.4 of the referenced paper is wrong. At a certain point, it performs a construction on an arbitrary algebra but the properties used in its development implicitly assume that the algebra is in fact a residuated lattice, which it needs not be. Here, we present a correct proof done by working only in the formula algebra, following the same ideas and performing essentially the same construction, modulo a certain crucial lemma that characterizes the supremum operation in the lattice of theories of one of the logics considered in [1].

Let us recall the necessary background. Let $K$ be an arbitrary variety of commutative, integral residuated lattices. The paper studies two finitary logics associated with each such $K$, which are denoted by their consequence relations: The first one, denoted by $\vdash$, is the truth-preserving logic determined by the algebras in $K$ when their maximum 1 (which is also the unit of the monoid structure of the fusion operation $*$) is taken as representing truth; the second one, denoted by $\leq_K$, is the logic preserving degrees of truth determined by the ordering relation of the algebras in $K$ (which are always lattices). More precisely, for any $n \geq 1$ and any formulas $\psi, \varphi_0, \ldots, \varphi_{n-1}$:

\[
\emptyset \vdash_K \psi \iff \forall A \in K, \forall \psi \in \text{Hom}(\text{Fm}, A), \psi(\psi) = 1.
\]

\[
\psi_0, \ldots, \psi_{n-1} \vdash_K \psi \iff \forall A \in K, \forall \psi \in \text{Hom}(\text{Fm}, A), \psi(\psi) = 1.
\]

The formula algebra is denoted by $\text{Fm}$, and its universe (i.e. the set of all formulas) by $\text{Fm}$.

The paper focuses on the less-known logic $\leq_K$. The logic $\leq_K$ is the one customarily associated with $K$, and has been extensively studied, see [3]; in particular, it is finitely and regularly algebraizable, having $K$ as its largest equivalent algebraic semantics. The properties needed here that already appear in [1] are summarized below.

1. The two logics have the same theorems, and the logic $\vdash$ is an extension of the logic $\leq_K$ by either of the following rules: Modus Ponens for $\rightarrow$ ($\alpha, \alpha \rightarrow \beta \vdash \beta$); Adjunction for $\star$ ($\alpha, \beta \vdash \alpha \star \beta$); or Squaring ($\alpha \vdash \alpha^2$); the exponential notation denotes iteration of the operation $\star$. Hence, every theory of $\vdash$ is in particular a theory of $\leq_K$.

2. For all $\alpha, \beta \in \text{Fm}$, $\vdash \alpha \rightarrow \beta \iff K \models \alpha \leq \beta \iff \alpha \leq_K \beta$. The second expression means that $\forall A \in K, \forall \psi \in \text{Hom}(\text{Fm}, A), \psi(\alpha) \leq \psi(\beta)$.

3. The logic $\vdash$ satisfies the Local Deduction-Detachment Theorem: For all $\Gamma \cup \{\alpha, \beta\} \subseteq \text{Fm}$, $\Gamma, \alpha \vdash_K \beta$ if and only if there is some $\alpha_0 \in \alpha$ such that $\Gamma \vdash_K \alpha_0 \rightarrow \beta$.

4. The Leibniz congruence $\Omega$ of a theory $T$ of $\vdash$ is defined from $T$ with the help of the equivalence connective $\equiv$ as follows: for any $\alpha, \beta \in \text{Fm}$, $(\alpha, \beta) \in \Omega T \iff \alpha \equiv \beta \in T$.

5. For any theory $T$ of $\leq_K$ there is a theory $T^+$ of $\vdash$ such that $T^+ \subseteq T$ and $\Omega T^+ = \Omega T$.

Moreover, we need the following properties, not explicitly mentioned in [1]:

6. For all $\alpha \in \text{Fm}$, $\vdash (\alpha \star 1) \equiv \alpha$ and $\vdash (1 \star \alpha) \equiv \alpha$. Recall that 1 is a constant term which is a theorem of both logics.

7. For all $\alpha_1, \ldots, \alpha_n \in \text{Fm}$, $\vdash (\alpha_1 \star \cdots \star \alpha_n) \equiv \alpha_i$ for each $i \in \{1, \ldots, n\}$.
All these properties are easily shown using the first equivalence in (2) and the corresponding properties of the ordering relation and the operations $\star$ and $\rightarrow$ in commutative, integral residuated lattices; for instance, (10) follows from the monotonicity of $\star$ with respect to order, (11) corresponds to its commutative character, and so on.

**Lemma.**
Let $S$ and $T$ be two theories of $\vdash_{\mathcal{K}}$. Then, the smallest theory of $\vdash_{\mathcal{K}}$ which contains $S \cup T$ is the set

$$\{ \varphi \in \mathcal{F}m : \exists \gamma \in S, \exists \delta \in T \text{ such that } \vdash_{\mathcal{K}}(\gamma \star \delta) \rightarrow \varphi \}.$$

**Proof.** Let us denote the displayed set by $L$. That this set contains both $S$ and $T$ is a consequence of (6). In order to show that $L$ is closed under the rules of $\vdash_{\mathcal{K}}$, let us assume that $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{K}} \varphi$ and that $\varphi_1, \ldots, \varphi_n \in L$; we have to prove that $\varphi \in L$. Using (7) one can see that the assumption implies that $\varphi_1 \star \cdots \star \varphi_n \vdash \varphi$. Now, by (8), there is some $\omega \in T$ such that $\vdash_{\mathcal{K}}(\varphi_1 \star \cdots \star \varphi_n) \rightarrow \varphi$, and by (9) this implies that $\vdash_{\mathcal{K}}(\varphi_1 \star \cdots \star \varphi_n) \rightarrow \varphi$.

Since for every $i \in \{1, \ldots, n\}$, $\varphi_i \in L$, there are formulas $\gamma_i \in S$ and $\delta_i \in T$ such that $\vdash_{\mathcal{K}}(\gamma_i \star \delta_i) \rightarrow \varphi_i$. Using (6), (8), (9) and (10) in succession we obtain

$$\vdash_{\mathcal{K}}(\gamma_1 \star \cdots \gamma_n \star \delta_1 \star \cdots \star \delta_n) \rightarrow (\varphi_1 \star \cdots \varphi_n).$$

We have already seen that $\vdash_{\mathcal{K}}(\varphi_1 \star \cdots \varphi_n) \rightarrow \varphi$, therefore

$$\vdash_{\mathcal{K}}(\gamma_1 \star \cdots \gamma_n \star \delta_1 \star \cdots \star \delta_n) \rightarrow \varphi.$$

By one of the properties in (11), the theories of $\vdash_{\mathcal{K}}$ are closed under fusion, hence $\gamma_1 \star \cdots \gamma_n \in S$ and $\delta_1 \star \cdots \delta_n \in T$. Thus, by definition, $\varphi \in L$. This shows that $L$ is a theory of $\vdash_{\mathcal{K}}$.

Finally, $L$ is the smallest such theory containing $S \cup T$: If $T'$ is a theory of $\vdash_{\mathcal{K}}$ such that $S \cup T \subseteq T'$, and $\varphi \in L$, by definition there are $\gamma \in S$ and $\delta \in T$ such that $\vdash_{\mathcal{K}}(\gamma \star \delta) \rightarrow \varphi$. These facts imply that $\gamma, \delta \in T'$ and also that $(\gamma \star \delta) \rightarrow \varphi \in T'$. Since by (12) the theories of $\vdash_{\mathcal{K}}$ are closed under fusion and Modus Ponens, we obtain first that $\gamma \star \delta \in T'$, and then that $\varphi \in T'$. This closes the proof.

Observe that, by (1), the property that $\vdash_{\mathcal{K}}(\gamma \star \delta) \rightarrow \varphi$ can be equivalently stated as $\gamma \star \delta \vdash_{\mathcal{K}}^2 \varphi$, so that, even if it is written using $\vdash_{\mathcal{K}}$, it tells us something about the theories of $\vdash_{\mathcal{K}}^2$. It is in this form that it will be used in a crucial step in the next proof. We can now give the right proof that corrects the original one in (1). This proof uses some standard facts on algebraizability and protoalgebraicity that can be found in (13).

**Theorem 4.4 of (1).**
Let $\mathcal{K}$ be a variety of residuated lattices. Then the following conditions are equivalent:

1. The logic $\vdash_{\mathcal{K}}^2$ is protoalgebraic.
Thus, all the conditions required in item 4 of the present theorem are satisfied, and the assumption
This completes the proof that
\[ \Omega \vdash \text{semantics of } \]
Proof of the claim. It is clear that \( \models S \models \) to the same effect; therefore, it will be
simpler to assume, without loss of generality, that \( S \) is a theory of \( \vdash \). Now, using (5) again, we can
consider the theory \( T^+ \) of \( \vdash \) associated with \( T \), and denote by \( L \) the smallest theory of \( \vdash \) that
contains \( S \cup T^+ \).

Claim: \( T^+ \subseteq L \subseteq T \).

Proof of the claim. It is clear that \( T^+ \subseteq L \), so we have to prove that \( L \subseteq T \). Using the previous
Lemma, it is enough to prove that if \( \gamma \in S, \delta \in T^+ \) and \( \Gamma (\gamma \ast \delta) \rightarrow \psi \), then \( \psi \in T \). Since \( T^+ \) is a
theory of \( \vdash \), by (2) and Modus Ponens it follows that \( \gamma \leftrightarrow (\gamma \ast \delta) \in T^+ \). Now by (3) this tells us that
(\( \gamma, \gamma \ast \delta \) \in \( \Omega T^+ = \Omega T \). But \( \gamma \in S \subseteq T \), so by the compatibility of \( \Omega T \) with \( T \) we get that \( \gamma \ast \delta \in T \).
Finally, by (2), the assumption that \( \Gamma (\gamma \ast \delta) \rightarrow \psi \) can be equivalently stated as \( \gamma \ast \delta \models \psi \), and this
allows us to conclude that \( \psi \in T \). This completes the proof of the claim.

By algebraizability of \( \vdash \), the Leibniz operator \( \Omega \) is monotonic over its theories. Since both \( T^+ \)
and \( L \) are theories of \( \vdash \), from the claim it follows that \( \Omega T = \Omega T^+ \subseteq \Omega L \); hence, \( \Omega T^+ \) is compatible
with \( L \), and it is also compatible with \( T \) by definition; so, it makes sense to factor out \( L \) and \( T \) by \( \Omega T^+ \) and we will obtain filters of the respective logics on the quotient algebra. Put \( A : = Fm / \Omega T^+ \),
\( \tilde{L} : = L / \Omega T^+ \), and \( \tilde{T} : = T / \Omega T^+ \). Then \( \tilde{L} \) is a filter of \( \vdash \) and hence by (1) a filter of \( \models \), while \( \tilde{T} \) is a
filter of \( \models \); thus, \( \tilde{L} \subseteq \tilde{T} \). Since \( K \) is the largest equivalent algebraic semantics of \( \vdash \), we know that \( A \in K \). And since \( \Omega T = \Omega T^+ \) we know that \( \Omega A \tilde{T} = \Omega T / \Omega T^+ = Id \).
Thus, all the conditions required in item 4 of the present theorem are satisfied, and the assumption
implies that \( \Omega A \tilde{L} = Id \). But since \( \Omega A \tilde{L} = \Omega L / \Omega T^+ \), it follows that \( \Omega T^+ = \Omega L \). Algebraizability of \( \vdash \) implies that \( \Omega \) is one-to-one over its theories, therefore \( T^+ = L \), which is the same as \( S \subseteq T^+ \).
Since both are theories of \( \vdash \), we can use monotonicity again and obtain that \( \Omega S \subseteq \Omega T^+ = \Omega T \).
This completes the proof that \( \Omega \) is monotonic over the theories of \( \models \).

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References

degrees of truth from varieties of residuated lattices. Journal of Logic and Computation, 19,
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