# Knot Theory And Categorification

Master's Thesis Mathematical Physics
under supervision of Prof. Dr. Robbert Dijkgraaf

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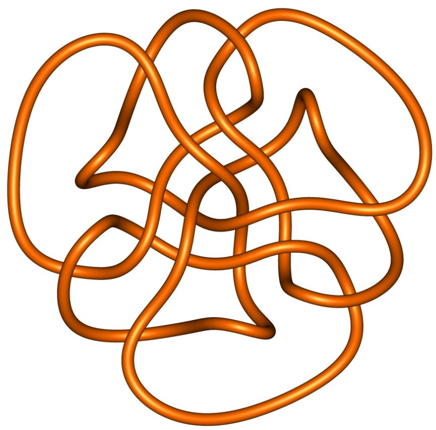

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**Abstract**

We will start out by giving a short introduction into the concept of categorification of natural numbers and polynomials with positive coefficients. Next we will give some examples of how categorification arises in several branches of physics and mathematics. Especially in topological string theory because here lies it's connection with knot theory. Also we will give a short introduction to knot theory and give a description of the most important knot invariants. Then we will go a little deeper into the theory of categorification and show how chain complexes are natural structures to categorify integers. Also we will show some properties of the category of chain complexes and do some calculations that will turn out to be crucial in the rest of this thesis. We will show how the Kauffman bracket can be constructed from a cube-construction. Then we will try to categorify everything in order to improve the Jones polynomial leading to Khovanov homology.

Instead of defining Khovanov homology straight away and then showing that it is Reidemeister invariant, we will try to construct it ourselves such that it becomes Reidemeister-invariant automatically. We will see that this works out for $RI$ moves and for most cases of $RII$ moves in a very natural way. This strongly improves our understanding of Khovanov homology. Having derived Khovanov homology we will then use it to calculate the Khovanov polynomial of the Trefoil knot as an example. Finally we give a very brief introduction to Khovanov-Rozansky theory, which is a categorification of the Homfly polynomial.

In Appendix A we will give a short review of homology theory in algebraic topology and show how this serves as a blueprint for other theories in which information is stored in chain complexes. Appendix B contains a review of the theory of Hopf-algebras and quantum groups and shows how knot-invariants are related to representations of quantum groups.

# Contents

# 1 Introduction

The central subject of this thesis will be Khovanov Homology. Khovanov Homology is a new topic in knot theory that emerged a few years ago with an article by Khovanov [9]. In this article Khovanov describes a clever way to improve the strength of the Jones Polynomial by using a trick called categorification. Categorification is a means of replacing certain objects of a theory by objects in some category so that also morphisms between them are defined.

Take for instance the set of natural numbers $\mathbb{N}$. It is wat we call a *commutative rig* that is: a set with to abelian operations: 'addition' and 'multiplication' denoted by $+$ and $\cdot$ . Addition and multiplication are both associative and multiplication acts distributive on a sum: $a \cdot (b + c) = a \cdot b + a \cdot c$. In other words it is a ring that does not necessarily have additive inverses.

Now the category of vector spaces over some field $k$ denoted by $Vect(k)$ contains a similar structure: it has a direct sum and a tensor product. We say $Vect(k)$ is a monoidal tensor category. So categorification of the natural numbers means replacing a theory which involves natural numbers by a theory which involves vector spaces (or ring modules). Every number $n$ is then replaced by an $n$-dimensional vector space. The expression $m + n$ is then replaced by the direct sum $V \oplus W$ where $V$ is an $m$-dimensional space and $W$ is an $n$-dimensional space. Also the expression $m \cdot n$ is replaced by $V \otimes W$. In this category the trivial vector space $\{0\}$ behaves like the number 0 since we have:

$$V \oplus \{0\} \cong V$$

and the ground field $k$ (which can be interpreted as a 1-dimensional vector space) behaves like the number 1:

$$V \otimes k \cong V$$

The opposite operation (going from a category to a set) is called decatorification. In this case decatorification is given by taking the dimension of a vector space.

There are some important subtleties however. For instance the tensor product is not associative. We don't have $(U \otimes V) \otimes W = U \otimes (V \otimes W)$. However we do have $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$. This is a very important aspect of categorification: certain expressions which are equal in one theory may lead to different, but isomorphic objects after categorification. This is however mostly an advantage. It means that we can store much more information in our new category. While in the set of natural numbers we can only decide wether two expressions are equal or not, in the category of vector spaces we can not only ask ourselves whether two objects are isomorphic or not but, if so, we can also wonder which isomorphisms we can establish between them. Moreover we also have non-invertible morphisms in the category of vector spaces, defining extra relationships between objects.

Especially in the case of knot-theory this is an advantage. The Jones polynomial can only describe knots, but after categorification we also have morphisms between them, describing (as we will see) 2-dimensional surfaces between knots.

It is important to notice that this construction works only because we are dealing with *positive* integers. Within the set of positive integers there is no well defined notion of subtraction since after subtraction of two such numbers we might end up with a negative integer. However in calculating the Kauffman bracket we do need to subtract integers. So in order to categorify the Kauffman bracket we need a way to categorify negative integers. Khovanov does this using chain complexes. An expression of the form $a - b$ can be replaced by a map $V^a \to V^b$. So any integer number can now be replaced by a sequence of maps between vector spaces. However, since we are now talking about sequences it is more natural to assign the sequence $0 \to V^a \to 0$ to $a$ instead of just $V^a$. And then we naturally assign $0 \to V^a \to V^b \to 0$ to $a - b$. We define the sequence $0 \to V^a \to V^b \to V^c \to 0$ to correspond to the expression $a - (b - c)$. This way we get back our original integer number by taking the Euler characteristic of the sequence (with 'Euler characteristic' we simply mean here the alternating sum of the dimensions of the spaces). Notice that the sequence we get from categorifying any integer number depends on the expression:

$$a + b \quad \Rightarrow \quad 0 \to V^a \oplus V^b \to 0$$

$$a - (0 - b) \quad \Rightarrow \quad 0 \to V^a \to 0 \to V^b \to 0$$

This is not a problem however. In fact, this is exactly what we want. We want to have more expressions then in our original category so we can distinguish more knots.

Khovanov states that we can categorify the Jones polynomial in this way, such that these sequences are actually chain complexes. Furthermore he states that we can do this such that not only the Euler characteristic of such a complex (which is the Jones polynomial) is an invariant for knots but even the homology. And since the homology contains strictly more information than the Euler characteristic, it is a stronger invariant then the Jones polynomial.

## 2 Graded Vector Spaces

Another example of categorification that will be very important to us is the categorification of polynomials. Notice that a polynomial can be described by an ordered set of numbers indexed by the natural numbers such that only a finite amount of these numbers is nonzero. These numbers are the coefficents of the polynomial and the indices are the degrees of the corresponding monomials:

$$a + bq + cq^3 \quad \cong \quad (a, b, 0, c, 0, 0, 0, ...)$$

If we look at Laurent-polynomials (polynomials in both $q$ and $q^{-1}$) we look at ordered sets indexed by the integer numbers.

$$aq^{-2} + b + cq + dq^3 \quad \cong \quad (..., 0, 0, 0, a, 0, b, c, 0, d, 0, 0, 0, ...)$$

Now we want to categorify the set of laurent-polynomials with positive integer coefficients: $\mathbb{N}[q, q^{-1}]$. Since we have seen in our previous example that we can categorify the natural numbers by replacing them with vectorspaces, we now get an ordered set of vectorspaces. For each of these vector spaces the dimension is equal to one of the coefficients of the categorified polynomial.

$$aq^{-2} + b + cq + dq^3 \quad \Rightarrow \quad (..., 0, 0, 0, V^a, 0, V^b, V^c, 0, V^d, 0, 0, 0, ...)$$

Here $V^n$ denotes a vectorspace with dimension $n$. We can put these vectorspaces all together in one so-called *graded* vectorspace.

$$aq^{-2} + b + cq + dq^3 \quad \Rightarrow \quad V^a_{-2} \oplus V^b_0 \oplus V^c_1 \oplus V^d_3$$

(The subscripts denote the grading.)

**Definition 1** *A graded vector space is a vector space which is the direct sum of subspaces which are all labelled by an integer number. A vector which is an element of one of these subspaces, labelled by $n$, is called homogenous of degree $n$.*

For tensor products we have by definition:

$$\deg(v \otimes w) = \deg(v) + \deg(w)$$

In general we define categorification of the rig of Laurent-polynomials with natural number coefficients as:

$$\sum_{i \in \mathbb{Z}} a_i q^i \quad \Rightarrow \quad \bigoplus_{i \in \mathbb{Z}} V_i^{a_i}$$

So we see that under categorification the laurent-polynomials are mapped to the category of graded vector spaces. The polynomial on the left-hand side is called the *graded dimension* of the space on the right-hand side.

This can easily be generalized to a categorification of $\mathbb{Z}[q, q^{-1}]$: replace a polynomial $P_1$ by a sequence $0 \to V^{P_1} \to 0$ such that the graded vector space $V^{P_1}$ has graded dimension $P_1$ Then the expression $P_1 - P_2$ becomes:

$$0 \to V^{P_1} \to V^{P_2} \to 0$$

The map in this sequence is chosen to be grading preserving (that is: it maps homogeneous subspaces of degree $n$ into homogeneous subspaces of degree $n$) so that it can be seen as the direct sum of graded sequences, each corresponding to one of the monomials in $P_1$. Moreover, we will define these maps such that these sequences are chain complexes.

# 3 Knot Theory and Categorification in Physics

## 3.1 Quantum Mechanics

An example of categorification that is very important for physicists is quantum mechanics. Suppose we have a classical particle which lives in a space that we divide in three sections. In other words: the coordinates of the particle can take on three different values. We call these sections $A$, $B$ and $C$. Now a quantummechanical particle does not have to be in exactly one of these three sections, but can be in a superposition. This means that the state of the particle is now described as: $aA+bB+cC$, where $a$, $b$ and $c$ are three complex numbers such that

$$|a|^2 + |b|^2 + |c|^2 = 1$$

So the state of the particle is now described by a vector in a three dimensional space, while the state of the classical particle was described by an element of a set with three elements. This is clearly an example of categorification.

In reality of course a classical particle lives in a continuum of coordinates. We could for instance say the particle lives in a one-dimensional infinite square well, with the walls located at $x = 0$ and $x = 1$ respectively. So the particle's position is described by an element of the set $[0, 1]$ a set with uncountable many elements. Categorification of this set would lead to vector space with an uncountable basis! A vector in such a space is simply a function from $[0, 1]$ to $\mathbb{C}$ (with value 0 on the boundary). Lucky for us the laws of quantummechanics demand that particles are described by wavefunctions that are continuous. This restriction severely lowers the amount of possible states. We know from fourier analysis that the space of continuous, normalizable complex functions on a real interval is a Hilbert space $V$ with countable basis. So we see that categorification of a set is here realized as going to its function space $V$.

Notice that in fact a classical particle is described not only by its position in coordinate space, but also by its 'position' in momentum space. The two spaces taken together are called the phase space of the particle (usually the phase space is defined as the co-tangent bundle $T^*M$ of coordinate space). However, the quantummechanical Hilbert space is the categorification of only the coordinate space. The classical positions form a basis in the quantummechanical space. The classical momenta form a different basis, so a momentum eigenvector is a superposition of position eigenvectors. Time evolution in classical mechanics is described by a map from $T^*M$ to $T^*M$. This becomes a linear operator from $V$ to $V$.

With every momentum eigenvector is associated a certain momentum, which is a real number. Since the momentum eigenvectors form a basis we see that we can use these momentum eigenvalues to interpret the Hilbert space as a graded vector space:

$$\deg(v) = \lambda \quad \text{where} \quad H(v) = \lambda v$$

Here $H$ is the hamiltonian which is defined by $H = \frac{P^2}{2m} + U$. $P$ is the momentum operator and $U$ is the potiential. In the case of the infinite square well for instance $U$ is zero on the interval $[0, 1]$ and infinite outside. We could also have taken another potential as long as it gives $H$ a countable spectrum, for instance the harmonic oscillator potential. Or, if we also have anti-particles we could define:

$$\deg(v) = q\lambda \quad \text{where} \quad Q(v) = qv \quad \text{and} \quad H(v) = \lambda v$$

With $Q$ the charge operator. The time evolution operator is then defined by:

$$\exp(QHt)$$

with $t$ the time. This means that we can define the graded dimension of $V$ as:

$$\text{qdim}(V) = \text{Tr}(e^{QHt}) = \sum_{\lambda} q^{Q\lambda}$$

with $q = \exp(t)$ and the summation is over all eigenvalues $\lambda$ of $H$. For instance if we look at a system consisting of a particle with charge 1 and energy 1 together with its corresponding anti-particle, then $\text{qdim}(V) = q + q^{-1}$ (although in quantum mechanics we cannot yet really speak of anti-particles).

The fact that a quantummechanical particle can indeed be in a superposition of energy-eigenstates follows from the fact that if we try to measure its position, it will collapse to a position-eigenfunction, which is a delta-function (we ignore here the fact that the dirac-delta is not a well defined function). We can consider the function $\psi$ as a superposition of delta functions:

$$\psi(x) = \int \psi(x')\delta(x - x')dx'$$

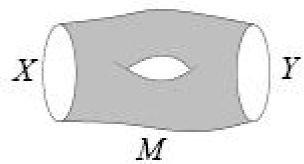
We see here that we can consider $\psi(x)$ as a linear combination of basis vectors $\delta(x - x')$ labeled by $x'$ with coefficients $\psi(x')$. In the same way we can view a delta function as a superposition of energy-eigenstates:

$$\delta(x) = \int e^{ipx} dp$$

Notice that if we are in an infinite square well $p$ can only take on a discrete set of values, since $\psi$ has to be zero at the boundary.

## 3.2 Topological Quantum Field Theory

A $d + 1$-dimensional Topological Quantum Field Theory (TQFT) is a functor that assigns vector spaces to closed oriented $d$-dimensional manifolds and linear maps to compact oriented $d + 1$-dimensional manifolds such that these vector spaces and maps are topological invariants.



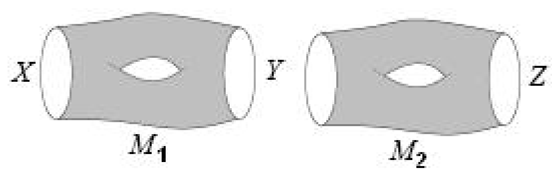

So if $X$ and $Y$ are two $d$-dimensional manifolds and $M$ is a cobordism between $X$ and $Y$ ($X$ is called the incoming boundary and $Y$ is called the outgoing boundary) then we have a functor $F$ such that:

$$F(X) = V_X, \quad F(Y) = V_Y \quad \text{and} \quad F(M) = f \tag{1}$$

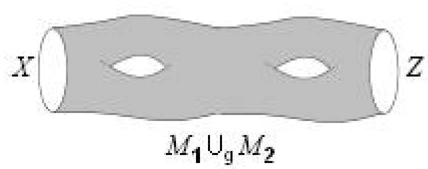
Where $V_X$ and $V_Y$ are vector spaces and $f$ is a linear map

$$f : V_X \to V_Y$$

A TQFT should satisfy 'gluing properties'. That is: suppose we have two manifolds $M_1$ and $M_2$ with corresponding linear maps $f_1$ and $f_2$ respectively.




Furthermore the outgoing boundary of $M_1$ is diffeomorphic to the incoming boundary of $M_2$ through a diffeomorphism $g$. If they are then glued together, the result $M_1 \cup_g M_2$ gets assigned the composition map $f_2 \circ f_1$.




$$F(M_1 \cup_g M_2) = F(M_2) \circ F(M_1) \tag{2}$$

The disjoint union of two manifolds goes to tensor product:

$$F(M_1 \sqcup M_2) = F(M_1) \otimes F(M_2) \tag{3}$$

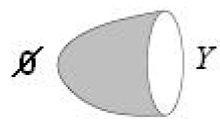
The linear map should be a topological invariant of the cobordism. So if two cobordisms are diffeomorphic the TQFT should assign to them the same linear map:

$$M_1 \cong M_2 \quad \Rightarrow \quad F(M_1) = F(M_2) \tag{4}$$

In particular this means that if $M$ is the trivial cobordism from $X$ to itself (that is: $M \cong X \times I$ with $I$ the interval $I = [0, 1]$) then the map $F(M)$ is the identity map:

$$F(M) \quad = \quad \mathrm{Id} : X \to X$$

It follows from (3) that if $X$ is the empty space $X = \emptyset$ then the functor assigns to it the ground field $k$.




$$F(\emptyset) = k$$

A closed $d+1$-dimensional manifold $M$ then gets assigned a linear map $k \to k$ which can be interpreted as an element of $k$ itself.

So (if for instance $d = 2$) we have the following pattern:

| **Category of manifolds** | **Category of vector spaces** |
|---|---|
| closed 2-manifold $\Sigma$ | Vector space $V_\Sigma$ |
| 3-manifold | Linear map $V_{\Sigma_1} \to V_{\Sigma_2}$ |
| closed 3-manifold $Y$ | Linear map $k \to k \quad = \quad$ number $Z_Y \in k$ |

In the theory of the Jones Polynomial, where we assign a number (polynomial) to a closed 1-manifold (knot), we have the same kind of pattern. We can consider it as a kind of 1-dimensional TQFT embedded in $S^3$:

| **Category of manifolds** | **Category of quantum group reps** |
|---|---|
| closed 0-manifold (points) | Representation of quantum group |
| 1-manifold (lines) | Morphism of representations |
| closed 1-manifold (link) | Morphism $k \to k \quad = \quad$ number $Z \in k$ |

Suppose now we have two manifolds $Y_1$ and $Y_2$ where $Y_1$ has only an outgoing boundary and $Y_2$ has only an incoming boundary. Then $Y_1$ has assigned to it a linear map $f_1 : k \to V_{\partial Y_1}$. Which can be seen as a vector $v$ in $V_{\partial Y_1}$ since $f$ is determined by its value $f(1)$ on the unit element $1 \in k$. In the same way $Y_2$ gets assigned a linear map $f_2 : V_{\partial Y_2} \to k$ which can be seen as a covector $w$ in ${V_{\partial Y_2}}^*$.

Suppose now that their respective boundaries are diffeomorphic: $\partial Y_1 \cong \partial Y_2$. Then $V_{\partial Y_2} = V_{\partial Y_1}$ and we can glue them together to create a closed 3-manifold $Y$ by identifying their boundaries. Then $Y$ gets assigned $f_2 \circ f_1$ which is $w(v)$ , or in bracket notation: $\langle w, v \rangle$.

### 3.3 Frobenius Algebras

Suppose $O$ is a 1-dimensional closed connected manifold without boundary (a cycle) and the functor $F$ defines a 2-dimensional TQFT. Then in order to satisfy (4) the vector space $F(O)$ should be a commutative Frobenius algebra (see [15]).

**Definition 2** *A Frobenius algebra is a vector space over some field $k$ which is equipped with the following structures:*

a multiplication $m$:

$$m: \ V \otimes V \to V$$

a co-multiplication $\Delta$:

$$\Delta: \ V \to V \otimes V$$

a unit $\eta$:

$$\eta: \ k \to V$$

a co-unit $\epsilon$

$$\epsilon: \ V \to k$$

The maps $m$ and $\Delta$ have to satisfy the frobenius condition:

$$(m \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta) = \Delta \circ m = (\mathrm{Id} \otimes m) \circ (\Delta \otimes \mathrm{Id}) \tag{5}$$

and also associativity and co-associativity:

$$m \circ (m \otimes \mathrm{Id}) = m \circ (\mathrm{Id} \otimes m) \tag{6}$$

$$(\mathrm{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{Id}) \circ \Delta \tag{7}$$

and the unit and co-unit condition:

$$m \circ (\mathrm{Id} \otimes \eta) \quad = \quad \mathrm{Id} \quad = \quad m \circ (\eta \otimes \mathrm{Id}) \tag{8}$$

$$(\epsilon \otimes \mathrm{Id}) \circ \Delta = \quad \mathrm{Id} \quad = (\mathrm{Id} \otimes \epsilon) \circ \Delta \tag{9}$$

A commutative Frobenius algebra is a Frobenius algebra that also contains a twist map:

$$\tau(v \otimes w) \ := \ w \otimes v$$

such that it satisfies the following two equivalent relations:

$$m \circ \tau \ = \ m$$

$$\tau \circ \Delta \ = \ \Delta$$

Whenever two cycles join like in the first picture of figure 1 this cobordism gets assigned the map $m$. When a cycles splits into two like in the second picture of figure 1 it gets assigned the map $\Delta$. The cobordism of the third picture gets $\eta$ and the fourth cobordism gets assigned $\epsilon$.

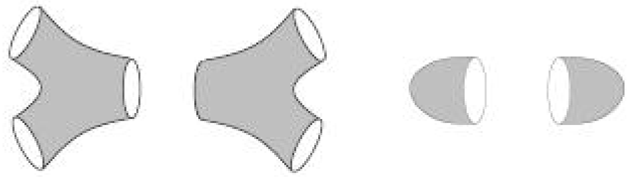


Figure 1: elementary cobordisms

**Lemma 1** *With these four elementary cobordisms and the cobordism that intercanges two cycles we can build up any 2-dimensional cobordism. The relations (5)-(9) make sure that (4) is always satisfied.*

When we have more then two incoming cycles and the map $m$ acts on the $i^{th}$ and the $i+1^{th}$ component we'll denote it by $m_i$. If the $i^{th}$ component splits into an $i^{th}$ and an $i+1^{th}$ component we'll call the corresponding map $\Delta_i$.

In other words, if we have $k$ cycles:

$$m_1 := m \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes ...$$

$$m_2 := \mathrm{Id} \otimes m \otimes \mathrm{Id} \otimes ...$$

$$m_{k-1} := \mathrm{Id} \otimes \mathrm{Id} \otimes ... \otimes m$$

And analogous for $\Delta$. We can then rewrite the conditions (5), (6) and (7) as:

$$m_{i+1}\Delta_i = \Delta_i m_i = m_i \Delta_{i+1}$$

$$m_{i+1}m_i = m_i m_{i+1}$$

$$\Delta_{i+1}\Delta_i = \Delta_i \Delta_{i+1}$$

### 3.4 Path Integration

A TQFT can be described using Quantum Field Theory and path integration. Suppose we have a closed 2-manifold $\Sigma$. We can look at the space of all gauge-fields on $\Sigma$. If we then consider the vector space generated by all classical fields we obtain the so-called physical Hilbert space $H_\Sigma$. If we consider 2-manifolds as space and 3-manifolds as space-time then we can use quantum field theory as a 3-dimensional TQFT since we already know that QFT satisfies (1), (2) and (3) if the linear map $F(M)$ plays the role as time-evolution operator.

We know from quantum field theory that an initial state evolves in time by acting on it with the time-evolution operator, which is given by:

$$\int \mathcal{D}\phi e^{-S(\phi)}$$

Here $\phi$ denotes a gauge-field on the space $\Sigma$ so this is an integral over the Hilbert space of all field-configurations (a path-integral). In order to make the theory topological (that is: it satisfies (4)) the action $S(\phi)$ in this expression is the Chern-Simons action (see for instance [19]).

A closed manifold $Y$ then defines a path integral from the vacuum to $H_\Sigma$ and back to the vacuum. This is a map $k \to k$ so this is again a number in $k$. This number is known as the partition function.

In Jones-Witten theory we include Wilson loops in our 3-manifolds and use this technique to find invariants of knots embedded in a 3-dimensional space.

| **Category of manifolds with punctures** | **Category of quantum group reps** |
|---|---|
| closed 2-manifold with dots | representation of quantum group |
| 3-manifold with lines | Morphism of representations |
| closed 3-manifold with link | Morphism $k \to k \quad = \quad$ number $J \in k$ |

## 3.5 Categorifying TQFT

So what if we now want to extend this to 4-manifolds? Extending the pattern of (3.2) would mean a 4-dimensional cobordism should correspond to something like a morphism between two numbers, which makes no sense. However, we have seen that this can be resolved using categorification. Categorification makes a number into a vector space. Also a vector space in turn can be categorified into a category and a linear map is then turned into a functor. So if we first categorify this theory, then a closed 3-manifold gets assigned a vector space (which has the partition function as its dimension) and then a 4-manifold can be assigned a linear map between two vector spaces.

| **Manifolds** | **Vector spaces** |
|---|---|
| closed 2-manifold $\Sigma$ | Category $C_\Sigma$ |
| 3-manifold | Functor $C_{\Sigma_1} \to C_{\Sigma_2}$ |
| closed 3-manifold $Y$ | Functor $C_\emptyset \to C_\emptyset \quad = \quad$ vector space $V_Y$ |
| 4-manifold | Linear map $V_{Y_1} \to V_{Y_2}$ |
| closed 4-manifold $X$ | Linear map $k \to k \quad = \quad$ number $Z_X \in k$ |

And maybe we can even go on like this infinitely, categorifying categories etc... to describe any $d$-manifold.

In the above diagram the vector space $V_Y$ is the categorification of the partition function $Z_Y$ of the 3-dimensional TQFT. Similarly the category $C_\Sigma$ is the categorification of the vector space $V_\Sigma$. And the category $C_\emptyset$ is the categorification of the ground field $k$. A vector space is decategorified by taking its dimension, that is: $\dim(V_Y) = Z_Y$. The decategorification of a category is defined by taking its Grothendieck group, tensored with $k$. So $V_\Sigma$ is the Grothendieck group (tensored with $k$) of $C_\Sigma$.

**Definition 3** *Let $S$ be a commutative semigroup (that is: it does not have inverses or a unit element) then we define its Grothendieck group as the set* *of pairs $(x,y)$ modulo the equivalence relation $(x,y) \sim (x+t, y+t)$ for any $x, y, t \in S$. This is indeed a group since it has a unit: $(x,x) \sim (0,0)$ and every element has an inverse: $(x,y) + (y,x) = (x+y, x+y) \sim (0,0)$.*

The construction of the Grothendieck group of a semigroup can thus be interpreted as 'making a semigroup into a group by adding inverses to it'. Notice that any monoidal category has the structure of a semigroup, so it makes sense to talk about the Grothendieck group of a monoidal category.

Since $\Sigma \times I$ is mapped to a functor, we see that a 4-manifold $\Sigma \times I \times I$ should be mapped to a natural transformation between two functors $F_0$ and $F_1$. Where $F_0$ is the functor corresponding to $\Sigma \times I \times \{0\}$ and $F_1$ corresponds to $\Sigma \times I \times \{1\}$. So we see that the diagram:

$$\begin{array}{ccccc} \Sigma & \rightarrow & \Sigma \times I & \rightarrow & \Sigma \times I \times I \\ & & \downarrow & & \downarrow \\ & & Y & \rightarrow & Y \times I \\ & & & & \downarrow \\ & & & & X \end{array}$$

gets assigned the following diagram after applying the TQFT:

$$\begin{array}{ccccc} \text{Category} & \rightarrow & \text{Functor} & \rightarrow & \text{Natural Transformation} \\ & & \downarrow & & \downarrow \\ & & \text{Vector space} & \rightarrow & \text{Linear map} \\ & & & & \downarrow \\ & & & & \text{Number} \end{array}$$

In these diagrams going one column to the right corresponds to increasing the dimension of the manifold, while going one row down corresponds to taking the trace. But how do we assign a category to a manifold? And how can a functor from the category $C_k$ to itself be interpreted as a vector space? We know that the Grothendieck group of $C_\Sigma$ should be the physical Hilbert space $H_\Sigma$. We have seen that a linear map $k \rightarrow V$ is determined by its value on the unit element $1 \in k$. Does this mean that a functor $C_k \rightarrow C_V$ for any category $C_V$ is also determined by its value on a unit object $I \in C_k$?

To make things a little easier we'll work with free $\mathbb{Z}$-modules instead of vector spaces, so that we don't have to worry about categorification of rational numbers. The decategorification of such a module is then its rank. The categorification of a natural number $n$ is a free $\mathbb{Z}$-module of rank $n$, which we denote by $\mathbb{Z}^n$. In particular the number 1 becomes $\mathbb{Z}$. The elements of $\mathbb{Z}^n$ will be denoted by:

$$\mathbb{Z}^n = \{a_1e_1 + a_2e_2 + ... + a_ne_n | a_1, a_2, ...a_n \in \mathbb{Z}\}$$

here $e_1, e_2, ...e_n$ denote the 'basis vectors'.

The categorification of $\mathbb{Z}^n$ is then a monoidal category of $\mathbb{Z}$-modules generated by modules $E_1,\ E_2 ...\ E_n$ which are all free $\mathbb{Z}$-modules of rank one (in other words they are all isomorphic to $\mathbb{Z}$). We'll call this category $C^n$. A $j$-fold direct sum of a module $E_m$ with itself will be denoted by $jE_m$. For instance:

$$3E_5 := E_5 \oplus E_5 \oplus E_5$$

$C^n$ can then be written as:

$$C^n = \{A_1 E_1 \oplus A_2 E_2 \oplus ... A_n E_n \mid A_1, A_2, ... A_n \in \mathbb{N}\}$$

We then see that $\mathbb{Z}^n$ is the Grothendieck group of $C^n$.

Notice that if we have a monoidal functor $F$ from $C^n$ to $C^m$ and we look only at its restriction to the objects of $C^n$, then we see that it is determined by its values on the 'basis modules' $E_1, ... E_n$. For instance a functor from $C^3$ to $C^5$:

$$E_1 \mapsto E_2, \quad E_2 \mapsto E_4, \quad E_3 \mapsto E_5$$

In particular this means for a functor from $C^1$ to $C^1$ that it can be written as $\mathbb{Z} \mapsto \mathbb{Z}^m$. Then we know that any other object $\mathbb{Z}^n$ in $C^1$ is mapped to $\mathbb{Z}^n \otimes \mathbb{Z}^m \cong \mathbb{Z}^{nm}$. In other words: we can interpret a functor from $C^1$ to $C^1$ as an element of $C^1$ (in this example this element would be $\mathbb{Z}^m$). This is of course completely analogous to the fact that any linear map from the field $k$ to itself can be seen as an element of $k$ itself.

Now in 3-d TQFT the vacuum gets assigned the ground field $k$, but since we now work only with $\mathbb{Z}$-modules it gets assigned the ring $\mathbb{Z}$. Then in 4-d TQFT the vacuum gets assigned the category $C^1$. A 4-dimensional cobordism gets assigned a functor from $C^n$ to $C^m$ and a closed 4-dimensional cobordism gets assigned a functor $C^1 \to C^1$ which is, as we have seen, a free $\mathbb{Z}$-module.

We can just as well interpret these monoidal categories $C^n$ as categorifications of vector spaces $k^n$. However to decategorify $C^n$ we should then take the Grothendieck group and then take its tensor product with $k$.

However we can also assign a module to a functor from $C_k$ to itself in a different way. To a 3-cycle $\Sigma$ we assign the category $C_\Sigma$ This means that to a 3-dimensional manifold $Y_1$ which has $\Sigma$ as its (outgoing) boundary, we assign a functor $C_\emptyset \Rightarrow C_\Sigma$ which can be interpreted as an object $A$ of $C_\Sigma$. In the same way we assign a functor $C_\Sigma \Rightarrow C_\emptyset$ to a manifold $Y_2$ which has $\Sigma$ as its incoming boundary. This functor assigns a module $\mathbb{Z}^m$ to any module $A$ in $C_\Sigma$. This functor can for instance be defined as: $\mathrm{Hom}(B, \cdot)$ for a given $B$ in $C_\Sigma$. Notice that this is a monoidal functor since

$$\mathrm{Hom}(B \oplus B,\ A) \quad \cong \quad \mathrm{Hom}(B, A) \oplus \mathrm{Hom}(B, A).$$

Notice also that $\mathrm{Hom}(B, A) \cong \mathbb{Z}^m$ for some $m$.

In Jones-Witten theory we have that $\Sigma$ is represented by the space $H_\Sigma$ generated by functions on $\Sigma$. Then $Y_1$ with $\partial Y_1 = \Sigma$ is an element of $H_\Sigma$.

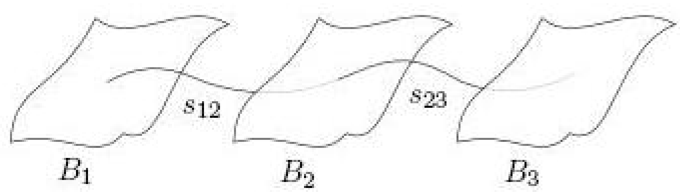
This is element is altered if we include Wilson-lines. Suppose we have a knot $K$ embedded in $Y$. This knot then determines the partition function (this partition function is an invariant of the knot, for instance the Jones polynomial). Since $Y$ is obtained by gluing together $Y_1$ and $Y_2$ we have two tangles $K_1 = K \cap Y_1$ and $K_2 = K \cap Y_2$. So the element of $H_\Sigma$ that represents $Y_1$ depends on the tangle $K_1$ that is embedded in $Y_1$. And a similar statement holds for $Y_2$ and $K_2$.

This means we can choose a particular element of $H_\Sigma$ by choosing a particular tangle in $Y_1$ or $Y_2$. In other words: any tangle in $Y_1$ (or $Y_2$) represents a vector in $H_\Sigma$.

A similar thing happens in Gukov-theory. This time $\Sigma$ is represented by a category $C_\Sigma$. Then $Y_1$ with $\partial Y_1 = \Sigma$ is represented by an object of $C_\Sigma$. When embedding a knot in $Y$ we should get a different vector space (this vector space is the categorification of the knot-invariant). This means that we can choose a particular object of $C_\Sigma$ by choosing a particular tangle in $Y_1$.

## 3.6 Topological String Theory

If we replace our classical particle of section 3.1 by a string, then we must specify the string's boundary conditions. For instance something like: the first boundary point of the string must lie on a submanifold $B_1 \subset X$ where $X$ is the space the string lives in. And the second boundary point lies on a submanifold $B_2 \subset X$. These submanifolds are also known as *branes*. After quantization the state of the string is an element of the Hilbert space $H_{B_1,B_2}$. Suppose now we also have a third brane $B_3$ and a pair of strings $s_{12}$ and $s_{23}$. $s_{12}$ runs from $B_1$ to $B_2$ and $s_{23}$ runs from $B_2$ to $B_3$. Then these two strings can also be considered as one string $s_{23}$ that runs from $B_1$ to $B_3$. Moreover we see that in this way the Hilbert spaces $H_{B_1,B_2}$ and $H_{B_1,B_2}$ together give rise to the Hilbert space $H_{B_1,B_3}$ by joining strings. This can be rephrased by saying that the branes are objects in a category and strings are the morphisms in this category. The Hilbert space $H_{B_i,B_j}$ is then equal to $\mathrm{Hom}(B_j, B_i)$




It turns out that the category assigned to $\Sigma$ by the TQFT in the previous sections can be interpreted in some sense as the category of branes in topological string theory. A 3-manifold $Y_1$ which has $\Sigma$ as its boundary then corresponds to an object of this category, a brane $B_1$. And the closed 3-manifold $Y$ which is constructed by gluing $Y_1$ and $Y_2$ along their common boundary is then the vector space $\mathrm{Hom}(B_2, B_1)$, which is the space of all strings running from $B_1$ to $B_2$. If we incorporate knots and tangles in the theory we see that every tangle embedded in $Y_1$ determines a different brane $B_1$. And every choice of knot a $K$ embedded in $Y$ determines a different space of string states.

We have the following pattern:

| **Manifolds** | | **Strings** |
|---|---|---|
| closed 2-manifold | category | collection of all branes |
| 3-manifold (with tangle) | object | brane |
| closed 3-manifold (with knot) | vectorspace $\mathrm{Hom}(B_2, B_1)$ | space of all strings between $B_1$ and $B_2$ |

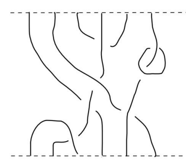
So a tangle represents a brane. Notice that this means there is an action of the braid group on the category of branes.

# 4 Knot Theory

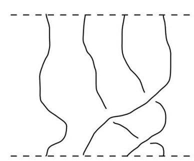
## 4.1 Knots, Braids and Tangles

Knot theory is a mathematical theory that tries to distinguish different knots. Mathematically a knot is a compact one-dimensional manifold without boundary embedded in a three-dimensional background space. Two knots are equivalent whenever we can deform the first knot into the second one continuously such that it remains a properly embedded one-manifold all the time.

**Definition 4** *A Tangle $L$ of type $(k, l)$ (with $k + l$ even) is a proper embedding of the disjoint union of of a finite number of arcs into the space $\mathbb{R}^2 \times [0, 1]$ such that $\partial L \in \mathbb{R}^2 \times \{0, 1\}$, $\partial L \cap \mathbb{R}^2 \times \{0\}$ is a set of $k$ points and $\partial L \cap \mathbb{R}^2 \times \{0\}$ is a set of $l$ points.*



**Definition 5** *A braid $L$ is a tangle of type $(k, k)$ such that for any $x \in [0, 1]$ $L \cap \mathbb{R}^2 \times \{x\}$ consist of exactly $k$ points.*



**Definition 6** *A link is a tangle of type $(0, 0)$.*

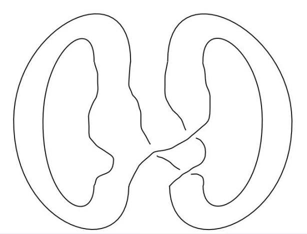
**Definition 7** *A knot is a link consisting of exactly one component. That is: it is a proper embedding of $S^1$ into $\mathbb{R}^3$.*

**Definition 8** *An isotopy of a space $X$ is map $h$ from $[0, 1] \times X$ to $X$ such that for any $t \in [0, 1]$ the mapping $h(t, \cdot)$ is a homeomorphism of $X$ and $h(0, \cdot)$ is the identity.*

*Let $L$ and $L'$ be two links embedded in $X \subset \mathbb{R}^3$. We say $L$ and $L'$ are isotopic if there exists an isotopy $h$ of $X$ such that $h(1, L) = L'$. We then write $L \sim L'$.*

Isotopy defines an equivalence relation for links. In the future whenever we say 'knot' or 'link' we actually mean 'isotopy class of knots' or 'isotopy class of links'.

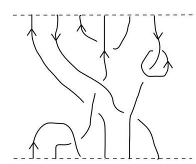
**Definition 9** *A closure of a braid is a link that is obtained by connecting the boundary points of a braid with each other.*



**Lemma 2** *Every link is equivalent to the closure of some braid.*

**Definition 10** *an oriented tangle is a tangle in which every arc is equipped with an orientation. The boundary of a tangle is a finite set of points in $R^2 \times \{0,1\}$ which are marked by either a $+$ or a $-$ sign. A point in $R^2 \times \{0\}$ is marked $+$ if it is the endpoint and it is marked $-$ if it is the starting point of an arc. For $R^2 \times \{1\}$ we define it the other way around: a point gets a $-$ if it is the endpoint for an arc.*

That is: arrows going up are going from a point marked $+$ to another point marked $+$ and arrows going down go from $-$ to $-$.

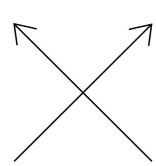


Oriented links and knots are defined in a similar way. From now on we will assume that all links and knots are oriented, so we will not explicitly call them oriented anymore.

**Definition 11** *A regular link projection is a projection of a link to a two-dimensional plane such that there are nowhere more then two points of the link projected to the same point in the plane. If $x$ is a point in the plane and there are exactly two points of the link projected to $x$ then $x$ is called a crossing point.*

**Lemma 3** *For every link L there is always an isotopic link L' such that the projection of L' is a regular link projection.*

If we equip a link with an orientation then in a regular link projection every crossing point has a neighborhood that looks like:

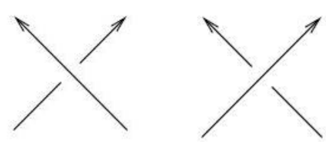


it consists of two diagonal lines (*'edges'*) which are respectively denoted by $e_1$ and $e_2$. For every crossing point $x$ we denote by $E_x$ the set $\{e_1, e_2\}$ consisting of these two edges.

**Definition 12** *A link diagram is a regular link projection for which every set $E_x$ is ordered. The first edge of $E_x$ with respect to this ordering is called the overcrossing edge and the other edge is called the undercrossing edge.*

Let $\pi$ be the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$: $(x, y, z) \mapsto (x, y)$. Suppose we have a link $L$ such that it is projected by $\pi$ to a regular link projection. Let $p = (x, y)$ be a crossing point of this projection and the two edges it lies on are called $e_1$ and $e_2$. The pre-image of $p$ under the projection $\pi$ is a set $\{q_1, q_2\}$ of two points of the link $L$. Then $q_1 = (x, y, z_1)$ lies on the pre-image of $e_1$ and $q_2 = (x, y, z_2)$ lies on the pre-image of $e_2$. We define $e_1$ to be the overcrossing edge if $z_1 > z_2$. We define $e_2$ to be the overcrossing edge if $z_2 > z_1$.

If $e_1$ is the overcrossing edge we say $x$ is a negative crossing point and if $e_2$ is the overcrossing edge we say $x$ is a positive crossing point. A link diagram can be drawn by replacing every crossing point of the projection by one of the following two pictures:



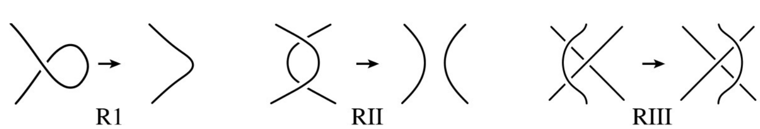
The left picture is a negative crossing point and the right one is a positive crossing point. Every link diagram represents a unique isotopy class of links. The other way around however is not true: a link can have many different link diagrams.

**Definition 13** *Two link diagrams $\Pi$ and $\Pi'$ are isotopic if there is an isotopy $h$ of $\mathbb{R}^2$ such that $h(1, \Pi) = \Pi'$. We say there is a diagram isotopy between $\Pi$ and $\Pi'$.*

Two isotopic link diagrams always represent isotopic links in $\mathbb{R}^3$.

## 4.2 Reidemeister Moves

It is not hard to see that two link diagrams $\Pi$ and $\Pi'$ also represent the same link if we can change $\Pi$ into $\Pi'$ by applying any of the following so called *Reidemeister moves* a finite number of times:



**Theorem 1** *Two link diagrams represent the same link if and only if they are related to eachother by diagram isotopies and applying Reidemeister moves a finite number of times.*

The ultimate goal of knot theory is to assign some algebraic quantity to every isotopy class of knots. One method to do this for instance, is to consider the pieces of string as linear maps so that we can replace the topological relations by algebraic relations. For instance Reidemeister II becomes:

$$R \circ R' = R' \circ R = \mathrm{Id}$$

And Reidemeister III becomes:

$$(R \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes R) \circ (R \otimes \mathrm{Id}) \quad = \quad (\mathrm{Id} \otimes R) \circ (R \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes R)$$

where $R$ and $R'$ denote the two types of crossing. This last equation is called the 'Yang-Baxter' equation. A map $R : V \otimes V \to V \otimes V$ that satisfies the Yang-Baxter equation is called an R-Matrix. It is clear now that knot theory is intimately related to finding solutions of the Yang-Baxter equation. This brings us to the theory of Hopf-Algebras and Quantum-groups. In order to describe knots we have to choose our maps and spaces such that they satisfy exactly the same relations as topological knots. In other words: we have to find solutions to the Yang-Baxter equation. In general we can find such a solution in so-called quasi-triangular Hopf-algebras. Or more precisely: these solutions are morphisms of representations of these algebras.

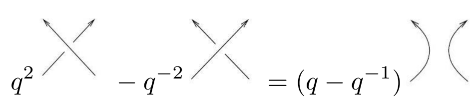
## 4.3 Invariants

So we can use quantum groups to define knot-invariants. The essential ingredient for a knot invariant is an $R$-matrix. In Appendix $B$ we show that we can find an $R$-matrix as a map from $V_1 \otimes V_1$ to itself that is linear over $U_q(sl(2))$ (it is an automorphism of a representation of a quantum group). It turns out that we can find an $R$-matrix in the same way for any representation and any quantum group $U_q(sl(n))$. The invariants constructed from quantum groups are called quantum invariants.

So we have an invariant for any choice of quantum group and any choice of representation.

### 4.3.1 The Jones Polynomial

Probably the most famous quantum invariant is the Jones polynomial $J(q)$. It is determined by the quantum group $U_q(sl(2))$ and its two-dimensional representation $V_1$. So this is exactly the case of the example above. Equation (71) determines its skein relations:



The skein relations are, together with its value on the unknot and its behavior under disjoint union, enough to determine the Jones polynomial for any knot. For the unknot it is defined as:

$$J_O(q) = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}$$

which is the trace of $K$ in the representation $V_1$. (To be precise: this is the *unnormalized* Jones polynomial. The actual Jones polynomial is obtained after dividing the unnormalized Jones polynomial by $q + q^{-1}$.)

If a knot $K$ has Jones Polynomial $J_K$ and $K'$ is the mirror image of $K$, then

$$J_{K'}(q) = J_K(q^{-1})$$

This follows directly from the skein relations. Replacing a knot by its mirror image is in fact the same as replacing all positive crossings by negative ones, and vice versa. In the skein relations we see that this is in turn the same as replacing $q$ by $q^{-1}$. In the same way we see that the Jones polynomial of a knot remains unchanged after a change of orientation.

For a link $L$ which is the disjoint, unknotted union of two knots $K_1$ and $K_2$ we have for its unnormalized Jones polynomial:

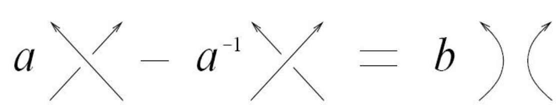
$$J_{K_1 \sqcup K_2} = J_{K_1} \cdot J_{K_2}$$

#### 4.3.2 The Colored Jones Polynomial

However we can choose any representation $V_n$ of $U_q(sl(2))$ to label the endpoints of a braid. This representation can differ even per endpoint. The resulting invariant is then what we call the colored Jones polynomial.

#### 4.3.3 The Homfly Polynomial

The Homfly polynomial $P_K(a, b)$ is defined by the following skein relations:




If we use the special case $a = q^n$, $b = q - q^{-1}$ then this is the quantum invariant $P_{n,K}(q)$ of the quantum group $U_q(sl(n))$ and its fundamental $n$-dimensional representation. Its value on the unknot is:

$$P_{n,O} = \frac{q^n - q^{-n}}{q - q^{-1}}$$

For the mirror image $K'$ of $K$ we have again:

$$P_{n,K'}(q) = P_{n,K}(q^{-1})$$

It is also invariant under orientation change, and is multiplicative with respect to disjoint unknotted union:

$$P_{n,K_1 \sqcup K_2} = P_{n,K_1} \cdot P_{n,K_2}$$

Notice that the $n = 2$ specialization of the homfly-polynomial is the Jones polynomial.

### 4.3.4 The Kauffman Bracket

A polynomial related to the Jones polynomial is the Kauffman bracket. The Kauffman bracket itself is not knot invariant, since it changes under Reidemeister moves. However, after a certain normalization depending only on the number of positive and negative crossing points it can be changed into the Jones polynomial. It uses the following skein relations:

These skein relations lead to a summation over $2^n$ terms where $n$ is the number of crossing points in the diagram. It is related to the unnormalized Jones polynomial in the following way:

$$J_K = (-1)^{n_-} q^{n_+ - 2n_-} \langle K \rangle$$

where $n_+$ is the number of positive crossings and $n_-$ the number of negative crossings. Since the Kauffman bracket plays an essential role in Khovanov homology we'll come back to it later.

# 5 Categorification

## 5.1 Product and Co-product

In their article Baez and Dolan [1] describe how one can categorify positive integers and positive rationals using finite sets and groupoids. Here follows a quick overview.

Categorification comes down to replacing elements of a set by objects and replacing relations between elements by (iso)morphisms in some kind of category. In reverse one can decategorify a category, which means replacing morphisms by relations such that isomorphisms are replaced by equalities. In other words: after decategorification one treats an isomorphism class of objects as one single object. As a consequence information is lost after decategorification. The authors make the important remark that one can view the set of positive integers as the decategorification of the category of finite sets, *by definition*. The natural numbers were invented as a means to compare finite sets (checking wether they are isomorphic or not), without actually establishing an isomorphism between them directly. Then later on one used the operations on natural numbers to invent negative, rational, and complex numbers. These can not be directly defined as the decategorification of objects in the category of finite sets. However, one can try to find categories which do behave like the categorifications of these non-natural numbers.

**Definition 14** *The category of finite sets $FinSet$ is the category consisting of finite sets as objects and maps between them as morphisms. In particular this means that two finite sets are isomorphic if and only if they have the same number of elements.*

Whenever we have a finite set $A$, we denote the cardinality of $A$ by $|A|$.

Of course the natural numbers do not just form a set, there are operations defined on it. With these operations (addition and multiplication) $\mathbb{N}$ forms a so called commutative rig. It is not a ring because it does not allow for subtraction.

**Definition 15** *A rig is a set that allows addition and multiplication, but does not necessarily allow for subtraction. In other words, it is a ring without additive inverses (negative numbers).*

Now these operations descend directly from operations that are already defined at the level of the category of finite sets. Namely taking the disjoint union of two sets and taking the cartesian product. If $A$ and $B$ are two finite sets then we denote the disjoint union by $A+B$. If we consider the natural numbers as the decategorification of the category of finite sets then we can define addition of natural numbers as the decategorification of disjoint union. In other words:

$$|A| + |B| \quad := \quad |A + B|$$

Multiplication can be defined in a similar way:

$$|A| \cdot |B| \quad := \quad |A \times B|$$

Where $\times$ denotes the cartesian product. The notion of disjoint union and cartesian product can be generalized to arbitrary categories.

**Definition 16** *Let $A$ and $B$ be two objects of an arbitrary category. The coproduct of $A$ and $B$ is an object $A + B$ equipped with inclusion morphisms $i : A \to A + B$ and $j : B \to A + B$ such that for any morphisms $f : A \to X$, $g : B \to X$ there exists a unique morphism $h : A + B \to X$ making the the following diagram commute:*

$$\begin{tikzcd}
A \arrow[dr, "i"'] \arrow[drr, "f"] & & \\
& A + B \arrow[r, "h"] & X \\
B \arrow[ur, "j"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

What does this mean? It means that a pair of objects $(A, B)$ can be viewed as an object itself (which we denote by $A + B$) and any pair of morphisms $(f, g)$, from $A$ and $B$ respectively, to $X$ can be viewed as one morphism from $A + B$ to $X$.

We can define a similar concept for pairs of morphisms *to* a pair of objects:

**Definition 17** *If $S$ and $T$ are two objects of an arbitrary category then their product is an object $S \times T$ together with projection morphisms $p : S \times T \to S$, $q : S \times T \to T$ such that for any morphisms $f : X \to S$ and $g : X \to T$ there is a unique morphism $h : X \to S \times T$ making the following diagram commute:*

$$\begin{tikzcd}
S & & \\
& S \times T \arrow[ul, "p"] \arrow[dl, "q"'] & X \arrow[l, "h"'] \arrow[ull, "f"'] \arrow[dll, "g"] \\
T & &
\end{tikzcd}$$

So a pair of objects $(S, T)$ can be viewed as one object $S \times T$ such that any pair of morphisms $(f, g)$ to $S$ and $T$ respectively can be viewed as one morphism to $S \times T$.

One can check that the co-product in the category of finite sets is disjoint union and that the product is given by the cartesian product. For vector spaces we have the peculiar fact that the product and the co-product have the same underlying object. They are both the direct sum:

$$A + B = A \times B = A \oplus B$$

The embedding morphisms are $i : x \mapsto (x, 0)$ and $j : x \mapsto (0, x)$. And the projection morphisms are: $p : (x, y) \mapsto x$ and $q : (x, y) \mapsto y$.

This is very important to us, because it means that if we have a pair of maps $(f, g)$ to $A$ and $B$ and a pair of maps $(h, k)$ from $A$ and $B$ we can compose them. The square

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ g \downarrow & & \downarrow h \\ B & \xrightarrow{k} & Y \end{array}$$

can be viewed as the composition

$$X \xrightarrow{f \oplus g} A \oplus B \xrightarrow{h+k} Y$$

(When $f$ and $g$ have different target spaces we will write their sum as $f \oplus g$, while we will write $f + g$ when they have the same target space.)

## 5.2 Groupoids

Baez and Dolan also show us a way to categorify the rational numbers. So how do they do this? Well, think about what division actually means. The equation $6/2 = 3$ means that if we have a set of six objects and we divide it in equal subsets of two elements then we are left with three such subsets. Stated more mathematically:

If we let the group $\mathbb{Z}/2$ act freely on a set of six elements then this set consists of three orbits.

So we might be able to categorify the rational numbers using finite sets and groups acting freely on them. We can consider such a set with group action as a category in which the elements serve as objects and the morphisms are the group actions. Since group elements are always invertible these morphisms are actually isomorphisms, making the category a groupoid.

**Definition 18** *A groupoid is a category in which all morphisms are isomorphisms.*

However there is not always a free action possible. For instance when we let the group of two elements act on a set of five elements. In that case at least one element will be mapped to itself under the nontrivial group element. So we need a way to count this element as 'half' an element. Therefore Baez & Dolan define the "weak quotient" :

**Definition 19** *the weak quotient $S//G$ of a set $S$ and a group $G$ is the groupoid whose objects are the elements of $S$ and with morphisms $g : s \to s'$ whenever there is a group element $g \in G$ such that $g(s) = s'$*

And they define the cardinality of the weak quotient (or any groupoid) as follows: for every isomorphism class we pick a representative object $x$ and compute the reciprocal of the number of automorphisms of $x$. This is then summated over all isomorphism classes.

**Definition 20** *Let $A$ be a groupoid. The cardinality $|A|$ of $A$ is defined as:*

$$|A| = \sum_{iso\ classes} \frac{1}{|aut(x)|}$$

A groupoid is called *tame* whenever this sum converges. The cardinality then satisfies the following nice relations:

$$\begin{array}{rcl} |S//G| & = & |S|\ /\ |G| \\ |A+B| & = & |A| + |B| \\ |A \times B| & = & |A| \cdot |B| \end{array}$$

Notice here that every positive rational number is now the decategorification of a finite groupoid and for every finite groupoid the cardinality is a positive rational number. So the set of positive rationals is categorified by the category of finite groupoids. This might cause a little confusion since a groupoid is itself already a category. Notice that isomorphic groupoids are mapped to the same rational number under decategorification. However, this map is not injective. A rational number can be categorified by multiple non-isomorphic groupoids. More generally the set of positive reals can be categorified in the same way by the category of tame (not necessarily finite) groupoids.

### 5.3 Vector Spaces Instead Of Sets

In the case of Khovanov Homology however, we are not working with finite sets but with vector spaces (or modules). A natural number $n$ can be categorified by a vector space of dimension $n$ over some ground field. Equality of two numbers is replaced by linear isomorphism of two vector spaces. Addition of numbers is categorified by direct summation, which is indeed the co-product in the category of vector spaces. Multiplication however is categorified by taking a tensor product which is not the category-theoretical product. This is not really a problem since after close examination it appears that the category of finite dimensional vector spaces contains all information (and more) that is also present at the category of finite groupoids. To clarify this we first define the 'restricted category of vector spaces' $rVect(k)$.

**Definition 21** *The objects of $rVect(k)$. are finite dimensional vector spaces that are equipped with a given basis. The morphisms in this category are the linear maps that map the basis of one vector space to the basis of another vector space. This means that an automorphism in $rVect(k)$ simply permutes the basis vectors.*

We can now define a functor from the restricted category of vector spaces to the category of finite sets. This functor maps a vector space to its basis. A morphism in $rVect(k)$ can be completely described as a map of one basis to another basis so under this functor such a morphism is a naturally mapped to its restriction on the basis. Moreover, the direct sum of two vector spaces is mapped to the disjoint union of its two bases and the tensor product of two vector spaces is mapped to the cartesian product of the two bases.

The category of finite vector spaces $Vect(k)$ is of course the same as the restricted category of vector spaces, only with more morphisms. And $rVect(k)$ is in fact equivalent to the category of finite sets. We see that $Vect(k)$ is just $FinSet$, with extra morphisms. These extra morphisms give us a more powerful category, however they are also responsible for the fact that the tensor product is no longer the category-theoretical product.

## 5.4 Negative Integers

The following consists of informal language. This will be rewritten however in a more formal way in the next section.

Inspired by Baez & Dolan's article I wondered how we could possibly categorify negative numbers. The problem lies in the interpretation of a 'set with negative cardinality' or a 'vector space of negative dimension'. To resolve this problem let's ask ourselves how we in daily life deal with negative numbers. What does it mean for instance when you say you have a negative bank account? You certainly do not literally have a negative amount of money. In fact it means that you have borrowed an amount of money from the bank which you'll have to pay back. You need to put money on your account to get it back to zero. The bank acts here as a 'bulk' from which money can be borrowed. This may inspire us to describe it in a mathematical way.

Instead of working directly with an $n$-dimensional vector space we will work with the direct sum of this space and a multidimensional 'bulk space' from which we can 'borrow' dimensions. Let's say for instance that this bulk space is 100-dimensional.

$$V^{3,eff} \quad := \quad B^{100} \oplus V^3$$

Here $V^3$ denotes a 3-dimensional vector space. $B^{100}$ denotes our bulk space. Then $V^{3,eff}$ is 103-dimensional, but we say it is *effectively 3-dimensional*. This means that in the definition of the effective dimension we do not count the dimensions of the bulk space (this will be formalized in the next section). In other words: the effective dimension is the co-dimension of the bulk space.

Now we want to calculate the effective dimension of the direct sum of two vector spaces:

$$V^{3,eff} \oplus V^{2,eff} \quad = \quad (B^{100} \oplus V^3) \oplus (B^{100} \oplus V^2) \quad \cong \quad B^{200} \oplus V^5$$

When we take the direct sum of $V^{3,eff}$ and $V^{2,eff}$ we get a 205-dimensional vector space of which the bulk space is 200-dimensional, so the effective dimension is 5:

$$V^{3,eff} \oplus V^{2,eff} \quad \cong \quad V^{5,eff}$$

The advantage of this is that we can now define vector spaces of negative effective dimension.

$$V^{-3,eff} \quad := \quad B^{97}$$

Here $B^{97}$ is a 97-dimensional bulk space. And since by our own definition the bulk space is supposed to be 100-dimensional we say $B^{97}$ has effective dimension $-3$. Now when we take the direct sum of a positive and a negative effective dimension vector space we get:

$$V^{-3,eff} \oplus V^{4,eff} \quad = \quad B^{97} \oplus B^{100} \oplus V^4 \quad \cong \quad B^{197} \oplus V^4$$

a vector space consisting of 197 bulk dimensions plus 4 ordinary dimensions. However, like in a bank account, we have to 'pay back' our borrowed dimensions. Remembering that we have taken the direct sum of *two* vector spaces we know that our bulk space is supposed to be a 200-dimensional space, so we make use of the following isomorphism:

$$B^{197} \oplus V^4 \quad \cong \quad B^{200} \oplus V^1 \quad = \quad V^{1,eff}$$

So we end up with a vector space of effective dimension 1:

$$V^{-3,eff} \oplus V^{4,eff} \quad \cong \quad V^{1,eff}$$

## 5.5 More Formal

Now this obviously works, however it is not very elegantly formulated. How for instance do we know which subspaces are 'bulk spaces' and which are the effective ones? In the previous section we only knew this from the notation. Moreover, how do we know how much dimensions have to be 'paid back' to the bulk? In the previous we knew this only from 'remembering' that we originally had two vector spaces so the direct sum should contain a 200-dimensional bulk.

We will resolve these problems right now. We will discriminate between the bulk dimensions and the effective dimensions by equipping $V^{3,eff}$ with an inclusion map that embeds the bulk space into the entire vector space:

$$V'^{3,eff} \quad := \quad B^{100} \hookrightarrow B^{100} \oplus V^3$$

So $V'^{3,eff}$ is now not just a vector space, but a vector space equipped with the inclusion of a subspace. We now don't even really have to make a distinction anymore between the bulk space and the rest. The bulk space is just a vector space embedded in a larger vector space. The orthoplement of this subspace determines the effective dimension. We could say the inclusion map 'points out' which subspace forms the bulk space. This then automatically works out for direct sums just as we would want it to:

$$V'^{3,eff} \oplus V'^{2,eff} \quad = \quad B^{100} \oplus B^{100} \hookrightarrow B^{100} \oplus V^3 \oplus B^{100} \oplus V^2$$

The inclusion map here is just the direct sum of the two inclusion maps of $V'^{3,eff}$ and $V'^{2,eff}$. This map then automatically embeds a 200-dimensional space into a 205-dimensional vector space so the orthoplement is 5-dimensional.

Moreover, we can do a similar thing for negative effective dimensions:

$$V'^{-3,eff} \quad := \quad B^{100} \to B^{97}$$

Here we have a projection map that projects a 100-dimensional space onto a 97-dimensional subspace, hence the kernel is 3-dimensional.

We can rewrite these maps more suggestively as chain complexes:

$$C^{3,eff} \quad := \quad 0 \to B^{100} \to B^{100} \oplus V^3 \to 0$$

$$C^{-3,eff} \quad := \quad 0 \to B^{100} \to B^{97} \to 0$$

Then we see that the 'effective dimension' is now nothing more then the Euler characteristic of the complex!

This becomes even clearer when we take the direct sum between a complex with positive effective dimension and one with a negative effective dimension:

$$C^{4,eff} \oplus C^{-3,eff} \quad = \quad 0 \to B^{100} \oplus B^{100} \to B^{100} \oplus V^4 \oplus B^{97} \to 0$$

then the Euler characteristic of this complex is the alternating sum of the dimensions of the chain spaces: $201 - 200 = 1$. So the effective dimension of this sum is indeed 1.

Notice that this construction is very similar to the construction of the Grothendieck group in section 3.5. However, the difference here is that we do not only have two vector spaces, but we also have a map between them which makes it possible to store more information.

We define the height of the first non-trivial chain space here as $i = 1$ and the height of the second non-trivial chain space as $i = 2$. Without such a definition the Euler characteristic would only be defined up to a minus sign.

Right now we can just as well forget about terms like 'bulk spaces' and 'effective dimensions'. Also, the dimension of the bulk space (which we took here to be 100 purely as an example) has become completely irrelevant. The only thing we need to remember is that any integer number $n$ can be categorified by a chain complex such that the Euler characteristic equals $n$. Taking sums can then be categorified by taking the direct sum between the complexes (although later on we will see that this does not necessarily have to be so).

We started these notes with the intention of categorifying the integers in a more natural, intuitive way then we did in the introduction. However, to my own surprise, we have ended up with exactly the same thing. Apparently the use of chain complexes and Euler characteristic to categorify integer numbers is a lot more natural then I originally expected.

## 5.6 The Category Of Chain Complexes.

In this section we will use the convention that a chain map between chain spaces $C^i$ and $C^{i+1}$ is denoted by $c^i$ and similarly the chain map between spaces $D^i$ and $D^{i+1}$ is denoted by $d^i$.

The objects of the category of chain complexes are, obviously, chain complexes.

**Definition 22** *A morphism between two chain complexes $C$ and $D$ is a set of linear maps $f^j$ between the respective chain spaces $C^j$ and $D^j$ such that they commute with the chain maps.*

In other words, the following diagram commutes:

$$\begin{array}{ccccccc} ... & C^i & \rightarrow & C^{i+1} & \rightarrow & C^{i+2} & ... \\ & \downarrow & & \downarrow & & \downarrow & \\ ... & D^i & \rightarrow & D^{i+1} & \rightarrow & D^{i+2} & ... \end{array}$$

Here the downward pointing arrows denote the respective maps $f^j$.

**Lemma 4** *The fact that the diagram commutes makes sure that the kernel of any chain map $c^j$ is mapped by $f^j$ into the kernel of $d^j$ and the image of any $c^j$ is mapped into the image of $d^j$. A morphism of chain complexes then induces a homomorphism between the respective homology groups.*

**Definition 23** *We say that two morphisms $f$ and $g$ are homotopic if there are morphisms $h^i : C^i \rightarrow D^{i-1}$ such that:*

$$f^i - g^i = h^{i+1} \circ d^i + d^{i+1} \circ h^i$$

*for every $i$.*

**Lemma 5** *If $f$ and $g$ are homotopic, then they induce the same homomorphism on homology.*

This means particularly that if a morphism $f$ is homotopic to the identity, it induces an isomorphism on the homology groups.

**Definition 24** *When a morphism is homotopic to the zero morphism it is called null-homotopic.*

In this category we can distinguish between several 'degrees of equality':

**Isomorphism.** Like in any other category we say two objects $x$ and $y$ (in this case chain complexes) are isomorphic whenever there are two morphisms $f : x \to y$ and $g : y \to x$ such that $fg = gf = \mathrm{Id}$.

**Quasi-Isomorphism.** Two chain complexes are quasi-isomorphic whenever there exists a morphism between them such that it induces an isomorphism of their homology groups.

**Equal Euler characteristic.** Two chain complexes might have the same Euler characteristic although they have completely different homologies.

Notice that isomorphism implies quasi-isomorphism, and quasi-isomorphism implies equal Euler characteristic. In Khovanov's theory to every knot-diagram there is assigned a unique (up to isomorphism) chain complex. Whenever two different diagrams represent the same knot, their respective complexes are quasi-isomorphic. In other words: the homology of the Khovanov complex is a knot-invariant. Whenever two diagrams belong to different knots that have the same Jones polynomial their complexes have equal Euler characteristic (to be more precise: equal *graded* Euler characteristic, since Khovanov works with graded vector spaces). In other words: Khovanov homology is a categorification of the Jones polynomial, where taking the Euler characteristic serves as the decategorification.

So in Khovanov homology we are not really interested in isomorphism, but rather in quasi-isomorphism. Moreover, the whole point of this categorification is that some chain-complexes are *not* quasi-isomorphic although they do have the same Euler characteristic.

### 5.6.1 Summation

One can easily check that the co-product in this category is given by direct summation. That is: the co-product of two complexes $C$ and $D$ consists of chain spaces $(C \oplus D)^j$ which are the direct sums of the respective chain spaces of $C$ and $D$ of height $j$:

$$(C \oplus D)^j \quad = \quad C^j \oplus D^j$$

The differentials are also just the direct summations of the respective chain maps:

$$(c \oplus d)^j(x, y) \quad = \quad (c^j(x), d^j(y))$$

Diagrammatically:

$$\begin{array}{ccccccc} \ldots & C^i & \to & C^{i+1} & \to & C^{i+2} & \ldots \\ & \oplus & & \oplus & & \oplus & \\ \ldots & D^i & \to & D^{i+1} & \to & D^{i+2} & \ldots \end{array}$$

The inclusion morphism $f$ that embeds $C$ into $C \oplus D$ is simply the set of inclusion maps that embed $C^j$ into $C^j \oplus D^j$:

$$f^j : C^j \hookrightarrow C^j \oplus D^j$$

And in the same way we have the inclusion morphism $g : D \hookrightarrow C \oplus D$ consisting of the inclusion maps

$$g^j : D^j \hookrightarrow C^j \oplus D^j$$

**Lemma 6** *If we denote the Euler characteristic of a chain complex $C$ by $|C|$, then we have:*

$$|C \oplus D| \quad = \quad |C| + |D|$$

### 5.6.2 Subtraction

In the category of chain complexes we can define the height shift operator. It maps any chain complex $C$ to another chain complex denoted $C[s]$.

**Definition 25** *the height shift operator $[s]$ shifts the height of the chain spaces by s, where $s$ is an integer. That is: $C[s]^j = C^{j-s}$.*

For the Euler characteristic we then have: $|C[s]| = (-1)^s |C|$. This can be used to categorify subtraction, because we have:

$$|C \oplus D[1]| \quad = \quad |C| - |D|$$

The direct sum of of chain complexes $C$ and $D[1]$ can be generalized by allowing cross maps $a^j$ from $C^j$ to $D^j$, that anti-commute with the chain maps, for every $j$.

$$\begin{array}{ccccccc} ... & C^i & \rightarrow & C^{i+1} & \rightarrow & C^{i+2} & ... \\ & \oplus & \searrow & \oplus & \searrow & \oplus & \\ ... & D^{i-1} & \rightarrow & D^i & \rightarrow & D^{i+1} & ... \end{array}$$

The fact that the cross maps anti-commute with the chain maps makes sure that the resulting sequence of maps is again a chain complex. This complex is also denoted by $C \rightarrow D$, or by:

$$\begin{array}{ccccccc} ... & C^i & \rightarrow & C^{i+1} & \rightarrow & C^{i+2} & ... \\ & \downarrow & & \downarrow & & \downarrow & \\ ... & D^i & \rightarrow & D^{i+1} & \rightarrow & D^{i+2} & ... \end{array}$$

It is indeed a generalization, since if we take the cross maps to be zero we retrieve direct summation. This operation is called *flattening*.

**Definition 26** *The flattening (or also the cone) of $C$ and $D$ with respect to a morphism $f$ is a chain complex with chain spaces $C^i \oplus D^{i-1}$ and chain maps $c^i \oplus (d^{i-1} + (-1)^i f^i)$. The height of a chain space $C^i \oplus D^{i-1}$ is defined to be $i$.*

It is important to notice that the Euler characteristic of $C \to D$ is totally independent of the cross maps because it can be calculated from the dimensions of the chain spaces. So we have:

$$|C \to D| \quad = \quad \sum_i (-1)^i \dim(C^i \oplus D^{i-1}) \quad = \quad |C| - |D|$$

By choosing the cross maps carefully Khovanov managed to make sure that not only the Euler characteristic, but also even the homology becomes a knot invariant. If we used ordinary direct summation we would get a theory equivalent to the Kauffman bracket.

**Definition 27** *If all homology groups of a chain complex $C$ are trivial ($H^i = \{0\}$) we say the chain complex is contractible*

**Lemma 7** *If $f$ is an isomorphism between chain complexes $C$ and $D$ then the flattening of $C$ and $D$ with respect to $f$ is contractible.*

Proof: Suppose $x \in C^i$ and $y \in D^{i-1}$ then $(x, y)$ is mapped to $(c^i(x),\ d^{i-1}(y) + (-1)^i f^i(x))$. Since $f$ is and isomorphism we know there is a $z \in C^{i-1}$ such that $(-1)^{i-1} f^{i-1}(z) = y$. Because $f$ anti-commutes with $c$ and $d$ we have:

$$d^{i-1}(y) \ = \ d^{i-1}((-1)^{i-1} f^{i-1}(z)) \ = \ -(-1)^i f^i(c^{i-1}(z))$$

If we now assume that $(x, y)$ is in the kernel of the flattening we have:

$$(\ c^i(x),\ d^{i-1}(y) + (-1)^i f^i(x)\ ) \ = \ (0, 0)$$

specifically this means: $d^{i-1}(y) = -(-1)^i f^i(x)$, so we have now:

$$-(-1)^i f^i(x) \ = \ -(-1)^i f^i(c^{i-1}(z))$$

Once again using the fact that $f$ is an isomorphism, we have:

$$x \ = \ c^{i-1}(z)$$

which means that $(z, 0)$ is mapped to $(\ c^{i-1}(z),\ (-1)^{i-1} f^{i-1}(z) + d^{i-2}(0)\ ) = (x, y)$. This means that $(x, y)$ is in the image, which is exactly what we wanted to prove. $\square$

Since $C \to D$ is itself again a chain complex, we can again take its flattening with another chain complex $A \to B$:

$$(C \to D) \to (A \to B)$$

which is a 'square' of chain complexes:

$$\begin{array}{ccc} C & \to & D \\ \downarrow & & \downarrow \\ A & \to & B \end{array}$$

By repeatedly doing this we can even create 'cubes' and 'hyper-cubes' (of arbitrary dimension) of chain complexes.

Notice that a vector space can be seen as a chain complex with only one non-trivial chain space and that the the Euler characteristic of this complex is then the dimension of the vector space.

So we can categorify a natural number $n$ by a vector space $V^n$ of dimension $n$, or equally by a complex $C$ with $|C| = n$:

$$C \quad = \quad 0 \rightarrow V^n \rightarrow 0$$

Then an expression like $n-m$ can be categorified by: $C \oplus D[1]$ or, more generally, by $C \rightarrow D$ for some morphism. We see that chain complexes can be used not only to categorify integer numbers, but even entire expressions like $(m-n)-(k-l)$ which becomes $(C \rightarrow D) \rightarrow (A \rightarrow B)$, with $|C| = m$, $|D| = n$, $|A| = k$ and $|B| = l$.

**Definition 28** *A cube is a collection of $2^n$ objects (known as the vertices of the cube) in some category, labelled by binary numbers $\alpha$ consisting of $n$ bits. The integer $n$ is called the dimension of the cube.*

**Definition 29** *Two vertices labelled by $\alpha$ and $\beta$ are called neighbors whenever $\alpha$ and $\beta$ differ at precisely 1 bit. If this bit has value $0$ in $\alpha$ and value $1$ in $\beta$ we write $\alpha \sim \beta$.*

**Definition 30** *The height $|\alpha|$ of a vertex labelled by $\alpha$ is the sum of all bits of $\alpha$.*

For example: if a vertex is labelled by $\alpha = 10110$ then its height is: $|\alpha| = 1+0+1+1+0 = 3$. If another vertex is labelled by $\beta = 10111$ then $\alpha$ and $\beta$ are neighbors.

**Definition 31** *An anti-commutative cube $C$ is a cube of vector spaces $C_\alpha$ with for every pair of neighbors $\alpha \sim \beta$ a linear map $c_{\alpha\beta} : C_\alpha \rightarrow C_\beta$ (known as an edge of the cube), such that all edges anti-commute.*

**Lemma 8** *An anti-commutative cube $C$ can be considered as as a chain complex with chain spaces:*

$$C^r = \bigoplus_{|\alpha|=r} C_\alpha$$

*and chain maps $c^r : C^r \rightarrow C^{r+1}$*

$$c^r \quad := \quad \bigoplus_{|\alpha|=r, \alpha\sim\beta} c_{\alpha\beta}$$

Proof: the proof goes by induction on $n$. For $n = 0$ and $n = 1$ this statement is obviously true. If $C$ and $D$ are anti-commutative cubes of dimension $n-1$ and we have maps $f_\alpha : C_\alpha \to D_\alpha$ such that they anti-commute with the edges of $C$ and $D$ then together they form an $n$-dimensional anti-commutative cube.

So we have for every pair of neighbors $\alpha \sim \beta$ an anti-commuting diagram:

$$\begin{array}{ccccc} & C_\alpha & \xrightarrow{c_{\alpha\beta}} & C_\beta & \\ f_\alpha & \downarrow & & \downarrow & f_\beta \\ & D_\alpha & \xrightarrow{d_{\alpha\beta}} & D_\beta & \end{array}$$

$$f_\beta \circ c_{\alpha\beta} \quad = \quad -d_{\alpha\beta} \circ f_\alpha \tag{10}$$

These maps $f_\alpha$ then form maps $f^r$ between the chain spaces: $f^r : C^r \to D^r$

$$f^r \quad := \quad \bigoplus_{|\alpha|=r} f_\alpha$$

So we have:

$$\begin{array}{ccccc} ...C^{i-1} & \xrightarrow{c^{i-1}} & C^i & \xrightarrow{c^i} & C^{i+1}... \\ f^{i-1}\downarrow & & \downarrow f^i & & \downarrow f^{i+1} \\ ...D^{i-1} & \xrightarrow{d^{i-1}} & D^i & \xrightarrow{d^i} & D^{i+1}... \end{array}$$

Since we know by induction that the cubes $C$ and $D$ are chain complexes we are only left to prove that the maps $f^r$ anti-commute with the chain maps $c^r$ and $d^r$:

$$f^{r+1} \circ c^r \quad = \quad -d^r \circ f^r$$

because if we can prove this, the $n$-dimensional cube is the flattening $C \to D$ so it is a chain complex. Notice that the chain maps of the flattening are then exactly the chain maps as defined in the lemma.

$$f^{r+1} \circ c^r = (\bigoplus_{|\gamma|=r+1} f_\gamma) \circ (\bigoplus_{\alpha\sim\beta} c_{\alpha\beta}) = \bigoplus_{\alpha\sim\beta} f_\beta \circ c_{\alpha\beta}$$

This last equation follows from the fact that the composition of $f_\gamma$ with $c_{\alpha\beta}$ is only nonzero if $\gamma = \beta$.

$$d^r \circ f^r = (\bigoplus_{\alpha\sim\beta} d_{\alpha\beta}) \circ (\bigoplus_{\alpha} f_\alpha) = \bigoplus_{\alpha\sim\beta} d_{\alpha\beta} f_\alpha$$

We now only have to prove:

$$\bigoplus_{\alpha\sim\beta} f_\beta \circ c_{\alpha\beta} \quad = \quad -\bigoplus_{\alpha\sim\beta} d_{\alpha\beta} f_\alpha$$

The summations on both sides of the equation are however over exactly the same set of pairs $(\alpha, \beta)$ so this follows directly from (10). $\square$

Notice that the two definitions of 'height' are in this case equivalent: the height $r$ of a chain space $C^r$ is exactly the height $|\alpha|$ of the vertices $\alpha$ that the chain space consists of.

### 5.6.3 Multiplication

We have seen that in the category of vector spaces we have the tensor product as a categorification of multiplication. In the same way we can 'multiply' chain complexes. First we define the multiplication of a chain complex $C$ with a vector space $V$, which is simply:

$$C \otimes V \quad := \quad ...C^1 \otimes V \to C^2 \otimes V \to C^3 \otimes V...$$

where the chain maps are given by $c^j \otimes \mathrm{Id}_V$. If we have a map $d : D^1 \to D^2$ we can define the tensor product of the complex $C$ with this map as the flattening of the two complexes $C \otimes D^1$ and $C \otimes D^2$ with respect to the morphism consisting of the chain maps $\mathrm{Id}_{C^j} \otimes d$.

$$\begin{array}{ccccccc} ... & C^1 \otimes D^1 & \xrightarrow{c^1 \otimes \mathrm{Id}} & C^2 \otimes D^1 & \xrightarrow{c^2 \otimes \mathrm{Id}} & C^3 \otimes D^1 & ... \\ & \downarrow & & \downarrow & & \downarrow & \\ ... & C^1 \otimes D^2 & \xrightarrow{c^1 \otimes \mathrm{Id}} & C^2 \otimes D^2 & \xrightarrow{c^2 \otimes \mathrm{Id}} & C^3 \otimes D^2 & ... \end{array}$$

If we generalize this to chain complexes $D$ which have more than two chain spaces we get a chain complex $C \otimes D$ with chain spaces:

$$(C \otimes D)^s \quad := \quad \bigoplus_i C^{s-i} \otimes D^i$$

and chain maps $f^s : (C \otimes D)^s \to (C \otimes D)^{s+1}$ defined by:

$$f^s \quad := \quad \bigoplus_i c^{s-i} \otimes \mathrm{Id} \ + (-1)^i \ \mathrm{Id} \otimes d^i$$

**Lemma 9** $|C \otimes D| \quad = \quad |C| \cdot |D|$

Proof:

$$\begin{aligned} |C \otimes D| &= \sum_s (-1)^s \dim(\ (C \otimes D)^s\ ) \\ &= \sum_s (-1)^s \sum_i \dim(\ C^{s-i} \otimes D^i\ ) \\ &= \sum_s (-1)^s \sum_i \dim(\ C^{s-i}\ ) \dim(\ D^i\ ) \\ &= \sum_i \dim(\ D^i\ ) \sum_s (-1)^s \dim(\ C^{s-i}\ ) \\ &= \sum_i \dim(\ D^i\ )(-1)^i \sum_s (-1)^{s-i} \dim(\ C^{s-i}\ ) \\ &= \sum_i \dim(\ D^i\ )(-1)^i \cdot |C| \\ &= |C| \cdot |D| \qquad \Box \end{aligned}$$

## 5.7 How homology transforms under flattening

We will assume all vector spaces in this section to be finite dimensional. Suppose we have a linear map $f : A \to C$ and a linear map $g : B \to C$.

$$\begin{array}{ccc} A & f & \\ & \searrow & \\ & & C \\ & \nearrow & \\ B & g & \end{array}$$

We have seen in section 5.1 that there is then a unique map:

$$f + g : \quad A \oplus B \to C$$

We would like to know the kernel and the image of this map now.

### 5.7.1 ker$(f + g)$

Notice that if we choose a vector $x \in \ker(f)$ and a vector $y \in \ker(g)$ then we certainly have

$$(f+g)(x,y) \quad = \quad f(x) + g(y) \quad = \quad 0 + 0 \quad = \quad 0$$

So $\ker(f) \oplus \ker(g) \subset \ker(f+g)$, but it is also possible that we have vectors $x$ and $y$ for which: $f(x) = -g(y) \neq 0$. So we have:

$$\ker(f+g) \quad = \quad \ker(f) \oplus \ker(g) \oplus V \tag{11}$$

with:

$$V \quad = \quad \{\ (x,y) \mid f(x) = -g(y) \neq 0\ \} \tag{12}$$

If we choose a vector $x \in A$ then we can find a pair $(x, y) \in V$ if $f(x) \in \mathrm{Im}(g)$. We then have $y = -g^{-1}(f(x))$. Notice however that $y$ is then not unique. After all we could replace $y$ by $y + y_0$ with $y_0 \in \ker(g)$. Therefore we have:

$$V \quad \cong \quad g^{-1}(\mathrm{Im}(f) \cap \mathrm{Im}(g))\ /\ \ker(g)$$

And from symmetry we see also:

$$V \quad \cong \quad f^{-1}(\mathrm{Im}(f) \cap \mathrm{Im}(g))\ /\ \ker(f)$$

it follows that:

$$\dim(V) \quad = \quad \dim(\mathrm{Im}(f) \cap \mathrm{Im}(g)) \tag{13}$$

Therefore we have:

$$\dim(\ker(f+g)) \ = \ \dim(\ker(f)) \ + \ \dim(\ker(g)) \ + \ \dim(\mathrm{Im}(f) \cap \mathrm{Im}(g)) \tag{14}$$

### 5.7.2 $\mathbf{Im}(f+g)$

For the image of $f+g$ we have:

$$\begin{aligned}\mathrm{Im}(f+g) &= \{v+w \mid v \in \mathrm{Im}(f), w \in \mathrm{Im}(g)\} && (15)\\ &\cong (\mathrm{Im}(f) \oplus \mathrm{Im}(g)) \;/\; (\mathrm{Im}(f) \cap \mathrm{Im}(g)) && (16)\end{aligned}$$

So:

$$\dim(\mathrm{Im}(f+g)) \;=\; \dim(\mathrm{Im}(f)) \;+\; \dim(\mathrm{Im}(g)) \;-\; \dim(\;\mathrm{Im}(f) \cap \mathrm{Im}(g)\;) \quad (17)$$

### 5.7.3 $\mathbf{ker}(u \oplus l)$

Suppose now we have two maps $u: X \to S$ and $l: X \to T$.

$$\begin{array}{ccc} & u & S \\ & \nearrow & \\ X & & \\ & \searrow & \\ & l & T \end{array}$$

Then there is a unique linear map

$$u \oplus l: \quad X \to S \oplus T$$

The kernel of this map is simply:

$$\ker(u \oplus l) \;=\; \ker(u) \cap \ker(l) \quad (18)$$

### 5.7.4 $\mathbf{Im}(u \oplus l)$

From basic linear algebra it then follows that

$$\mathrm{Im}(u \oplus l) \;\cong\; X \;/\; (\;\ker(u) \cap \ker(l)\;)$$

This can also be seen from the fact that two vectors $x$ and $x'$ in $X$ are mapped to the same vector if and only if $x - x' \in \ker(u) \cap \ker(l)$. So:

$$\dim(\mathrm{Im}(u \oplus l) \;=\; \dim(X) - \dim(\;\ker(u) \cap \ker(l)\;) \quad (19)$$

### 5.7.5 Calculating Betti-numbers after flattening

Suppose we are given two chain complexes:

$$0 \to C^1 \xrightarrow{c} C^2 \to 0$$

$$0 \to D^1 \xrightarrow{d} D^2 \to 0$$

And we want to calculate the Betti-numbers of the flattening $C \to D$ with respect to any morphism $f$. We denote the homology groups of $C$ and $D$ by $H_c^i$ and $H_d^i$ respectively.

$$\begin{array}{ccc} C^1 & \xrightarrow{c} & C^2 \\ f^1 \downarrow & & \downarrow f^2 \\ D^1 & \xrightarrow{d} & D^2 \end{array}$$

$$0 \rightarrow C^1 \xrightarrow{c \oplus f^1} D^1 \oplus C^2 \xrightarrow{d + f^2} D^2 \rightarrow 0$$

Then

$$H^1 \quad = \quad \ker(c \oplus f^1) \quad = \quad \ker(c) \cap \ker(f^1) \quad = \quad H_c^1 \cap \ker(f^1)$$

$$\dim(H^1) \quad = \quad \dim(H_c^1 \cap \ker(f^1)) \tag{20}$$

The third homology group is:

$$H^3 = D^2 \ / \ \mathrm{Im}(d + f^2)$$

$$\begin{aligned} \dim(H^3) &= \dim(D^2) - \dim(\mathrm{Im}(d + f^2)) \\ &= \dim(D^2) - \dim(\mathrm{Im}(d)) - \dim(\mathrm{Im}(f^2)) + \dim(\mathrm{Im}(d) \cap \mathrm{Im}(f^2)) \\ &= \dim(H_d^2) - \dim(\mathrm{Im}(f^2)) + \dim(\mathrm{Im}(d) \cap \mathrm{Im}(f^2)) \\ &= \dim(H_d^2) - \dim(\ \mathrm{Im}(f^2) \ / \ \mathrm{Im}(d) \ ) \end{aligned}$$

So we see that, in order to calculate the Betti-numbers, we need the following two variables:

$$\dim(\ \ker(c) \cap \ker(f^1)\ )$$

$$\dim(\ \mathrm{Im}(f^2) \ / \ \mathrm{Im}(d)\ )$$

The second homology group is:

$$H^2 = \ker(d + f^2) \ / \ \mathrm{Im}(c \oplus f^1)$$

However, we know that the alternating sum of the dimensions of the chain spaces is equal to the alternating sum of the dimensions of the homology groups, so we can calculate $\dim(H^2)$ as follows:

$$dim(H^2) \quad = \quad -\dim(C^1) + \dim(C^2) + \dim(D^1) - \dim(D^2) - \dim(H^1) - \dim(H^3)$$

# 6 The Kauffman Bracket

In this section we will try to derive the Kauffman bracket ourselves. In their articles ([2], [9]) Khovanov and Bar-Natan take the usual top-down approach: first they present the 'cube construction' then they use this to define a knot-invariant (which appears to be just falling from the sky). And finally they prove that this invariant indeed (miraculously) turns out to be invariant under Reidemeister moves. To better understand this we will however take a bottom-up approach. We will do this is the following way: first we will assume that our invariant depends only on the resolutions of the knot. To every resolution we will assign a polynomial. We then have a set of $2^n$ polynomials that correspond to a knot with $n$ crossing points. Next, we will examine how this set transforms under Reidemeister moves. We will introduce some new notation to write these transformations in an algebraic way. In the end we want to combine these polynomials into one polynomial that remains invariant under Reidemeister moves. But first we will look for a combination that transforms in a certain way that is independent of the chosen link (we will call this a *universal transformation*). This combination is called 'The Bracket'. Then after a normalization depending on the number of crossing points this will become an invariant, known as the Jones polynomial. It is striking to see that in this way the Jones polynomial turns out to be in some sense the most obvious possible invariant. In the next chapter we will then try to extend this discussion to Khovanov-homology.

## 6.1 The Cube Construction

Whenever we remove a crossing point from a link diagram in one of the two following ways:

$0 - \mathit{smoothing}$ $\qquad$ $1 - \mathit{smoothing}$

we say it is *'smoothed out'*. The first way is called a 0-smoothing and the second way is called a 1-smoothing.

**Definition 32** *A link diagram in which every crossing point is smoothed out is called a resolution. A resolution consists of a finite number of unknots.*

A link diagram with $n$ crossing points then has $2^n$ possible resolutions.

**Assumption 1** *We will try to obtain a knot-diagram invariant (let's call it 'The Bracket') that is only obtained from the resolutions that result from all the possible choices of smoothing out the crossing points.*

**Assumption 2** *The bracket will assign to every diagram an element of a (polynomial) ring R with 1. For the diagram of the trivial knot without crossing points we will denote this element by O.*

**Assumption 3** *The bracket of an unknotted disjoint union of several links will be the product of the brackets of the individual links.*

**Assumption 4** *The bracket of a link will be a linear combination of the brackets of its resolutions. This means that it can be written as $aO^{k_1} + bO^{k_2} + cO^{k_3}$... In other words: the bracket will have a state-sum presentation.*

For now we will not specify the ring we use because we want our discussion to be as general as possible. But we do assume it is a polynomial ring. This is not a severe restriction since any commutative ring is isomorphic to a polynomial ring modulo some relations. Maybe it would be better if we also allowed non-commutative rings, but let's keep things simple for now. Since every resolution is a disjoint unknotted union of trivial knots we already have an expression for every resolution, namely $O^k$ (where $k$ is the number of unknots in that specific resolution). For a diagram with $n$ crossing points this gives us a set of $2^n$ polynomials. Notice that this set is already a link-diagram invariant since the resolutions do not change under diagram isotopies. So all we need to do is find an algorithm to take linear combinations of these resolutions that are invariant under Reidemeister moves.

First let's start with some notation. Whenever we have an indexed set of $2^n$ polynomials (a cube) we will denote it by:

$$\{p_\alpha ...\}$$

Here $\alpha$ denotes an $n$-tuple of binary digits: $\alpha \in \{0,1\}^n$. For instance: $\alpha = (0,0,1,0,1)$. Which we will simply denote as: $\alpha = 00101$ (just like in section 5.6.2). So for $n = 2$ we have:

$$\{p_\alpha ...\} = \{p_{00}, p_{10}, p_{01}, p_{11}\}$$

Where $p_{00}, p_{10}, p_{01}$ and $p_{11}$ are polynomials. Furthermore, with $\{p_\alpha ...\}_0$ we denote the same set, only with a 0 added at the end of the indices:

$$\{p_\alpha ...\}_0 = \{p_{000}, p_{100}, p_{010}, p_{110}\}$$

In the same way we define $\{p_\alpha ...\}_1$. This way we can define the disjoint union between two cubes. Suppose we have:

$$\{p_\alpha ...\} = \{p_0, p_1\} \text{ and } \{p_\beta\} = \{p'_0, p'_1\}$$

then:

$$\{p_\gamma ...\} \quad := \quad \{p_\alpha ...\}_0 \sqcup \{p_\beta ...\}_1 \quad = \quad \{p_{00}, p_{10}, p'_{01}, p'_{11}\}$$

which we can just as well write as:

$$\{p_\gamma ...\} \quad = \quad \{p_{00}, p_{10}, p_{01}, p_{11}\}$$

When all elements of the cube are multiplied by the same polynomial $p'$ we 'take the $p'$ outside of the brackets', so we write this as:

$$p' \cdot \{p_\alpha ...\} \quad := \quad \{p' \cdot p_{00}, \quad p' \cdot p_{10}, \quad p' \cdot p_{01}, \quad p' \cdot p_{11}\}$$

Now to every knot-diagram we will assign such a cube of polynomials. First number the crossing points of the diagram from 1 to $n$ in any random way. We can replace every crossing point by either a 0-smoothing or a 1-smoothing so that it becomes a disjoint union of unknots (a resolution). This gives us $2^n$ possible resolutions of the diagram. To each resolution we will assign the polynomial $O^{k_\alpha}$ where $k_\alpha$ is the number of unknots in that resolution. So we have assigned an $n$-dimensional cube of polynomials to the diagram. We give them indices corresponding to the chosen smoothings. For instance: say we have three crossing points and we choose a 0-smoothing for the first crossing point and a 1-smoothing for the other two crossings. Then we denote the polynomial corresponding to this resolution by: $p_{011}$. If this resolution contains $k$ unknots we then have: $p_{011} = O^k$. Notice that what we have done up to now is exactly the same as the 'cube construction' that Khovanov and Bar-Natan use. We have only written it down in a different notation which will turn out to be convenient later. Let's now look for example at the Hopf-link:

**Example: the Hopf Link**

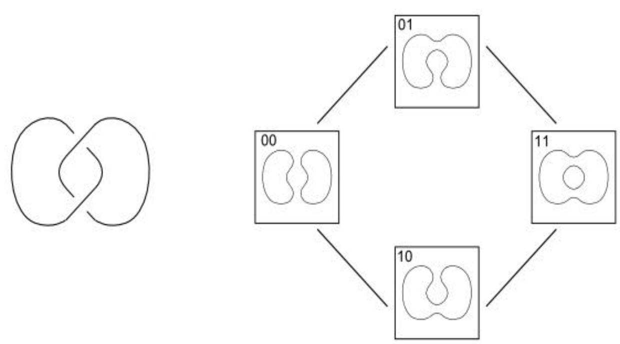



Figure 2: Hopf Link

As we can see from the picture we have: $p_{00} = O^2$, $p_{10} = p_{01} = O$, $p_{11} = O^2$. We write this as:

{} $= \{O^2, O, O, O^2\}$

**Definition 33** *We define the 'height' $r_\alpha$ of a resolution to be the sum of the binary digits of the index $\alpha$ (just like in section 5.6.2).*

We will sometimes denote the cube assigned to a diagram $D$ by $\{D\}$ , or by some diagrammatic symbol like above. This gives us an easy way of denoting the two Reidemeister I moves. Say we have a link-diagram $K$. If we perform an $RI$ move on $K$ we obtain the diagram $K'$. For $K$ we will denote its cube by: $\{\ |\ \}$ and for $K'$ it will be denoted by $\{\ \}$ or $\{\ \}$ (The first one we will call the $RIa$ move and the second one the $RIb$ move). If $K''$ is the disjoint, unknotted union of $K$ with an unknot we denote the cube of $K''$ by: $\{\ |O\ \}$. Then we see we have the following relations:

$$\{\ \} = \{\ |\ \}_0 \sqcup \{\ |O\ \}_1 = \{\ |\ \}_0 \sqcup O \cdot \{\ |\ \}_1 \tag{21}$$

$$\{\ \} = \{\ |O\ \}_0 \sqcup \{\ |\ \}_1 = O \cdot \{\ |\ \}_0 \sqcup \{\ |\ \}_1 \tag{22}$$

We see that under a Reidemeister I move the number of polynomials is doubled. This is obvious since $RI$ adds another crossing point to the diagram so the $2^n$ cube becomes a $2^{n+1}$ cube. We also see that the newly added polynomials are just copies of the original polynomials multiplied by $O$. It is not hard to verify that this equation also works the other way around:

**Lemma 10** *whenever the cube can be written like the right-hand side of (21) or (22) the diagram contains an RI twist.*

For Reidemeister II however things are a little more complicated. A Reidemeister II move adds two crossing points to the diagram. This means that if we had $2^n$ resolutions for the original diagram then the new diagram has $2^{n+2}$ resolutions. Every resolution is replaced by four new ones. The problem is that the number of unknots that are added or removed depend on the specific link and the specific resolution. So we cannot write down a general equation like (21) or (22) for the whole cube.

However, if we smooth out all crossing points that are not involved in the $RII$ move and we ignore all unknots in the resolution that are not involved we see that luckily there are only two possibilities, which we call *case 1* and *case 2*. See figures 3 and 4.

For case 1 we have:

$$p_\alpha \Rightarrow \{p'_{\alpha 00},\quad p'_{\alpha 10},\quad p'_{\alpha 01},\quad p'_{\alpha 11}\} = \{O \cdot p_\alpha,\quad O^2 \cdot p_\alpha,\quad p_\alpha,\quad O \cdot p_\alpha\} \tag{23}$$

While for case 2 we have:

$$p_\alpha \Rightarrow \{p'_{\alpha 00},\quad p'_{\alpha 10},\quad p'_{\alpha 01},\quad p'_{\alpha 11}\} = \{O^{-1} \cdot p_\alpha,\quad p_\alpha,\quad p_\alpha,\quad O^{-1} \cdot p_\alpha\} \tag{24}$$

**Lemma 11** *Under an RII move for every resolution $\alpha$ the polynomial $p_\alpha$ is replaced by either the four polynomials of (23) or the four polynomials of (24)*
□

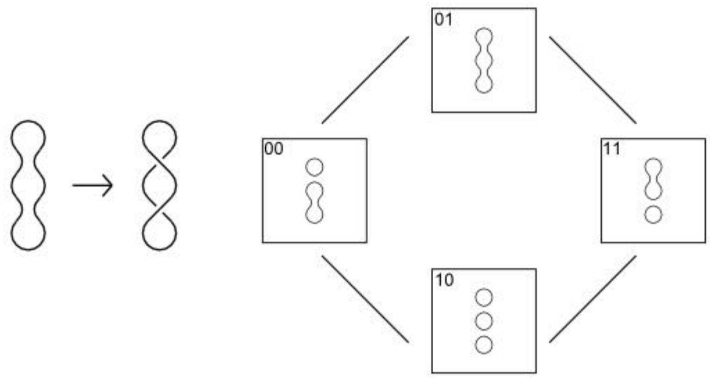



Figure 3: Reidemeister II, case 1

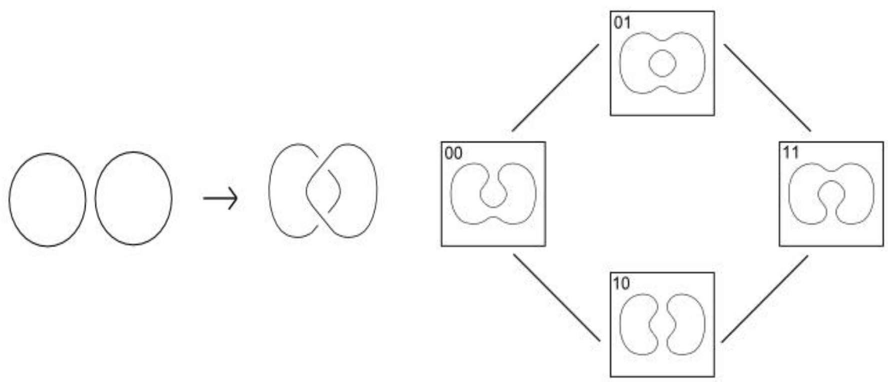



Figure 4: Reidemeister II, case 2

## 6.2 The Bracket

We now want to assign a unique polynomial to every knot-diagram. This means in particular that we want an algorithm to obtain a polynomial from the cube $\{p_\alpha ...\}$. This polynomial will be denoted by $\langle\{p_\alpha ...\}\rangle$ or $\langle p_\alpha\rangle$ and will be called 'The Bracket'.

**Definition 34** *We say that an operation that transforms a link-diagram $K$ into another link-diagram $K'$ induces a transformation of $\langle K\rangle$ into $\langle K'\rangle$.*

**Definition 35** *A universal transformation is a transformation of link-diagrams such that the induced transformation on the bracket is the same for every link.*

**Example:** any diagram-isotopy is a universal transformation. The bracket is invariant under diagram-isotopy, so in this special case the induced transformation is 'multiplication by 1'. Since this holds for any link, diagram-isotopy is indeed universal.

**Example:** If we use crossing number as a diagram invariant then the Reidemeister moves are universal transformations for this invariant. This is true since the crossing number decreases by 1 for $RIa$, it increases by 1 for $RIb$ and it stays the same for $RII$ and $RIII$. This holds for any link.

Some more examples will follow.

**Assumption 5** *We want the bracket to be invariant under the Reidemeister moves RIa, RIb, RII and RIII up to a multiplicative factor. This multiplicative factor depends only on the performed Reidemeister moves and not on the particular link. In other words: we want the Reidemeister moves to be universal transformations.*

The reason for us to make this last assumption is that it makes it possible for us to change the link-diagram invariant into a link invariant by a simple normalization.

**Example:**One could for instance try to define the bracket like this:

$$\langle\{p_\alpha ...\}\rangle \;:=\; \sum_\alpha p_\alpha \tag{25}$$

We can derive a few calculation rules from this definition:

$$\langle\{p_\alpha ...\}_0 \sqcup \{p_\beta ...\}_1\rangle \;=\; \langle\{p_\alpha ...\}\rangle + \langle\{p_\beta ...\}\rangle$$

$$\langle p' \cdot \{p_\alpha ...\}\rangle \;=\; p' \cdot \langle\{p_\alpha ...\}\rangle$$

From this and formulas (21) and (22) we then conclude:

$\langle$  $\rangle \quad = \quad \langle|\rangle + \langle|O\rangle \quad = \quad \langle|\rangle + \; O \cdot \langle|\rangle$

$\langle$  $\rangle \quad = \quad \langle|O\rangle + \langle|\rangle \quad = \quad O \cdot \langle|\rangle + \langle|\rangle$

in other words:

$\langle$  $\rangle \quad = \quad \langle$  $\rangle \quad = \quad (1+O) \cdot \langle|\rangle$

We see that this bracket gives a polynomial that is invariant up to a multiplicative factor of $(1+O)$ for both $RIa$ and $RIb$. Since we did not specify $\{\ |\ \}$ this holds for any link so we see that $RIa$ and $RIb$ are indeed universal transformations for this bracket. In the following we will see however that we will need to use a different bracket.

Since we assumed that the bracket is a linear combination of the resolutions $p_\alpha = O^{k_\alpha}$ the general expression for the bracket is:

$$\langle\{p_\alpha ...\}\rangle \quad = \quad \sum_\alpha Q_\alpha O^{k_\alpha} \tag{26}$$

For some not yet defined set of elements $Q_\alpha$ in some ring $R$. So every choice of $2^n$ elements $Q_\alpha$ defines a different bracket. However, according to assumption 1, the bracket should only be dependent of the resolutions so it should certainly be independent of the numbering of the crossing points. So not every choice is possible. Therefore we now specialize to a specific choice:

**Extra Assumption 1** *Let's try:* $Q_\alpha = Q^{r_\alpha}$ *for some element* $Q \in R$. *So we have:*

$$\langle\{p_\alpha ...\}\rangle \quad = \quad \sum_\alpha Q^{r_\alpha} O^{k_\alpha} \tag{27}$$

The above choice is manifestly invariant under renumbering, since the height $r_\alpha$ (which was defined as the sum of the binary digits of $\alpha$) is invariant under renumbering.

(We will stick with this choice, however we could ask ourselves what other choices are possible that leave the bracket invariant under renumbering. )

So now we have left only the freedom to choose $Q$. If we would take for instance $Q = 1$ then we'd have (25) again as the definition of our bracket. The next step is to determine what other values for $Q$ we can or should use. For general $Q$ we have:

$$\begin{aligned}
\langle\{p_\alpha ...\}_1\rangle &= Q \cdot \langle\{p_\alpha ...\}\rangle \\
\langle\{p_\alpha ...\}_0\rangle &= \langle\{p_\alpha ...\}\rangle \\
\langle\{p_\alpha ...\}_0 \sqcup \{p_\beta ...\}_1\rangle &= \langle\{p_\alpha ...\}\rangle + Q\langle\{p_\beta ...\}\rangle
\end{aligned}$$

From which follows:

$\langle$  $\rangle$

$$= \quad \langle|\rangle + Q\langle|O\rangle \quad = \quad \langle|\rangle + QO\langle|\rangle \quad = \quad (1 + QO)\ \langle|\rangle \tag{28}$$

$\langle$  $\rangle$

$$= \quad \langle|O\rangle + Q\langle|\rangle \quad = \quad O\langle|\rangle + Q\langle|\rangle \quad = \quad (O + Q)\ \langle|\rangle \tag{29}$$

Since these relations hold for any link we see that $RIa$ and $RIb$ are universal transformations for every bracket of the form (27).

In order to satisfy assumption 5 we want $RII$ to be universal as well. This can only hold if transformations (23) and (24) lead to the same multiplicative factor in the bracket.

Transformation (23) induces a transformation of the bracket (27) as follows:

$$\begin{aligned}
\langle\{p_\alpha ...\}\rangle &\Rightarrow O\langle\{p_\alpha ...\}_{00}\rangle + O^2\langle\{p_\alpha ...\}_{10}\rangle + \langle\{p_\alpha ...\}_{01}\rangle + O\langle\{p_\alpha ...\}_{11}\rangle \\
&= O\langle\{p_\alpha ...\}\rangle + QO^2\langle\{p_\alpha ...\}\rangle + Q\langle\{p_\alpha ...\}\rangle + Q^2 O\langle\{p_\alpha ...\}\rangle \\
&= (O + QO^2 + Q + Q^2 O)\langle\{p_\alpha ...\}\rangle
\end{aligned}$$

And (24) leads to:

$$\begin{aligned}
\langle\{p_\alpha ...\}\rangle &\Rightarrow O^{-1}\langle\{p_\alpha ...\}_{00}\rangle + \langle\{p_\alpha ...\}_{10}\rangle + \langle\{p_\alpha ...\}_{01}\rangle + \ O^{-1}\langle\{p_\alpha ...\}_{11}\rangle \\
&= O^{-1}\langle\{p_\alpha ...\}\rangle + \ Q\langle\{p_\alpha ...\}\rangle + \ Q\langle\{p_\alpha ...\}\rangle + \ Q^2O^{-1}\langle\{p_\alpha ...\}\rangle \\
&= (O^{-1} + 2Q + Q^2O^{-1})\langle\{p_\alpha ...\}\rangle
\end{aligned}$$

Now we want $RII$ to be universal, so we have to solve:

$$O + QO^2 + Q + Q^2O \quad = \quad O^{-1} + 2Q + Q^2O^{-1} \tag{30}$$

We will refer to this equation as "The $RII$-equation" One can easily verify that it has the following three solutions (it's a polynomial equation of degree 3): $O = 1$, $O = -1$ and $O = -(Q + Q^{-1})$.

Notice that the first two solutions are not very suitable because for instance they wouldn't even make a distinction between the link consisting of $k$ unknots and the link consisting of $k + 2$ unknots. So the third solution seems to be the only reasonable one. Moreover, if we define $q := -Q$ we get a very familiar equation: $O = q + q^{-1}$. (Remember that $O$ was defined as the polynomial that is assigned to the trivial knot). The bracket now automatically takes its values in the ring $\mathbb{Z}[q, q^{-1}]$. Also the bracket is now written as:

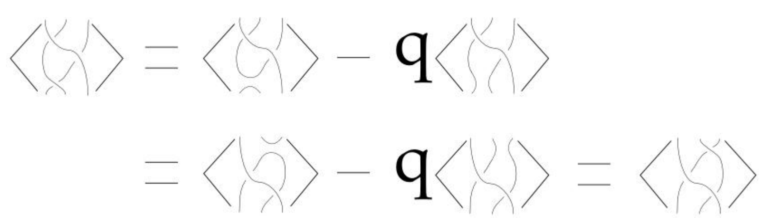
$$\langle\{p_\alpha ...\}\rangle \quad = \quad \sum_\alpha (-q)^{r_\alpha}(q + q^{-1})^{k_\alpha} \tag{31}$$

Which defines exactly the Kauffman Bracket! Notice how the bracket (26) followed from only a few very general assumptions and how this specialized to the Kauffman bracket by one simple extra assumption (27).

Now we still need to check if $RIII$ is universal. However we can prove that, because we are working with a state-sum presentation, this automatically follows from the fact that it $RI$ and $RII$ are universal.

**Lemma 12** *For the bracket RIII is also universal.*

Proof: this can be seen in the following equations:



The second equality follows from the universality of $RII$. □

We will now try to derive a link-invariant from the bracket. We first use the fact that $RIa$ adds a left-handed crossing to de diagram and that it is universal. We see from (28) that it multiplies the bracket by $1 + QO$. If we combine these facts we see that $(1+QO)^{-n_-}\langle\{p_\alpha...\}\rangle$ is invariant under $RIa$ ($n_-$ denotes the number of left-handed crossings in the diagram). Since $RIb$ multiplies the bracket with $(O+Q)$ and adds a right-handed crossing to the diagram we see that

$$(O+Q)^{-n_+}(1+QO)^{-n_-}\langle\{p_\alpha...\}\rangle \tag{32}$$

is invariant under both $RI$ moves ($n_+$ is the number of right-handed crossings).

What about $RII$? Well, if we look at the case of figure 3, we see that this move is in fact equal to performing an $RIa$ move and an $RIb$ move on the respective crossing points. So invariance of (32) under this type of move is implied by $RIa$ and $RIb$ invariance. And since $RII$ is universal we get the same multiplication factor for the case of figure 4 so (32) is automatically also invariant under $RII$. In the case of the Kauffman-Bracket we have $O=q+q^{-1}$ and $Q=-q$ If we fill in these values in (32) we get:

$$q^{n_+}(-q^2)^{-n_-}\langle\{p_\alpha\}\rangle \quad = \quad (-1)^{n_-}q^{-2n_-+n_+}\sum_\alpha(-q)^{r_\alpha}(q+q^{-1})^{k_\alpha} \tag{33}$$

Which is the unnormalized Jones polynomial.

(In section 9.1 we will see that there is also a kind of state sum presentation for the Homfly polynomial. However for this one the polynomial is not a *linear* combination of its resolutions)

**Conclusion:** in order to find a link-invariant, we do not directly demand Reidemeister-invariance. We demand first that we have a link-diagram invariant that is allowed to transform under Reidemeister moves. However we do demand that these transformations are universal. This means that they are the same for all links. We are then able to force our invariant to be Reidemeister-invariant by applying a normalization that cancels out the Reidemeister transformations.

**Suggestions for improvement:** We suggest two ways to possibly find different link-invariants: firstly we could replace the polynomials $p_\alpha$ by elements of some *non-commutative* ring. Secondly, we could drop our 'Extra Assumption' that $Q_\alpha=Q^{r_\alpha}$. And replace it by some other choice of $Q_\alpha$. We will not go into this however.

# 7 Khovanov Homology

In this chapter we will try to improve the Jones Polynomial by using categorification. We will replace the polynomials of the previous chapter by graded vector spaces and we replace an alternating sum by a chain complex. The trick is to choose the chain maps such that the homology transforms universally under Reidemeister moves. We will show that if we directly categorify the calculations of the previous chapters this then leads to Khovanov homology.

We will follow the following procedure: first we categorify the Kauffman bracket of a diagram with only one crossing point, that is: a trivial knot on which we have performed an $RI$ move. Then we generalize this to any knot on which we have performed an $RI$ move and then to any knot with multiple $RI$ twists. Subsection 7.5.1 then summarizes the main ideas of this chapter and in fact this whole thesis. Next we investigate how to define the chain maps of a chain-complex of a diagram on which an $RII$ move was performed.

## 7.1 Categorifying the Kauffman Bracket

We have seen that the Kauffman bracket is defined by:

$$\langle\{p_\alpha ...\}\rangle \quad = \quad \sum_\alpha O^{k_\alpha} Q^{r_\alpha} \quad = \quad \sum_\alpha (q+q^{-1})^{k_\alpha}(-q)^{r_\alpha}$$

where the summation is over all resolutions $\alpha$, where $k_\alpha$ is the number of unknots in the resolution and $r_\alpha$ is the height of the resolution. Now the problem of the Kauffman bracket (and the Jones polynomial) is that it does not make enough distinction between different knots, because it has too many symmetries. We can show this in the following way:

Notice that the Kauffman bracket can be written as an alternating sum over the index $r$:

$$\langle\{p_\alpha ...\}\rangle \quad = \quad \sum_r (-1)^r P_r$$

where we have defined:

$$P_r := \sum_{\{\alpha | r_\alpha = r\}} (q+q^{-1})^{k_\alpha} q^r \tag{34}$$

So whenever we have a diagram of a knot this diagram determines a set of polynomials $\{P_0, P_1, ...P_n\}$ for which the alternating sum is the Kauffman bracket. We can arrange these polynomials in one big polynomial in two variables $t$ and $q$:

**Definition 36** *We define the $t$-polynomial of a link diagram $D$ as follows:*

$$P_D(t) \quad = \quad \sum_r P_r t^r$$

Then we obtain the Kauffman bracket by filling in $t = -1$ in the $t$-polynomial:

$$\langle \{p_\alpha ...\} \rangle \quad = \quad \sum_r P_r \cdot (-1)^r \quad = \quad P_D(-1)$$

If we would perform a Reidemeister move on our diagram, we would get a different $t$-polynomial but the alternating sum would remain (up to normalization) invariant. However if we would have two different knots $K_1$ and $K_2$ with respective diagrams $D_1$ and $D_2$ such that $D_1$ has $t$-polynomial:

$$P_{D_1} \quad = \quad P_0 + P_1 t + P_2 t^2 + P_3 t^3 ...$$

and to $D_2$ has $t$-polynomial:

$$P_{D_1} \quad = \quad P_0 + (P_1 + P')t + (P_2 + P')t^2 + P_3 t^3 ...$$

then we see that after filling in $t = -1$ the two expressions become equal so $D_1$ and $D_2$ have the same Kauffman bracket. Therefore it is possible that two completely different knots can have the same Jones polynomial. What we actually want is an invariant that remains the same *only* under Reidemeister moves. In other words: the Jones polynomial has too many symmetries.

So the $t$-polynomial changes under Reidemeister moves, while after filling in $t = -1$ the $t$-polynomial is invariant (up to normalisation) under a much larger class of 'transformations'. (With a transformation here we mean a change of one diagram to another one, possibly corresponding to an entirely different knot.) In other words: The $t$-polynomial contains too much information, the Kauffman bracket contains too little information.

Khovanov had the following idea to solve this problem: if we use categorification then the $t$-polynomial becomes a chain complex and the Kauffman bracket becomes its Euler characteristic. Now the convenient thing about this is that (as we have seen in section 5.6) there is an extra 'level of information' between these quantities, namely the homology of the chain complex.

| **before categorification:** | **after categorification:** |
|---|---|
| $P_0 + P_1 t + P_2 t^2 + ...$ | $0 \to V^{P_0} \to V^{P_1} \to ...$ |
| | $\{H^0, H^1, H^2, ...\}$ |
| $P_0 - P_1 + P_2 - ...$ | $P_0 - P_1 + P_2 - ...$ |

So if we have a link-diagram $D$ with $t$-polynomial

$$P_0 + P_1 t + P_2 t^2 + ...$$

then after categorification it becomes some complex $C(D)$:

$$0 \to V^{P_0} \to V^{P_1} \to V^{P_2} \to ... \to 0$$

(here $V^{P_i}$ denotes a vector space with graded dimension $P_i$) and maybe we can define its chain maps such that not just its Euler characteristic but even its homology groups transform universally under Reidemeister moves. Then it might be possible that some chain complexes corresponding to different diagrams have the same Euler characteristic, but have different homologies. The homology would then be a strictly stronger invariant then the Jones polynomial.

**Assumption 6** *For any knot-diagram $D$ we define a chain complex $C(D)$ with vector spaces $V^{P_i}$. Where the polynomials $P_i$ are defined as in (34)*

**Assumption 7** *We want to define the chain maps of these complexes such that the Homology groups transform universally under Reidemeister moves.*

Suppose we have a link-diagram $D$ and after performing an $RIb$ move it becomes the diagram $D'$. We have seen that under an $RIb$ move the Kauffman bracket is multiplied by $q^{-1}$. What does this mean for the homology?

The Reidemeister move replaces the chain complex $C(D)$ by the complex $C(D')$. Since the Kauffman bracket of $D$ is the Euler chararacteristic of $C(D)$ we see that the Euler characteristic of $C(D')$ is $q^{-1}$ times the Euler characteristic of $C(D)$.

$$\chi(C(D')) \quad = \quad q^{-1} \cdot \chi(C(D))$$

This also means that the homology $H(D)$ of $C(D)$ is replaced by a the homology $H(D')$ of $C(D')$ with different homology-groups, with different graded dimensions such that the alternating sum of the graded dimensions of the groups $H^i(D')$ is a factor of $q^{-1}$ times the alternating sum of the graded dimensions of the original homology groups $H^i(D)$:

$$\sum_i (-1)^i \mathrm{qdim}(H^i(D')) \quad = \quad q^{-1} \cdot \sum_i (-1)^i \mathrm{qdim}(H^i(D))$$

There might be a lot of ways in which the homology could transform in such a way. One way is when the graded dimension of every homology-group itself is multiplied by $q^{-1}$:

$$\mathrm{qdim}(H^i(D')) \quad = \quad q^{-1} \cdot \mathrm{qdim}(H^i(D))$$

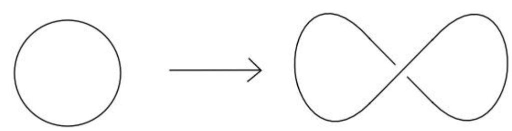
The nice thing about this is that every single homology group then transforms in exactly the same way as the Kauffman bracket. Then we can use a normalization similar to that of the bracket to make it a knot-invariant.

**Assumption 8** *We want the graded dimensions of the homology groups to transform exactly like the Kauffman bracket.*

Let's make this explicit.

## 7.2 Example: $RIb$ on the Unknot

Say we have two diagrams of the trivial knot. The first one ($D$) is the trivial diagram. The second diagram ($D'$) contains a left-twisted curl, so it can be obtained from the trivial diagram by applying an $RIb$ move. We will now try to assign chain complexes to these diagrams. The chain maps should be chosen in such a way that assumptions 6-8 are satisfied. This means that the graded dimension of the homology $H(D')$ should by $q^{-1}$ times the graded dimension of the homology of $H(D)$.



For the trivial diagram we have the cube: $\{p_\alpha ...\} = \{O\}$ so the $t$-polynomial is $P_D(t) = P_0 = O = q + q^{-1}$. It follows from assumption 6 that the corresponding chain complex is:

$$C(D) \quad = \quad 0 \to V^O \to 0$$

The homology of this complex is obviously: $H^0 = V^O$ and $H^i = \{0\}$ if $i \neq 0$ The cube for the second diagram is: $\{p_0, p_1\} = \{O^2, O\}$. So the $t$-polynomial is:

$$P_{D'}(t) \quad = \quad P_0 + P_1 t \quad = \quad O^2 + qOt \tag{35}$$

and the corresponding chain complex is:

$$C(D') \quad = \quad 0 \to V^{O^2} \xrightarrow{m} V^{qO} \to 0 \tag{36}$$

Where $V^{O^2}$ and $V^{qO}$ are two vector spaces with respective graded dimensions $O^2$ and $qO$. The chain map $m$ between them will be defined later. Notice the similarity between the $t$-polynomial (35) and the chain complex (7.2).

So how does this factor of $q^{-1}$ in the Kauffman bracket arise exactly? Remember that we had $O = q + q^{-1}$ so the first coefficient of the $t$-polynomial can be re-written in two terms: $O^2 = qO + q^{-1}O$. So (35) becomes:

$$P_{D'}(t) \quad = \quad qO + q^{-1}O + qOt$$

If we fill in $t = -1$ then the first term and the third term cancel each other, and the term that is left is $q^{-1}$ times the polynomial of the trivial diagram:

$$P_{D'}(-1) \quad = \quad qO + q^{-1}O + qO \cdot (-1) \quad = \quad q^{-1}O$$

Now we want the behavior of the complex and the homology to mimic this relation. So we want a same kind of re-writing and a same kind of cancellation. In other words: we want to categorify the above formulas. This is possible.

Say we have a graded vector space $V^O$ with graded dimension $O = q + q^{-1}$. This means we can write it as: $\text{span}\{v_+, v_-\}$ with $\deg(v_+) = 1$ and $\deg(v_-) = -1$. Then $V^O \otimes V^O$ has graded dimension $O^2$, so we define:

$$V^{O^2} \quad := \quad V^O \otimes V^O \quad = \quad \text{span}\{v_+ \otimes v_+,\ v_+ \otimes v_-,\ v_- \otimes v_+,\ v_- \otimes v_-\}$$

Just like we can write $O^2$ as $qO + q^{-1}O$ we can also split up $V^{O^2}$ as a direct sum of the following two subspaces:

$$W^{qO} = \text{span}\{v_+ \otimes v_+, \quad v_+ \otimes v_- + v_- \otimes v_+\}$$

$$W^{q^{-1}O} = \text{span}\{v_+ \otimes v_- - v_- \otimes v_+, \quad v_- \otimes v_-\}$$

such that:

$$\text{qdim}(W^{qO}) = q^2 + 1 = qO$$

and

$$\text{qdim}(W^{q^{-1}O}) = 1 + q^{-2} = q^{-1}O$$

So we have:

$$V^O \otimes V^O = W^{qO} \oplus W^{q^{-1}O}$$

Since $V^{qO}$ and $W^{qO}$ have the same graded dimension they are isomorphic. The chain complex (7.2) is then isomorphic to some chain complex:

$$C \quad = \quad 0 \to W^{qO} \oplus W^{q^{-1}O} \to W^{qO} \to 0$$

So we have a direct sum of two vector spaces on the left-hand side of the chain map now (just like we had a sum of two terms in the first coefficient of the $t$-polynomial in (35)). Moreover, one of the vector spaces on the left-hand side of the chain map equals the vector space on the right-hand side. Now we want these to cancel each other just like the two terms of the $t$-polynomial cancelled each other. So how can we make sure that the equal vector spaces 'cancel out' each other after taking the homology? This is very easy. If we define the chain map $m'$ of this complex:

$$m' : W^{qO} \oplus W^{q^{-1}O} \to W^{qO}$$

the homology group $H'^0$ is defined as the kernel of $m'$ and $H'^1$ is the target space of $m'$ modulo the image. So if we define $m'$ such that its kernel is exactly $W^{q^{-1}O}$. Then $m'$ acts as a linear isomorphism on $W^{qO}$ and so the image of $m'$ is $W^{qO}$, which is the entire target space. (We could for instance define $m'$ to be the projection map onto $W^{qO}$. In that case $m'$ acts as the identity on $W^{qO}$. However any automorphism of $W^{qO}$ would do, so we could for instance just as well take $m'$ to be *minus* the projection onto $W^{qO}$.) We have split up the complex into a direct sum of two complexes:

$$C' \quad = \quad 0 \to W^{q^{-1}O} \to 0 \to 0$$

$$C'' \quad = \quad 0 \to W^{qO} \to W^{qO} \to 0$$

Since $C''$ contains an isomorphism, its homology is trivial. This means that the homology of $C$ equals the homology of $C'$ which is obviously equal to: $H'^0 = W^{q^{-1}O}$, $H'^i = \{0\}$ if $i \neq 0$.

**Definition 37** *Let $V^{P_r}$ be a graded vector space with graded dimension $P_r$. Then $V^{P_r}\{n\}$ is as a vector space identified with $V^{P_r}$. Only the grading of all vectors is increased by $n$. This means that* $\mathrm{qdim}(V^{P_r}\{n\}) = q^n \cdot P_r$. *The operator $\cdot\{n\}$ is called the grading shift operator.*

Notice that for any graded vector space $A$ we then have:

$$V^O \otimes A \quad \cong \quad A\{1\} \oplus A\{-1\} \tag{37}$$

which is just the categorification of the equation:

$$(q + q^{-1}) \cdot p \quad = \quad q \cdot p + q^{-1} \cdot p$$

Now we can define the vector space $V^{qO}$ of (7.2) as:

$$V^{qO} := V^O\{1\}$$

Also we define a grading preserving isomorphism $\phi$ from $W^{qO}$ to $V^O\{1\}$:

$$\phi : \quad v_+ \otimes v_+ \quad \mapsto \quad v_+$$

$$\phi : \quad v_+ \otimes v_- + v_- \otimes v_+ \quad \mapsto \quad v_-$$

(notice that $\phi$ is indeed grading preserving, because the grades on the right-hand side are increased by 1 by the grading shift operator). Finally, we can then define the chain map $m$ of () as: $m = \phi \circ m'$.

$$m : \quad W^{qO} \oplus W^{q^{-1}O} \quad \to \quad V^O\{1\}$$

which can also be written as:

$$m : \quad V^O \otimes V^O \quad \to \quad V^O\{1\}$$

From this we see that:

$$H^0 = \ker(m) \cong \ker(m') = H'^0 = W^{q^{-1}O}$$

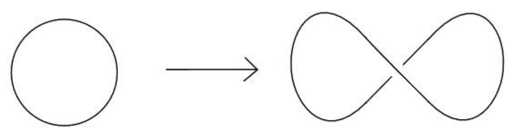
So we see that $\mathrm{qdim}(H^0) = q^{-1} \cdot O$. Exactly what we wanted.

Notice that we have taken here an approach slightly different from Khovanov's. He starts out by defining a multiplication $m$ from $V^O \otimes V^O$ to $V^O$ and then makes it into a grading preserving map by composing it with the canonical isomorphism $V^O \to V^O\{1\}$ (which is an isomorphism of vector spaces, but not of graded vector spaces, since it has degree 1). And then proves that the resulting homology miraculously turns out to be invariant for $RIb$.

We however began by looking for a map that creates an invariant homology. The most obvious one turned out to be the projection map $m'$, which is already grading preserving by construction. Then in order to make calculations easier and to make sure that the result is the same as Khovanov's, we composed it with the canonical grading preserving isomorphism $W^{qO} \to V^{O}\{1\}$. This results in the same map $m$.

## 7.3 Next example: $RIa$ on the Unknot

We will now do exactly the same thing for a diagram with an $RIa$ twist. We have seen before that the Kauffman bracket of this diagram is $-q^2$ times the Kauffman bracket of the trivial diagram.



We can define a map $\Delta$ in a way analogous to the way we defined $m$ in the previous section. The cube for the trivial knot after an $RIa$ move is: $\{O, O^2\}$ So the $t$-polynomial is: $O + O^2qt$. We can take for instance:

$$0 \to V^O \to V^O \otimes V^O\{1\} \to 0$$

as its chain-complex. Once again we split-up $V^O \otimes V^O$ into two subspaces:

$$Y^{qO} := \text{span}\{v_+ \otimes v_+, \quad v_+ \otimes v_- - v_- \otimes v_+\}$$

$$Y^{q^{-1}O} := \text{span}\{v_+ \otimes v_- + v_- \otimes v_+, \quad v_- \otimes v_-\}$$

We can define:

$$V^O \otimes V^O\{1\} \quad = \quad Y^{qO}\{1\} \oplus Y^{q^{-1}O}\{1\}$$

which can be re-written as:

$$V^O \otimes V^O\{1\} \quad = \quad Y^{q^2O} \oplus Y^O$$

where $Y^{q^2O} := Y^{qO}\{1\}$ and $Y^O := Y^{q^{-1}O}\{1\}$. We can also define a map:

$$\Delta' : \quad Y^O \to Y^{q^2O} \oplus Y^O$$

Which we split-up as a direct sum of:

$$Y^O \to Y^O \quad \text{and} \quad 0 \to Y^{q^2O}$$

where the first map is a linear isomorphism. The map $\Delta'$ can be interpreted as the inclusion of $Y^O$ into $Y^{q^2O} \oplus Y^O$. From this we see: $H'^1 \cong Y^{q^2O}$ and $H'^i = \{0\}$ if $i \neq 1$. Then we see that the graded dimension of the homology of the trivial diagram is multiplied by $q^2$. Also, the only nonzero homology group is now $H^1$ instead of $H^0$ so the alternating sum of graded dimensions is multiplied by $-q^2$. This time we need not only a grading shift to obtain an invariant, but also a shift in the homological degree.

The definition of $Y^{q^{-1}O}$ and $Y^{qO}$ was chosen such that $m$ and $\Delta$ satisfy the Frobenius condition. This is necessary so that we can combine them into a chain complex later.

### 7.3.1 Conclusions

From assumption 6 it follows that the trivial diagram has homology:

$$H^0 \cong V^O$$

the diagram of the trivial knot with one left-twisted curl has homology:

$$H^0 \cong V^{q^{-1}O} = V^O\{-1\}$$

and the diagram of the trivial knot with one right-twisted curl has homology:

$$H^1 \cong V^{q^2O} = V^O\{2\}$$

All other homology groups are trivial. We now know how the homology of the trivial diagram transforms under an $RIa$ or an $RIb$ move. Since we ultimately want a knot-invariant we want these moves to be universal so we want these transformations to hold for all link diagrams. That is: suppose we have a link with diagram $D$. After adding a left-twisted curl to this diagram we obtain a diagram $D'$ and after adding a right-twisted curl to $D$ we obtain diagram $D''$. Then we want the homology to satisfy the following relations:

$$\begin{aligned} H^i(D') &= H^i(D)\{-1\} \\ H^i(D'') &= H^{i-1}(D)\{2\} \end{aligned}$$

## 7.4 Transformations of Chain-Complexes

Now we have only shown yet that we can define homologies for diagrams of the trivial knot with one crossing point that behave in the desired way. But how can we extend this to general diagrams of general links? We will now show that everything works out in the same way just as well if we do not assume that the original diagram $D$ is the trivial knot. As long as we assume that the chain maps of the complex $C(D)$ commute with $m$ and $\Delta$.

Suppose we have two chain complexes:

$$V \quad = \quad 0 \to V^0 \to V^1 \to V^2 \to 0$$

$$A \quad = \quad 0 \to A^0 \to A^1 \to A^2 \to 0$$

and maps $f^i : V^i \to A^i$ that commute with the chain maps (so the collection of maps $f^i$ forms a morphism $f$ of chain-complexes). Then we can use these maps to make the two complexes into one:

$$\begin{array}{ccccccc} 0 \to & V^0 & \to & V^1 & \to & V^2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & A^0 & \to & A^1 & \to & A^2 & \to 0 \end{array}$$

as we have seen in section 5.6 (the downward pointing arrows denote the maps $(-1)^i f^i$). We take here direct sums between every pair $V^i$ and $A^{i-1}$. We call this the 'flattening' or the 'cone' of the complexes $V$ and $A$ with respect to $f$ and is denoted by: $V \to A$.

Suppose now we can write every $V^i$ as the direct sum of vector spaces $A^i$ and $B^i$: $V^i = A^i \oplus B^i$ and that every chain map $d^i : V^i \to V^{i+1}$ can be written as the direct sum of two maps:

$$d^i = a^i \oplus b^i$$

$$a^i : A^i \to A^{i+1} \quad \text{and} \quad b^i : B^i \to B^{i+1}$$

$$\begin{array}{ccccccc} 0 \to & A^0 \oplus B^0 & \to & A^1 \oplus B^1 & \to & A^2 \oplus B^2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & A^0 & \to & A^1 & \to & A^2 & \to 0 \end{array} \tag{38}$$

In shorter notation:

$$V \to A \quad = \quad A \oplus B \to A$$

Furthermore suppose that $f^i$ is the projection map from $V^i$ onto $A^i$. Then we can write this complex as the direct sum of two complexes:

$$\begin{array}{ccccccc} 0 \to & A^0 & \to & A^1 & \to & A^2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & A^0 & \to & A^1 & \to & A^2 & \to 0 \end{array} \tag{39}$$

and

$$0 \to \quad B^0 \quad \to \quad B^1 \quad \to \quad B^2 \quad \to 0 \tag{40}$$

The homology of (38) is then the direct sum of the homologies of (39) and (40). But since $f^i$ restricted to $A^i$ is an isomorphism we see that (39) is contractible (see lemma 7). So the homology of (38) equals the homology of (40).

Suppose we have a knot diagram with chain complex

$$V \quad = \quad ... \to V^0 \to V^1 \to V^2 \to ... \tag{41}$$

and we assume that after an $RIb$ move this becomes: $V^O \otimes V \to V\{1\}$

$$\begin{array}{ccccccc} ... \to & V^O \otimes V^0 & \to & V^O \otimes V^1 & \to & V^O \otimes V^2 & \to ... \\ & \downarrow & & \downarrow & & \downarrow & \\ ... \to & V^0\{1\} & \to & V^1\{1\} & \to & V^2\{1\} & \to ... \end{array} \tag{42}$$

We can write $V^O \otimes V^i$ as $V^i\{1\} \oplus V^i\{-1\}$ so this flattening equals:

$$\begin{array}{ccccccccc}
... \to & V^0\{1\} \oplus V^0\{-1\} & \to & V^1\{1\} \oplus V^1\{-1\} & \to & V^2\{1\} \oplus V^2\{-1\} & \to ... \\
 & \downarrow & & \downarrow & & \downarrow & \\
... \to & V^0\{1\} & \to & V^1\{1\} & \to & V^2\{1\} & \to ...
\end{array}$$

$$= \quad V\{1\} \oplus V\{-1\} \to V\{1\}$$

Notice that the chain maps of $V^O \otimes V$ are defined as $\mathrm{Id} \otimes d$ and therefore the map

$$\mathrm{Id} \otimes d^i : \quad V^i\{1\} \oplus V^i\{-1\} \quad \longrightarrow \quad V^{i+1}\{1\} \oplus V^{i+1}\{1\}$$

splits up as a direct sum of two maps:

$$a^i : \ V^i\{1\} \to V^{i+1}\{1\} \quad \text{and} \quad b^i : \ V^i\{-1\} \to V^{i+1}\{-1\}$$

If we furthermore assume the downward pointing arrows denote $(-1)^i$ times the projection maps onto $V^i\{1\}$ then we see the homology of this flattening equals the homology of:

$$V\{-1\} \quad = \quad ... \to V^0\{-1\} \to V^1\{-1\} \to V^2\{-1\} \to ...$$

That means that the homology transforms under $RIb$ as:

$$H^i \Rightarrow H^i\{-1\}.$$

Notice that our example of an $RIb$ move performed on the trivial diagram is just a special case of this. Only we have replaced the chain complex $0 \to V^O \to 0$ of the trivial diagram by a general chain complex $V$.

In the same way we can assume that under an $RIa$ move the complex (41) becomes:

$$\begin{array}{ccccccc}
... \to & V^0 & \to & V^1 & \to & V^2 & \to ... \\
 & \downarrow & & \downarrow & & \downarrow & \\
... \to & V^O \otimes V^0\{1\} & \to & V^O \otimes V^1\{1\} & \to & V^O \otimes V^2\{1\} & \to ...
\end{array} \tag{43}$$

From which we conclude that under RIa the homology transforms as:

$$H^i \Rightarrow H^{i+1}\{2\}.$$

Which can also be denoted as:

$$H^i \Rightarrow H^i[1]\{2\}.$$

If we could find a way to assign a complex to any link diagram such that it transforms like (42) under $RIb$ and like (43) under $RIa$, then the homology of such a complex would transform universally under $RI$ moves. And after a proper grading shift it would be an $RI$ invariant.

**Assumption 9** *Every link diagram $D$ gets assigned a chain complex $C(D)$ with chain maps that commute with $m$ and $\Delta$. If the diagram $D'$ is obtained from $D$ by an RIb move then we have:*

$$C(D') \quad = \quad V^O \otimes C(D) \xrightarrow{m} C(D)\{1\}$$

*And if $D''$ is obtained form $D$ by an $RIa$ move we have:*

$$C(D'') \quad = \quad C(D) \xrightarrow{\Delta} V^O \otimes C(D)\{1\}$$

## 7.5 Multiple $RI$ moves

We have seen that when we add an $RI$-twist to a diagram, then the resulting complex will be the flattening of two complexes. Thus subsequently adding $RI$-twists to the unknot results in the flattening of the flattening of... etc. of complexes. This can be seen as a cube of vector spaces where the edges are anti-commuting maps and the $r^{th}$ chain space is then the direct sum of all vertices of height $r$.

$$V^{P_r} := \bigoplus_{\{\alpha | r_\alpha = r\}} V^{p_\alpha}\{r\}$$

This means that we have a linear map $\pm f_{\alpha\beta}$ from $V^{p_\alpha}\{r_\alpha\}$ to $V^{p_\beta}\{r_\beta\}$ whenever the binary numbers $\alpha$ and $\beta$ are differ at only one digit (in other words: $\alpha$ and $\beta$ are neighbors) and this particular digit has value 0 in $\alpha$ and value 1 in $\beta$.

This map $f_{\alpha\beta}$ is then:

$$\begin{array}{l} m_{ij} \text{ if } k_\beta = k_\alpha - 1 \\ \Delta_{ij} \text{ if } k_\beta = k_\alpha + 1 \end{array}$$

Here $k_\alpha$ is the number of unknots in resolution $\alpha$. $m_{ij}$ acts as multiplication on the $i^{th}$ and $j^{th}$ tensor factor and as the identity on all other tensor factors. For instance:

$$m_{24}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = v_1 \otimes m(v_2 \otimes v_4) \otimes v_3$$

And a similar definition for $\Delta_{ij}$, for instance:

$$\Delta_{24}(v_1 \otimes v_2 \otimes v_3 \otimes v_5) = (v_1 \otimes v_a \otimes v_3 \otimes v_b \otimes v_5)$$

where $\Delta(v_2) = v_a \otimes v_b$ (this will be made more clear in section 7.7). Then the complex transforms exactly like (42) under an $RIb$ move and like (43) under $RIa$. Just the way we want it.

It is important here to notice that since $m$ and $\Delta$ satisfy the Frobenius condition we are sure that the chain maps of such a chain complex (anti-)commute with $m$ and $\Delta$. Therefore it is indeed possible to define a chain complex as subsequent flattenings with respect to $m$ and $\Delta$.

Notice furthermore that the vertices of such a cube are $V^{O^{\otimes k_\alpha}}\{r_\alpha\}$ which is exactly the categorification of the Kauffman bracket where the vertices are polynomials $O^{k_\alpha} q^{r_\alpha}$. Therefore we will from now on assume that this holds for *any* link-diagram (not just for the unknot).

**Assumption 10** *For any diagram $D$ the vertices of the cube $C(D)$ are $V^{O\otimes k_\alpha}\{r_\alpha\}$.*

### 7.5.1 important remarks

So what has just happened? We have seen that integer numbers can be categorified by chain complexes. We can then take the Euler characteristic as the decategorification. When two chain complexes $C$ and $D$ have the same Euler characteristic we'll denote this by: $C \sim D$. We have seen that subtraction can be categorified by flattening. If we have two integers $c$ and $d$ then an expression like $c + d - c = d$ becomes:

$$C \oplus D \xrightarrow{f} C \quad \sim \quad D$$

This is completely independent of the map $f$. Now we can make our theory strictly stronger by not using the Euler characteristic, but the dimensions of the homology groups as decategorification. Whenever two chain complexes have the same homology we'll denote this by $C \cong D$. In general the equation

$$C \oplus D \xrightarrow{f} C \quad \cong \quad D$$

does not hold. However, we have not specified $f$ yet. So it will hold if we choose $f$ carefully. This is exactly what we have done above. If $f^i$ is the projection map from $C^i \oplus D^i$ onto $C^i$ for each $i$ then we have:

$$C \oplus D \xrightarrow{f} C \quad = \quad (C \xrightarrow{\mathrm{Id}} C) \oplus (D \to 0) \quad \cong \quad D$$

Also if $f^i$ is the inclusion map $C^i \to C^i \oplus D^i$ then we have:

$$C \xrightarrow{f} C \oplus D \quad = \quad (C \xrightarrow{\mathrm{Id}} C) \oplus (0 \to D) \quad \cong \quad D[1]$$

Two special cases of this are:

$$V^O \otimes C \xrightarrow{m} C\{1\} \quad = \quad C\{1\} \oplus C\{-1\} \xrightarrow{m} C\{1\} \quad \cong \quad C\{-1\} \tag{44}$$

$$C \xrightarrow{\Delta} V^O \otimes C\{1\} \quad = \quad C \xrightarrow{\Delta} C\{2\} \oplus C \quad \cong \quad C\{2\}[1] \tag{45}$$

(remember that $V^O$ is a graded vector space and $C$ is a chain complex.) These are simply the categorifications of the following two equations, which play an important role in calculating the Kauffman bracket:

$$(q + q^{-1})x - qx \quad = \quad q^{-1}x$$

$$x - (q + q^{-1})qx \quad = \quad -q^2x$$

## 7.6 Another Example: RII, case 2

Now we have seen in the previous sections how a chain complex can be transformed under an $RI$ move such that its homology transforms universally. This means that if we have a chain complex for some knot-diagram $D$, then we also know how to construct a chain complex for all diagrams which are obtained by adding left- and right- twisted curls to $D$. So now its time to think of a way to transform a chain complex after performing an $RII$ move, in such a way that the homology remains invariant.

Notice that if we start out with the trivial diagram, and we perform an $RII$ move, then this is exactly the same as performing both an $RIa$ and an $RIb$ move. This means that the homology transforms as:

$$H^i \Rightarrow H^i[1]\{1\} \tag{46}$$

But if we want the homology to transform this way for *any* diagram we have to deal we with the fact that, just like in section 6.1, we cannot write down how the cube transforms under an $RII$ move in general. It depends on the particular knot and it even differs per resolution. But we do demand that the homology transforms universally. So equation (46) should always hold.

Suppose we have a knot-diagram $D$ and after performing an $RII$ move it becomes the diagram $D'$. In section 6.1 we saw that the cube was replaced by a four times bigger cube. Every resolution is replaced by a 'small cube' consisting of four new resolutions. These new resolutions look either like the four of figure 3 (case 1) or like the four of figure 4 (case 2). We have seen that the Kauffman bracket of such a small cube equals up to normalization the polynomial of the original resolution. This normalization factor is the same for every vertex and therefore the entire Kauffman bracket transforms universally for $RII$.

We want a similar thing to happen for Khovanov homology. That is: every vertex $V^\alpha$ is replaced by four new vertices. These four new vertices form a small complex $W^\alpha$, so the new chain complex is a cube which has at every vertex a small cube, consisting of four spaces, itself. These small cubes look either like the complex for case 1 or like the complex for case 2.

**Definition 38** *A 'small cube' or 'small complex' is a cube consisting of four vertices. In other words: it is the flattening of two chain complexes which both have two chain spaces.*

Notice that we already know what the complex for case 1 should look like since it is equivalent to performing both an $RIa$ and an $RIb$ move on the unknot and we have seen that in the previous section. What we now want is that the homology of the small complex for case 2 also transforms like (46). This means the vertex $V^\alpha$ of $C(D)$ is replaced by a small cube $W^\alpha$ of four spaces which has homology $H'^1 \cong V^\alpha\{1\}$. And then we must still show that this also leads to a homology for the entire cube that transforms universally. This is however much harder because the relation between the homology of the entire complex and the homology of the small complexes is not so obvious.

So how can we construct a cube such that its homology is invariant under $RII$ moves? We have a diagram $D$ with cube $C(D)$ and after an $RII$ move it becomes the diagram $D'$ with cube $C(D')$.

Let's first consider the case where $C(D)$ has only one vertex. So $D$ is a diagram with no crossing points. Since in an $RII$ move only one or two unknots are involved we can assume without loss of too much generality that $D$ consists of only one or two unknots. The $RII$ move is either of the type of case 1 or of the type of case 2. So we'll now take a look at what happens if we have two unknots and we perform a case 2 $RII$ move. Then:

$$C(D) \quad = \quad 0 \to V^O \otimes V^O \to 0$$

$$H^0 \;=\; V^O \otimes V^O, \quad H^i \;=\; \{0\}$$

According to assumption 10 we have: $C(D') =$

$$\begin{array}{ccc} V^O & \stackrel{\phi_1}{\to} & V^O \otimes V^O\{1\} \\ \phi_2 \downarrow & & \downarrow \phi_3 \\ V^O \otimes V^O\{1\} & \stackrel{\phi_4}{\to} & V^O\{2\} \end{array}$$

Where the maps $\phi_i$ are yet to be defined. (This follows from figure 4.) And according to (46) we must have:

$$H'^1 = V^O \otimes V^O\{1\} \quad \text{and} \quad H'^i = \{0\} \text{ if } i \neq 1$$

This means that $\phi_1 \oplus \phi_2$ should be injective and $\phi_3 + \phi_4$ should be surjective. Also it means that we can write $C(D')$ as:

$$C(D') \quad = \quad 0 \to V^O \stackrel{\phi_1 \oplus \phi_2}{\longrightarrow} A \oplus B \stackrel{\phi_3 + \phi_4}{\longrightarrow} V^O\{2\} \to 0 \tag{47}$$

where

$$A \quad \cong \quad B \quad \cong \quad V^O \otimes V^O\{1\}.$$

and $C(D')$ splits up as a direct sum of a contractible complex:

$$0 \to V^O \to A \to V^O\{2\} \to 0$$

and

$$0 \to B \to 0$$

Now we want this to generalize to diagrams $D$ which have more then one resolution (that is: diagrams that have one or more crossing points). We have seen that every vertex $V^\alpha$ of the cube $C(D)$ is replaced by a small complex consisting of four spaces. Let's first take a look at two neighboring vertices of $C(D)$ and the edge between them. The two vertices are labelled by binary numbers $\alpha$ and $\beta$. The vector spaces on the vertices are called $V^\alpha$ and $V^\beta$ and we have a linear map $d^{\alpha\beta} : V^\alpha \to V^\beta$. After an $RII$ move we have a cube consisting of eight vector spaces:

$$\begin{array}{ccccccc} V^{\alpha 00} & \stackrel{\phi_1^\alpha}{\rightarrow} & V^{\alpha 01} & & V^{\beta 00} & \stackrel{\phi_1^\beta}{\rightarrow} & V^{\beta 01} \\ \phi_2^\alpha \downarrow & & \downarrow \phi_3^\alpha & \xrightarrow{\psi} & \phi_2^\beta \downarrow & & \downarrow \phi_3^\beta \\ V^{\alpha 10} & \stackrel{\rightarrow}{\phi_4^\alpha} & V^{\alpha 11} & & V^{\beta 10} & \stackrel{\rightarrow}{\phi_4^\beta} & V^{\beta 11} \end{array}$$

The left four spaces form a complex we will denote by $W^\alpha$ and the right four form a complex $W^\beta$. The arrow in the middle, labelled by $\psi$, represents four maps:

$$\psi^{00} : V^{\alpha 00} \rightarrow V^{\beta 00} \qquad \psi^{01} : V^{\alpha 01} \rightarrow V^{\beta 01}$$
$$\psi^{10} : V^{\alpha 10} \rightarrow V^{\beta 10} \qquad \psi^{11} : V^{\alpha 11} \rightarrow V^{\beta 11}$$
$$\psi = \psi^{00} \oplus \psi^{01} \oplus \psi^{10} \oplus \psi^{11}$$

So the spaces $V^\alpha$ and $V^\beta$ are replaced by two small cubes $W^\alpha$ and $W^\beta$ and the linear map $V^\alpha \stackrel{d^{\alpha\beta}}{\rightarrow} V^\beta$ is replaced by the flattening $W^\alpha \stackrel{\psi}{\rightarrow} W^\beta$. This can also be written as:

$$\begin{array}{ccccc} & V^{\alpha 00} & \xrightarrow{\psi^{00}} & V^{\beta 00} & \\ g^{\alpha 1} & \downarrow & & \downarrow & g^{\beta 1} \\ & V^{\alpha 01} \oplus V^{\alpha 10} & \xrightarrow{\psi^{01} \oplus \psi^{10}} & V^{\beta 01} \oplus V^{\beta 10} & \\ g^{\alpha 2} & \downarrow & & \downarrow & g^{\beta 2} \\ & V^{\alpha 11} & \xrightarrow{\psi^{11}} & V^{\beta 11} & \end{array}$$

with:

$$g^{\alpha 1} = \phi_1^\alpha \oplus \phi_2^\alpha$$
$$g^{\alpha 2} = \phi_3^\alpha + \phi_4^\alpha$$

and the same for $\beta$ instead of $\alpha$. We see from figures 3 and 4 in section 6.1 that $V^{\alpha 01} \cong V^\alpha$ and $V^{\beta 01} \cong V^\beta$. Notice that both $W^\alpha$ and $W^\beta$ can only be either of the form of case 1 or of the form of case 2.

We are now left with defining the maps $\psi$ and $\phi$ such that we indeed obtain a chain complex that has transformed like (46) under $RII$. This means that we want $C(D)$ to be quasi-isomorphic to $C(D')[-1]\{-1\}$.

**Lemma 13** *If the following three points hold, then $C(D)$ is quasi-isomorphic to $C(D')[-1]\{-1\}$*

**1)** For every vertex $\alpha$ of $C(D)$ we want the sequence $W^\alpha$ to split up as the direct sum of $M^\alpha = V^{\alpha 00} \rightarrow A^\alpha \rightarrow V^{\alpha 11}$ and $N^\alpha = 0 \rightarrow B^\alpha \rightarrow 0$ where $M^\alpha$ is contractible and $B^\alpha \cong V^\alpha\{1\}$. (This means that $g^{\alpha 1}$ should be injective and $g^{\alpha 2}$ should be surjective for every $\alpha$.)

**2)** For every pair of neighbors $\alpha \sim \beta$ we want that $\psi^{01} \oplus \psi^{10} = a \oplus b$ where $a$ is a map $A^\alpha \rightarrow A^\beta$ and $b$ a map $B^\alpha \rightarrow B^\beta$.)

**3)** For every pair of neighbors $\alpha \sim \beta$ we want isomorphisms $\gamma^\alpha$ and $\gamma^\beta$ such that the following diagram commutes:

$$\begin{array}{ccc} V^\alpha\{1\} & \stackrel{d^{\alpha\beta}}{\rightarrow} & V^\beta\{1\} \\ \gamma^\alpha \downarrow & & \downarrow \gamma^\beta \\ B^\alpha & \stackrel{b}{\rightarrow} & B^\beta \end{array}$$

Proof: points 1) and 2) say that we can split up $C(D')$ as a direct sum of a contractible complex $Y$ (which is the flattening of contractible complexes $M^\alpha$) and a cube $X$ with vertices $B^\alpha$. Point 1) says that the $n^{th}$ chain space of $X$ is isomorphic to the $(n-1)^{th}$ chain space of $C(D)\{1\}$ and point 3) says that the edges of $X$ are isomorphic to the edges of $C(D)\{1\}$. Therefore the entire chain complex $X$ is isomorphic to $C(D)[1]\{1\}$. And since $C(D')$ was split up as the direct sum of $X$ and a contractible complex this means that $C(D')$ is quasi-isomorphic to $X$ and therefore also to $C(D)[1]\{1\}$. This is equivalent to saying that $C(D')[-1]\{-1\}$ is quasi-isomorphic to $C(D)$ $\Box$

The definitions of the maps $\phi$ and $\psi$ should be consistent with the previous sections. That is: sometimes an $RII$ move is equivalent to an $RIa$ plus an $RIb$ move and for $RI$ moves we already know how the cube transforms. Notice for instance that if we perform the $RII$ move on the unknot, then the maps $\phi$ and $\psi$ are of the form $\pm m_{ij}$ or $\pm\Delta_{ij}$. This inspires us to look what happens if we assume that $\phi$ and $\psi$ are *always* of this form.

**Assumption 11** *We will assume that the chain complex $C(D)$ of a diagram $D$ is a cube for which all edges are $\pm m_{ij}$ or $\pm\Delta_{ij}$, depending on the number of unknots in resolutions $\alpha$ and $\beta$.*

Notice that the fact that point 1) is then satisfied can easily be proven since there are only two cases. certainly for case 1 this must be true since in sections 7.2 and 7.3 $m$ and $\Delta$ were *defined* to satisfy this. For case 2 this can be seen since $\Delta$ is injective and $m$ is surjective. Therefore $g^{\alpha 1} = -\Delta \oplus \Delta$ is injective and $g^{\alpha 2} = m + m$ is surjective.

Suppose that all four resolutions of $W^\alpha$ have exactly one more unknot than their corresponding resolutions in $W^\beta$. Then all maps $\psi^{ij}$ are multiplications:

$$\begin{array}{ccl} \psi^{ij} & : & V^{\alpha ij} \rightarrow V^{\beta ij} \quad = \\ \psi^{ij} & : & V^O \otimes V \stackrel{m}{\rightarrow} V\{1\} \end{array}$$

This happens for instance in figure 5. $W^\alpha$ corresponds to a 0-smoothing of the crossing point on the left and $W^\beta$ corresponds to a 1-smoothing of this crossing point. We see then that $W^\alpha \cong V^O \otimes W^\beta\{-1\}$ and therefore:

$$\begin{array}{rcll} W^\alpha \rightarrow W^\beta & \cong & V^O \otimes W^\beta\{-1\} \quad \stackrel{m}{\rightarrow} & W^\beta \\ & \cong & W^\beta \oplus W^\beta\{-2\} \quad \stackrel{m'}{\rightarrow} & W^\beta \\ & = & M^\beta \oplus N^\beta \oplus M^\beta\{-2\} \oplus N^\beta\{-2\} \quad \stackrel{m'}{\rightarrow} & M^\beta \oplus N^\beta \end{array}$$

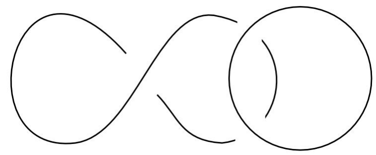


Figure 5: RI and RII commute

It can be split up as a direct sum of :

$$M^\beta \oplus M^\beta\{-2\} \quad \rightarrow \quad M^\beta \tag{48}$$

And

$$N^\beta \oplus N^\beta\{-2\} \quad \rightarrow \quad N^\beta \tag{49}$$

The first of these is contractible, since $M^\beta$ is contractible. We know that we can split it up like this because we know that $m'$ acts as the identity on $W^\beta$ and as the zero-map on $W^\beta\{-2\}$. Since $A^\beta$ is the second chain space of $M^\beta$ and $B^\beta$ is the second chain space of $N^\beta$ we see that (48) and (49) define maps:

$$\begin{aligned} a : A^\alpha \rightarrow A^\beta \quad &= \quad A^\beta \oplus A^\beta\{-2\} \rightarrow A^\beta \\ b : B^\alpha \rightarrow B^\beta \quad &= \quad B^\beta \oplus B^\beta\{-2\} \rightarrow B^\beta \end{aligned}$$

Therefore point 2) is also satisfied. Notice that it doesn't matter what choice we make for $B^\alpha$, for any other choice it would still hold. Because $b$ acts as the identity on $B^\beta$ and is zero on $B^\beta\{-2\}$ and also we have $B^\beta \cong V^\beta\{1\}$ we see that $b$ is isomorphic to the map:

$$m' : V^\beta\{1\} \oplus V^\beta\{-1\} \rightarrow V^\beta\{1\} \quad = \quad V^\alpha\{1\} \rightarrow V^\beta\{1\}$$

proving that also point 3) is satisfied. The case in which all maps $\psi^{ij}$ are co-multiplications is completely analogous. The essence of this is that the resolutions of two neighboring vertices are always related to each other by an $RI$ move. So we can consider the edge $V^\alpha \rightarrow V^\beta$ as corresponding to a diagram with one crossing point. After performing an $RII$ move this map becomes $W^\alpha \rightarrow W^\beta$.

Now we see clearly that in figure 5 the operation of performing the $RI$ move commutes with the operation of performing the $RII$ move. This means that we can also first do $RII$ which changes $V^\alpha$ into $W^\alpha$ and then we make it into the complex $W^\alpha \rightarrow W^\beta$ by performing an $RI$ move. We have seen that an $RIb$ move can be seen as a projection map, so just as in section 7.5.1 this makes sure that $\psi$ maps $M$ into $M$ and $N$ into $N$:

$$M' \oplus N' \oplus M'' \oplus N'' \overset{m}{\rightarrow} M' \oplus N' \quad = \quad (M' \oplus M'' \overset{m}{\rightarrow} M') \oplus (N' \oplus N'' \overset{m}{\rightarrow} N')$$

However, $RI$ and $RII$ do not always commute. See figure 6.

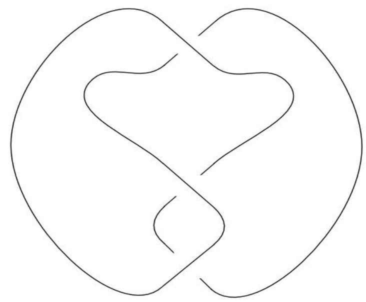


Figure 6: RI and RII do not commute

For this case $W^\alpha \to W^\beta$ would look like:

$$\begin{array}{ccccc} & V^{O\otimes 2} & \xrightarrow{m} & V^{O} & \\ \Delta_1 \oplus m & \downarrow & & \downarrow & \Delta \oplus \Delta \\ & V^{O\otimes 3} \oplus V^{O} & \xrightarrow{-m_2 \oplus -\Delta} & V^{O\otimes 2} \oplus V^{O\otimes 2} & \\ m_2 \oplus -\Delta & \downarrow & & \downarrow & m \oplus -m \\ & V^{O\otimes 2} & \xrightarrow{m} & V^{O} & \end{array}$$

Notice that both $W^\alpha$ and $W^\beta$ in this figure satisfy point 1) because $\Delta_1 \oplus m$ and $\Delta \oplus \Delta$ are both injective and $m_2 \oplus -\Delta$ and $m \oplus -m$ are both surjective.

Unfortunately, it turns out that for this diagram point 2) and 3) are not satisfied. The entire kernel of $g^{\alpha 2}$ is mapped into the image of $g^{\beta 1}$ so there is no map $b : B^\alpha \to B^\beta$. However, Khovanov shows in [9] that the entire complex $W^\alpha \to W^\beta$ can still be split up in a contractible complex and one that is isomorphic to $C(D)$. Although it is clear (after tedious computation) that his approach works, it is not so obvious to see *why* it works.

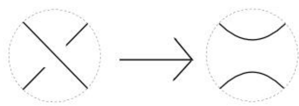
Also it turns out that Khovanov homology is also invariant under $RIII$ (of course), but we will not go into this.

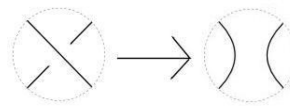
## 7.7 How to Calculate the Khovanov Homology

Now that we have seen how to build up a homology theory that categorifies the Kauffman bracket, and hence the Jones polynomial, let's summarize all this. We forget about all conceptual stuff now, and just give a recipe to explicitly calculate the Khovanov homology.

**Step 1)** First we choose a diagram that represents the knot. We assign a positive integer to every arc.

**Step 2)** Every crossing point can be removed in one of the two following ways:





$0 - smoothing$ $\quad$ $1 - smoothing$

We then say the crossing point is 'smoothed out'. We call the first one a 0-smoothing and the second one a 1-smoothing. If all crossing points are smoothed out then we have a diagram consisting of a finite number of unknotted cycles. Such a diagram is called a resolution. So for a diagram with $n$ crossing points there are $2^n$ resolutions. These resolutions can then be labelled by binary numbers $\alpha$ consisting of $n$ bits. Also we can label the unknots with a number. Every unknot is made up out of one or more arcs of the original diagram, so we have a finite amount of numbers assigned to every unknot, which are the numbers we assigned to the arcs the unknot was made up of. We choose the smallest of these numbers to label the particular unknot.

**Definition 39** *The height of a resolution is the number of 1-smoothings performed on the diagram to obtain the resolution.*

**Step 3)** To every unknot in every resolution we assign a graded vector space (or module) $V_l$ of graded dimension $q + q^{-1}$ (here $l$ is the label of the particular unknot). If a resolution consists of $k$ cycles the entire resolution then corresponds to the vector space $V^{\otimes k}$, the tensor factors are placed in increasing order of their labels $l$.

So if for instance we have a resolution consisting of three unknots labelled by 1, 2 and 5 respectively, then the resolution gets assigned the vector space $V_1 \otimes V_2 \otimes V_5$.

The graded vector space $V_l$ has two basis vectors: $v_+$ and $v_-$ with $\deg(v_+) = 1$ and $\deg(v_-) = -1$. This space comes equipped with a multiplication $m$ and a linear map $\Delta$ for which we have:

$$m(v_+ \otimes v_+) = v_+ \tag{50}$$
$$m(v_+ \otimes v_-) = v_- \tag{51}$$
$$m(v_- \otimes v_+) = v_- \tag{52}$$
$$m(v_- \otimes v_-) = 0 \tag{53}$$

and:

$$\Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+ \tag{54}$$
$$\Delta(v_-) = v_- \otimes v_- \tag{55}$$

These maps satisfy the Frobenius condition.

**Definition 40** *A map of degree $n$ is a linear map that maps homogeneous subspaces into homogeneous subspaces, such that for any integer $m$ the subspace of degree $m$ is mapped into the subspace of degree $m+n$. A map of degree 0 is also called a grading preserving map.*

Notice that the maps $\Delta$ and $m$ above are both of degree $-1$.

**Step 4)** For every resolution $\alpha$ we shift the grading of its corresponding vector space $V_\alpha$ by $r_\alpha$, where $r_\alpha$ is the height of the resolution $\alpha$. So every resolution now has a corresponding graded vector space $V^{\otimes k}\{r_\alpha\}$.

**Step 5)** We now have $2^n$ vector spaces and we want to apply linear maps between them. This goes as follows. Suppose we have two resolutions labelled by binary numbers $\alpha$ and $\beta$ with corresponding vectors spaces $A$ and $B$. If $\beta$ is obtained from $\alpha$ by changing one 0-smoothing into a 1-smoothing (in other words: if the binary number $\beta$ can be obtained from the binary number $\alpha$ by changing one bit from 0 to 1 so $\alpha$ and $\beta$ are neighbors) then there will be a linear map from $A$ to $B$. Such a change from $\alpha$ to $\beta$ is always either a splitting of one unknot into two, or a joining of two unknots into one. If two unknots are joined we will apply the multiplication $m$ to their corresponding spaces and if an unknot splits into two we apply the map $\Delta$ to its corresponding space. To the unknots that do not participate we apply the identity map. So we have a map from $A$ to $B$ of the form:

$$m_{ij} \text{ or } \Delta_{ij}$$

Because of the degree shift in step 4 these maps are grading preserving maps.

**Step 6)** Some of these maps will be multiplied by $-1$. Say we have a map $f$ from $V_\alpha$ to $V_\beta$ and the binary numbers $\alpha$ and $\beta$ are equal except in the $j^{th}$ bit. Then we add up the last $n-j$ bits of $\alpha$ and call the result $x$. Then $f$ will be multiplied by $(-1)^x$.

**Step 7)** We take the direct sum between all vector spaces of the same height. We also take the sum of the maps between these vector spaces. This means that we are now left with a sequence of $n$ maps between $n+1$ vector spaces. We have chosen minus signs in step 6 such that this sequence is a chain complex.

**Step 8)**The chain complex $C$ that we have now obtained still needs some degree shifting. That is: $C$ is replaced by $C[-n_-]\{n_+ - 2n_-\}$. Here $n_+$ denotes the number of positive crossing points in the diagram and $n_-$ denotes the number of negative crossing points.

**Step 9)**Finally we take the homology of this chain complex. This is now Khovanov's knot invariant. The *Khovanov polynomial* of a link $L$ is defined by:

$$Kh(L) := \sum_r t^r \mathrm{qdim}(H^r(L))$$

where $H^r$ is the $r^{th}$ homology group. This polynomial contains however slightly less information than the homology, because the actual homology groups form the invariants and not just their isomorphism classes. (in other words: if we had chosen a different diagram we would have obtained the same homology groups, not just isomorphic ones.)

## 7.8 Functoriality

Besides the fact that Khovanov Homology makes it possible to make a better distinction between knots, it has another big advantage. Khovanov homology can be extended to a functor. That is: it can be used to describe 'morphisms' between knots. With this we mean two-dimensional cobordisms embedded in $\mathbb{R}^4$.

We want Khovanov homology to be a functor from the category of links and link-cobordisms to the category of abelian groups. So if a link is described by homology groups, we can describe a 2-manifold $S$ embedded in $\mathbb{R}^4$ as a collection of homomorphisms between these homology groups. The morphisms in the category of link-cobordisms are diffeomorphism classes of 2-manifolds so if two such surfaces are diffeomorphic the corresponding homomorphisms should be equal. It turns out however, that these homomorphisms are only well-defined up to a minus sign.

To a trivial cobordism (that is: $S = K \times [0,1]$ for some link $K$) we naturally assign the identity. This means that if a surface is diffeomorphic to a trivial cobordism then its corresponding homomorphisms should be plus or minus the identity.

**Definition 41** *A link cobordism $S$ is an oriented compact surface properly embedded in $\mathbb{R}^3 \times [0,1]$. The boundary of $S$ is then a disjoint union:*

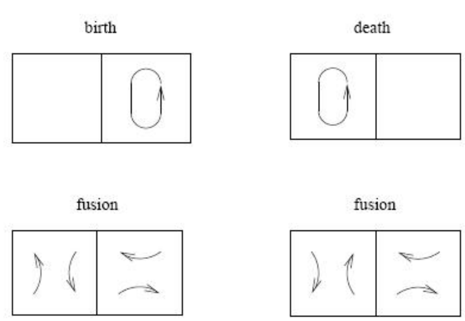
$$\partial S = \partial_0 S \sqcup -\partial_1 S$$

*of the intersection of $S$ with the two boundary components of $\mathbb{R}^3 \times [0,1]$:*

$$\partial_0 S = S \cap \mathbb{R}^3 \times \{0\}$$
$$-\partial_1 S = S \cap \mathbb{R}^3 \times \{1\}$$

Such a surface is always equivalent to a link cobordism for which every 'time-slice' $S \cap \mathbb{R}^3 \times \{i\}$ is a link. So we can represent the entire surface by a series of link diagrams, with every diagram corresponding to a slice.

Such a series of diagrams $J_i$ is called a representation. If we make enough of these slices two consecutive diagrams in a representation will differ only by either a Reidemeister move, a 'birth-' or 'death-'move or a so called 'fusion' move.

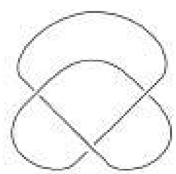


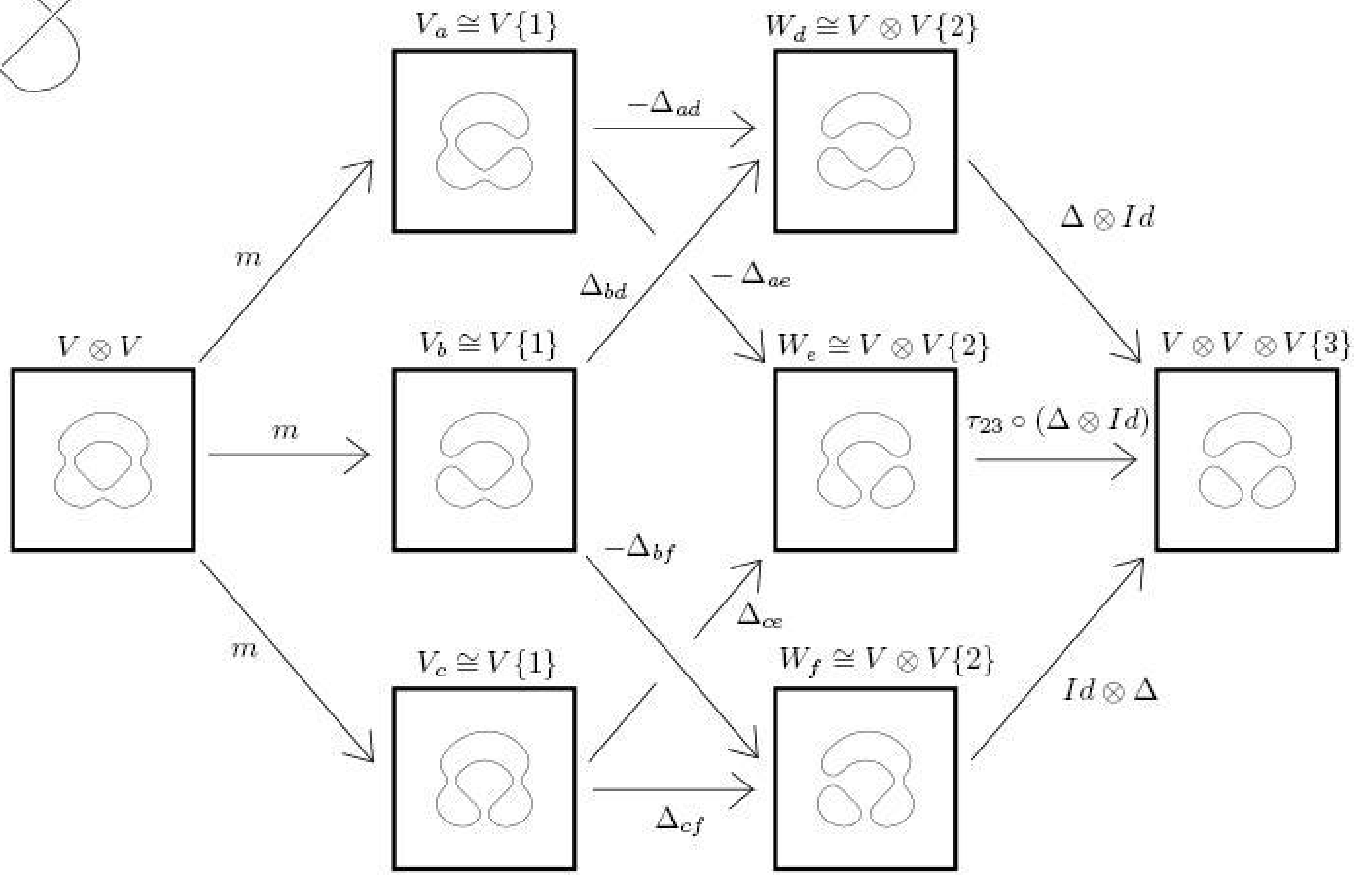
We can assign a Khovanov-complex $C_i$ to every such diagram $J_i$, so to a representation corresponds a series of chain complexes. We want to assign a morphism $f_i$ to every pair of consecutive diagrams $(J_i\ ,\ J_{i+1})$. We can then compose these morphisms to assign a morphism to the entire representation:

$$(J_0\ ,\ ...\ J_{n-1}\ ,\ J_n) \quad \Rightarrow \quad f_n \circ f_{n-1} \circ ... \circ f_0$$

Morphisms $f_i$ of chain complexes induce homomorphisms $F_i^j : H^j(C_i) \to H^j(C_{i+1})$ on the homology groups.

Notice that if two surfaces embedded in $\mathbb{R}^4$ are isotopic, then they are certainly diffeomorphic as manifolds without any embedding. This means that if we can assign Frobenius maps to these surfaces they lead to topological invariants. But we already know that $m$ and $\Delta$ satisfy the Frobenius condition. Moreover, they commute with the chain maps of $C(D)$. For a fusion move we see that at every vertex of the cube either a circle is added or removed, so we can just use these maps. And if for the birth move we use the map $i : 1 \mapsto v_+$ and for the death move we use the map $\epsilon : v_- \mapsto 1, v_+ \mapsto 0$. We see that the chain spaces are indeed Frobenius algebras.



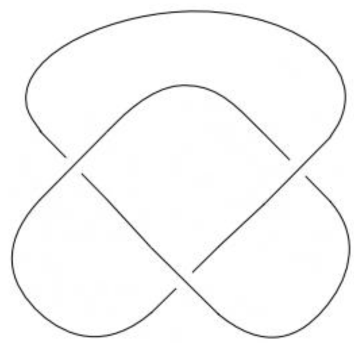



$$0 \longrightarrow V \otimes V \xrightarrow{d^0} V_a \oplus V_b \oplus V_c \xrightarrow{d^1} W_d \oplus W_e \oplus W_f \xrightarrow{d^2} V \otimes V \otimes V\{3\} \xrightarrow{d^3} 0$$

# 8 Example: The Trefoil Knot

## 8.1 Ker($d^0$)

As an example let's calculate the Khovanov polynomial of the trefoil-knot, which has three crossing points. We will use the following diagram:



To simplify notation we will in this section write $V$ instead of $V^O$. The first chain space consists of two unknots as we can see in the picture on the previous page, so its corresponding space is $V \otimes V$. There are three resolutions of height one. They all consist of one unknot so they all have the same vector space: $V\{1\}$.

The first chain map is then a map:

$$V \otimes V \quad \xrightarrow{d^0} \quad V\{1\} \oplus V\{1\} \oplus V\{1\} \tag{56}$$

$$v \otimes w \quad \mapsto \quad (vw,\ vw,\ vw) \tag{57}$$

The first homology group $H^0$ is the kernel of this map, which equals the kernel of $m$. The kernel is:

$$H^0 = \ker(m) = \operatorname{span}\{v_- \otimes v_- \quad , \quad v_- \otimes v_+ - v_+ \otimes v_-\}$$

The fact that this space is in the kernel is checked easily. We also see that both $v_+$ and $v_-$ are in the image of $m$ and since $V \otimes V$ is 4-dimensional we know that the kernel must be 2-dimensional so these vectors indeed span the kernel. We have $\deg(v_- \otimes v_-) = -1 + -1 = -2$ and $\deg(v_- \otimes v_+) = \deg(v_+ \otimes v_-) = -1 + 1 = 0$. The kernel is thus spanned by a homogenous element of degree -2 and a homogenous element of degree 0, so the graded dimension of $H^0$ is:

$$\operatorname{qdim}(H^0) \quad = \quad q^{-2} + q^0 \quad = \quad q^{-2} + 1$$

## 8.2 Im($d^0$)

To calculate $H^1$ we first need to know the image of $d^0$. The image of $m$ is the entire space $V$ and from (57) it follows that the image of $d^0$ is isomorphic to the image of $m$. It is the space that consists of all elements of the form $(x, x, x)$ with $x$ any element in $V$.

## 8.3 $\mathbf{Ker}(d^1)$

The kernel of $d^1$ is a little harder to calculate. We label the three resolutions of height 1 by $a$, $b$ and $c$ and their corresponding vectorspaces by: $V_a$, $V_b$ and $V_c$. The resolutions of height 2 are labelled $d$, $e$ and $f$ respectively. Their corresponding vector spaces are labelled in a similar way. The resolutions of height 2 all consist of two unknots so we have

$$W_d \cong W_e \cong W_f \cong V \otimes V\{2\}$$

As we can see in the picture there are six maps involved:

$$\begin{aligned}
-\Delta_{ad} : V_a &\to W_d \\
-\Delta_{ae} : V_a &\to W_e \\
\Delta_{bd} : V_b &\to W_d \\
-\Delta_{bf} : V_b &\to W_f \\
\Delta_{ce} : V_c &\to W_e \\
\Delta_{cf} : V_c &\to W_f
\end{aligned}$$

These make up $d^1$:

$$\begin{aligned}
d^1 &= -\Delta_{ad} \oplus -\Delta_{ae} \oplus 0 \quad + \quad \Delta_{bd} \oplus 0 \oplus -\Delta_{bf} \quad + \quad 0 \oplus \Delta_{ce} \oplus \Delta_{cf} \\
&= (\Delta_{bd} - \Delta_{ad}) \oplus (\Delta_{ce} - \Delta_{ae}) \oplus (\Delta_{cf} - \Delta_{bf})
\end{aligned}$$

In the following we will denote all these six maps simply by $\Delta$ without causing confusion.

Suppose $d^1(x, y, z) = (0, 0, 0)$, that is:

$$(\ \Delta(y) - \Delta(x), \quad \Delta(z) - \Delta(x), \quad \Delta(z) - \Delta(y)\ ) \quad = \quad (0, 0, 0)$$

Form this we have: $\Delta(x) = \Delta(y) = \Delta(z)$ and because $\Delta$ is injective we have $x = y = z$. Notice that $x$, $y$ and $z$ actually live in different spaces, but there is a canonical isomorphism between them so we can identify them with each other. This means the kernel consists of all elements of the form $(x, x, x)$ and we have already seen that this is exactly the image of $d^0$. Therefore we conclude:

$$H^1 = \{0\}$$

## 8.4 $\mathbf{Im}(d^1)$

For the second chain space $V_a \oplus V_b \oplus V_c \cong V\{1\} \oplus V\{1\} \oplus V\{1\}$ We have the following basis:

$$\{ \quad (v_+, 0, 0) \quad , \quad (v_-, 0, 0) \quad , \quad (0, v_+, 0) \quad , \quad (0, v_-, 0) \quad , \quad (0, 0, v_+) \quad , \quad (0, 0, v_-) \quad \}$$

Let's see how $d^1$ acts on it.

$$\begin{array}{rcl} d^1(v_+,0,0) & = & ( \quad \Delta(0)-\Delta(v_+) \quad , \quad \Delta(0)-\Delta(v_+) \quad , \quad \Delta(0)-\Delta(0) \quad ) \\ & = & ( \quad -\Delta(v_+) \quad , \quad -\Delta(v_+) \quad , \quad 0 \quad ) \\ & = & ( \quad -v_+\otimes v_- - v_-\otimes v_+ \quad , \quad -v_+\otimes v_- - v_-\otimes v_+ \quad , \quad 0 \quad ) \end{array}$$

We see that the span of this basis vector is mapped to a homogenous subspace of degree 2. In the same way we calculate:

$$\begin{array}{rcl} d^1(v_-,0,0) & = & ( \quad -\Delta(v_-) \quad , \quad -\Delta(v_-) \quad , \quad 0 \quad ) \\ & = & ( \quad -v_-\otimes v_- \quad , \quad -v_-\otimes v_- \quad , \quad 0 \quad ) \\ d^1(0,v_+,0) & = & ( \quad v_+\otimes v_- + v_-\otimes v_+ \quad , \quad 0 \quad , \quad -v_+\otimes v_- - v_-\otimes v_+ \quad ) \\ d^1(0,v_-,0) & = & ( \quad v_-\otimes v_- \quad , \quad 0 \quad , \quad -v_-\otimes v_- \quad ) \\ d^1(0,0,v_+) & = & ( \quad 0 \quad , \quad v_+\otimes v_- + v_-\otimes v_+ \quad , \quad v_+\otimes v_- + v_-\otimes v_+ \quad ) \\ d^1(0,0,v_-) & = & ( \quad 0 \quad , \quad v_-\otimes v_- \quad , \quad v_-\otimes v_- \quad ) \end{array}$$

These six vectors in the image are not linearly independent, they span a 4-dimensional space. One can verify that $d^1(v_-,0,0)$, $d^1(0,v_-,0)$ and $d^1(0,0,v_-)$ span a 2-dimensional space of degree 0 and that $d^1(v_+,0,0)$, $d^1(0,v_+,0)$ and $d^1(0,0,v_+)$ span a 2-dimensional space of degree 2. So we can write the image of $d^1$ as a direct sum of its homogenous subspaces:

$$\mathrm{Im}(d^1) \quad = \quad \mathrm{Im}(d^1)_0 \oplus \mathrm{Im}(d^1)_2$$

The subscripts on the right-hand side denote the grading. We have:

$$\dim(\mathrm{Im}(d^1)_0) = 2 \quad \text{and} \quad \dim(\mathrm{Im}(d^1)_2) = 2$$

## 8.5 Im$(d^2)$

We will not yet calculate $\ker(d^2)$ right now, because it turns out to be easier if we calculate $\mathrm{Im}(d^2)$ first.

The third chain space $W = W_d \oplus W_e \oplus W_f$ is 12-dimensional, where $W_d$, $W_e$ and $W_f$ are three copies of $V \otimes V\{2\}$. It has the following basis:

$$\{v_{\pm d}\otimes v_{\pm d} \quad , \quad v_{\pm e}\otimes v_{\pm e} \quad , \quad v_{\pm f}\otimes v_{\pm f}\}$$

Here, for instance $v_{+e}\otimes v_{-e}$, means $(0 \quad , \quad v_+\otimes v_- \quad , \quad 0)$. The fourth chain space corresponds to a resolution with three unknots, so it is a tensor product of three copies of $V$. We label these three tensor factors by $V_1$, $V_2$ and $V_3$. So the fourth chain space is denoted by:

$$V_1 \otimes V_2 \otimes V_3\{3\}$$

Furthermore we have maps:

$$\Delta_d : \quad W_d \to V_1 \otimes V_2 \otimes V_3\{3\}$$

$$\Delta_e : \quad W_e \to V_1 \otimes V_2 \otimes V_3\{3\}$$

$$\Delta_f : \quad W_f \to V_1 \otimes V_2 \otimes V_3\{3\}$$

defined by:

$$\Delta_d = \Delta \otimes \mathrm{Id}$$
$$\Delta_e = \tau_{23} \circ (\Delta \otimes \mathrm{Id})$$
$$\Delta_f = \mathrm{Id} \otimes \Delta$$

where $\tau_{23}$ denotes the 'flip operator on the second and third tensor factor', that is: $\tau_{23}(x \otimes y \otimes z) = x \otimes z \otimes y$.

We have:

$$d^2 \quad = \quad \Delta_d + \Delta_e + \Delta_f$$

$$\begin{array}{rccccc}
\Delta_d : & v_{+d} \otimes v_{\pm d} & \mapsto & v_+ \otimes v_- \otimes v_\pm & + & v_- \otimes v_+ \otimes v_\pm \\
\Delta_d : & v_{-d} \otimes v_{\pm d} & \mapsto & v_- \otimes v_- \otimes v_\pm & & \\
\Delta_e : & v_{+e} \otimes v_{\pm e} & \mapsto & v_+ \otimes v_\pm \otimes v_- & + & v_- \otimes v_\pm \otimes v_+ \\
\Delta_e : & v_{-e} \otimes v_{\pm e} & \mapsto & v_- \otimes v_\pm \otimes v_- & & \\
\Delta_f : & v_{\pm f} \otimes v_{+f} & \mapsto & v_\pm \otimes v_+ \otimes v_- & + & v_\pm \otimes v_- \otimes v_+ \\
\Delta_f : & v_{\pm f} \otimes v_{-f} & \mapsto & v_\pm \otimes v_- \otimes v_- & &
\end{array}$$

We would now like to know the dimension of the image of $d^2$ (the space spanned by the above vectors). It is convenient to use a different notation now. We will omit the tensor symbol $\otimes$, and the symbols $v_+$ and $v_-$ will be replaced by $p$ and $m$ respectively. So for example $v_+ \otimes v_- \otimes v_+$ becomes $pmp$.

Once again because we are dealing with grading preserving maps we can split-up the image in homogenous subspaces:

$$\mathrm{Im}(d^2) = \mathrm{Im}(d^2)_0 \oplus \mathrm{Im}(d^2)_2 \oplus \mathrm{Im}(d^2)_4$$

If we search above for all vectors of degree 2 in the image of $d^2$ we find:

$$\mathrm{Im}(d^2)_2 = \mathrm{span}\{pmm + mpm \quad , \quad mmp \quad , \quad pmm + mmp \quad , \quad mpm \quad , \quad mpm + mmp \quad , \quad pmm\}$$

But we can easily see that these six vectors are not linearly independent so the subspace of degree 2 is actually a 3-dimensional space. We have:

$$\begin{array}{lcccccl}
\mathrm{Im}(d^2)_0 = \mathrm{span}\{ & mmm & & & & & \} \\
\mathrm{Im}(d^2)_2 = \mathrm{span}\{ & mmp & , & mpm & , & pmm & \} \\
\mathrm{Im}(d^2)_4 = \mathrm{span}\{ & pmp + mpp & , & ppm + mpp & , & ppm + pmp & \}
\end{array} \tag{58}$$

Conclusion:

$$\begin{array}{rcl}
\dim(\mathrm{Im}(d^2)_0) & = & 1 \\
\dim(\mathrm{Im}(d^2)_2) & = & 3 \\
\dim(\mathrm{Im}(d^2)_4) & = & 3
\end{array}$$

### 8.6 Ker($d^2$)

We would now like to calculate the kernel of $d^2$, which is quite difficult. However we can calculate its graded dimension right away. We know that for the third chain space $W$ we have subspaces of degree 4, 2 and 0, which have respective dimensions 3, 6 and 3. Furthermore, using the fact that $d^2$ is a graded map, we know:

$$d^2 : W_4 \to \mathrm{Im}(d^2)_4$$

is a map from a 3-dimensional space ($W_4$ denotes the degree 4 subspace of W) onto a 3-dimensional space, so

$$\dim(\ker(d^2)_4) = 0$$

In the same way we have:

$$d^2 : W_2 \to \mathrm{Im}(d^2)_2$$

is a map from a 6-dimensional space onto a 3-dimensional space so

$$\dim(\ker(d^2)_2) = 3$$

And

$$d^2 : W_0 \to \mathrm{Im}(d^2)_0$$

is a map from a 3-dimensional space onto a 1-dimensional space so

$$\dim(\ker(d^2)_0) = 2$$

Conclusion:

$$\dim(H_0^2) = \dim(\ker(d^2)_0) - \dim(\mathrm{Im}(d^1)_0) = 2 - 2 = 0 \tag{59}$$

$$\dim(H_2^2) = \dim(\ker(d^2)_2) - \dim(\mathrm{Im}(d^1)_2) = 3 - 2 = 1 \tag{60}$$

$$\dim(H_4^2) = \dim(\ker(d^2)_4) - \dim(\mathrm{Im}(d^1)_4) = 0 - 0 = 0 \tag{61}$$

Therefore:

$$\mathrm{qdim}(H^2) \quad = \quad 0q^0 + 1q^2 + 0q^4 \quad = \quad q^2 \tag{62}$$

We have now only calculated the graded dimension of $H^2$. It would be nicer if we knew the actual space itself, since this carries more information. We know however that $\mathrm{Im}(d^1) \subset \ker(d^2)$ so we see from (59), (60) and (61) that the space for us to determine, which is in the kernel but not in the image, is only 1-dimensional. Moreover, we see that it must be a subspace of degree 2. After some puzzling around one sees that

$$v_{+d} \otimes v_{-d} + v_{-e} \otimes v_{+e} + v_{+f} \otimes v_{-f}$$

(or in shorter notation: $(pm, mp, pm)$) is in the kernel of $d^2$ and it is linearly independent of $\mathrm{Im}(d^1)_2$ so we can say:

$$\ker(d^2) = \mathrm{Im}(d^1) \oplus k \cdot (pm, mp, pm)$$

where $k$ is the ground field. So we have:

$$H^2 \cong k \cdot (pm, mp, pm)$$

## 8.7 Ker($d^3$)

Since $V_1 \otimes V_2 \otimes V_3\{3\}$ is the last non-trivial chain space, the kernel of $d^3$ is the entire space itself. It is spanned by all vectors of the form $v_\pm \otimes v_\pm \otimes v_\pm$. The fourth homology group is:

$$\begin{array}{rcccl} H^3 & = & \ker(d^3) & / & \mathrm{Im}(d^2) \\ & = & \ker(d^3) & / & \mathrm{Im}(d^2)_0 \oplus \mathrm{Im}(d^2)_2 \oplus \mathrm{Im}(d^2)_4 \end{array}$$

Dividing out $\mathrm{Im}(d^2)_0$ and $\mathrm{Im}(d^2)_2$ is easy. We see from (58):

$$\begin{aligned} H^3 &= \mathrm{span}\{mpp, pmp, ppm, ppp\} \quad / \quad \mathrm{Im}(d^2)_4 \\ &= \mathrm{span}\{mpp, pmp, ppm, ppp\} \quad / \quad \mathrm{span}\{pmp + mpp,\ ppm + mpp,\ ppm + pmp\} \end{aligned}$$

It is clear that $\mathrm{Im}(d^2)_4$ is equal to $\mathrm{span}\{mpp, pmp, ppm\}$. Therefore we have:

$$H^3 \cong \mathrm{span}\{ppp\}$$

Conclusion:

$$\mathrm{qdim}(H^3) = q^6$$

because $ppp$ is an element of degree 6. We have now calculated all homology groups, but we still need an extra degree shift by an amount:

$$[-n_-]\{n_+ - 2n_-\} = [0]\{3\}$$

This amounts to multiplying all graded dimensions of the groups by $q^3$ We can now put together the Khovanov polynomial:

$$q^3 \cdot (\ \mathrm{qdim}(H^0)t^0 + \mathrm{qdim}(H^1)t^1 + \mathrm{qdim}(H^2)t^2 + \mathrm{qdim}(H^3)t^3\ ) \quad =$$

$$q^1 + q^3 + q^5t^2 + q^9t^3$$

# 9 Khovanov-Rozansky Theory

Now that we have categorified the Jones polynomial the question is if we can also categorify other knot invariants. Especially the Homfly polynomial would be interesting since this is the $sl(n)$ generalization of the Jones polynomial. The problem however is that the Homfly polynomial doesn't have such a simple state sum presentation like the Kauffman bracket. A state-sum presentation for the Homfly polynomial does exist however and it is known how to categorify this. We will not attempt to derive or prove this categorification, we will just simply present it.

## 9.1 State Sum Presentation Of Homfly

For the Homfly polynomial we use the following skein relations:

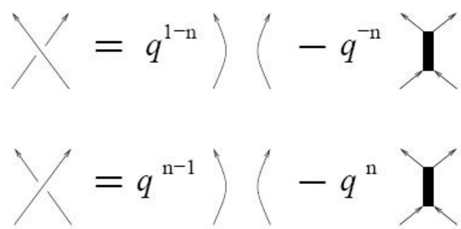
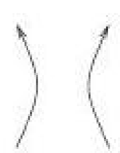
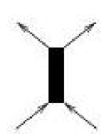



Figure 7: Skein relations

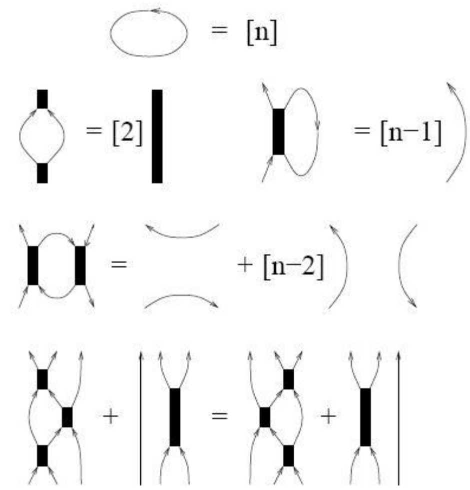
This means that a resolution is now no longer a set of disjoint unknots, but a graph with oriented edges and so-called 'wide' edges. To these graphs we assign polynomials that satisfy the following rules:

These rules determine polynomials $p_\alpha$ for every resolution $\alpha$. (Although at this point it is not really clear that the above skein relations indeed lead to unique polynomials) Then these polynomials $p_\alpha$ can be used to construct the Homfly-polynomial.

The following 'elementary' graphs will be called $\Gamma^0$ and $\Gamma^1$ respectively:

$\Gamma^0 =$  $\Gamma^1 =$ 

In the catergorified version of Homfly, we will assign a chain complex wit two chain spaces to every resolution instead of just a vector space, and the specific link will hence get assigned a cube of chain complexes.




$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}}$$

Figure 8: Skein relations

## 9.2 Matrix Factorizations

Let $R$ be the commutative ring $\mathbb{Q}[x]$ for some set of variables $x = \{x_1, x_2, ...x_k\}$ and $w$ an element of $R$. We define $R$ to be a graded ring by giving each variable $x_i$ degree 2.

**Definition 42** *A matrix factorization $M_w$ is a pair of $R$-modules $(M_0, M_1)$ together with maps $d^0 : M_0 \to M_1$ and $d^1 : M_1 \to M_0$ such that $d^0 d^1 = d^1 d^0 = w$. That is: for any $x \in M_0$ (or $x \in M_1$) we have $d^1 d^0(x) = w \cdot x$ (or $d^0 d^1(x) = w \cdot x$)*

The matrix factorization $M_w$ will often be denoted as:

$$M_0 \xrightarrow{d^0} M_1 \xrightarrow{d^1} M_0$$

Also, a matrix factorization $M_w$ will sometimes be called a $w$-factorization. Notice that a matrix factorizations with $w = 0$ is just a chain complex with two chain spaces.

**Definition 43** *For matrix factorizations $M_{w_1}$ and $N_{w_2}$ we can define the tensor product $M_{w_1} \otimes N_{w_2}$ as :*

$$M_0 \otimes N_0 \oplus M_1 \otimes N_1 \xrightarrow{D_0} M_0 \otimes N_1 \oplus M_1 \otimes N_0 \xrightarrow{D_1} M_0 \otimes N_0 \oplus M_1 \otimes N_1$$

*with chain maps:*

$$d \otimes \mathrm{Id} + (-1)^i \mathrm{Id} \otimes d$$

*where $i = 0$ for $M_0$ and $i = 1$ for $M_1$.*

If $w_1$ and $w_2$ are both in $R$, then the tensor product is with respect to $R$. This is called the internal tensor product. If $w_1$ and $w_2$ are not in the same ring, for instance $w_1 \in \mathbb{Q}[x]$ and $w_2 \in \mathbb{Q}[y]$ then the tensor product is only with respect to $\mathbb{Q}$. This is then called the external tensor product.

**Lemma 14** *$M_{w_1} \otimes N_{w_2}$ is a $w_1 + w_2$-factorization.*

Proof: suppose we have $e_0 \in M_0$ then we can define $d(e_0) = a_m e_1$ with $e_1 \in M_1$ and $d(e_1) = b_m e_0$ for $a_m, b_m \in R$ and $a_m b_m = w_1$. In the same way we can define: $d(f_0) = a_n f_1$ with $f_1 \in N_1$ and $d(f_1) = b_n f_0$ Then from the above definition of the tensor product we see:

$$\begin{aligned}
e_0 \otimes f_0 &\mapsto a_m e_1 \otimes f_0 + a_n e_0 \otimes f_1 \\
e_0 \otimes f_1 &\mapsto a_m e_1 \otimes f_1 + b_n e_0 \otimes f_0 \\
e_1 \otimes f_0 &\mapsto b_m e_0 \otimes f_0 - a_n e_1 \otimes f_1 \\
e_1 \otimes f_1 &\mapsto b_m e_0 \otimes f_1 - b_n e_1 \otimes f_0
\end{aligned}$$

Then we can write the tensor product in matrix form:

$$D_0 = \begin{pmatrix} a_n & b_m \\ a_m & -b_n \end{pmatrix} \quad , \quad D_1 = \begin{pmatrix} b_n & b_m \\ a_m & -a_n \end{pmatrix}$$

it follows that:

$$D_0 D_1 = D_1 D_0 = \begin{pmatrix} a_n b_n + a_m b_m & 0 \\ 0 & a_n b_n + a_m b_m \end{pmatrix} = \begin{pmatrix} w_1 + w_2 & 0 \\ 0 & w_1 + w_2 \end{pmatrix}$$

from which we see that $M_{w_1} \otimes N_{w_2}$ is indeed a $w_1 + w_2$-factorization. $\square$

Let $\mathbf{m}$ denote the maximal ideal of $R$, that is: the ideal generated by the variables $x_i$. In the following we will always assume that $w \in \mathbf{m}$ so that the following definition makes sense:

**Definition 44** *the homology $H^i(M)$ of a factorization is the homology of the chain complex:*

$$M_0/\mathbf{m}M_0 \xrightarrow{d^0} M_1/\mathbf{m}M_1 \xrightarrow{d^1} M_0/\mathbf{m}M_0 \xrightarrow{d^0} M_1/\mathbf{m}M_1$$

Because $w \in \mathbf{m}$ this is indeed a chain complex.

## 9.3 Categorification of Homfly

We now want to assign matrix factorizations to graphs. Suppose we have an arc with endpoints labelled by the two variables $x_i$ and $x_j$ respectively. Let $R$ be the polynomial ring $\mathbb{Q}[x_i, x_j]$. Then to this arc we assign the following $w$-factorization $L^i_j$:

$$R \xrightarrow{\pi_{ij}} R \xrightarrow{x_i - x_j} R$$

where

$$\pi_{ij} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j}$$

so we see that

$$w \quad = \quad x_i^{n+1} - x_j^{n+1}$$

To a tangle of the form $\Gamma^0$ with boundary points labelled by $x_i, x_j, x_k$ and $x_l$ we then assign the external tensor product $C(\Gamma^0) := L^i_j \otimes L^k_l$. Notice that $L^i_j$ is a factorization over the ring $\mathbb{Q}[x_i, x_j]$, while $L^k_l$ is a factorization over $\mathbb{Q}[x_k, x_l]$ so the tensor product is with respect to the field $\mathbb{Q}$.

To a graph of the form $\Gamma^1$ we assign the tensor-product $C(\Gamma^1) = M \otimes N\{-1\}$ of the following two factorizations:

$$M \quad = \quad R \xrightarrow{u_1} R\{1-n\} \xrightarrow{x_1+x_2-x_3-x_4} R$$

and

$$N \quad = \quad R \xrightarrow{u_2} R\{3-n\} \xrightarrow{x_1x_2-x_3x_4} R$$

With:

$$u_1 = \frac{x_1^{n+1} + x_2^{n+1}}{x_1 + x_2 - x_3 - x_4}$$

and

$$u_2 = \frac{-x_3^{n+1} - x_4^{n+1}}{x_1x_2 - x_3x_4}$$

So this is a $x_1^{n+1} + x_2^{n+1} - x_3^{n+1} - x_4^{n+1}$-factorization.

Suppose we have two graphs $T_1$ and $T_2$ with common boundary points $x_i$ and $x_j$. Their corresponding factorizations are denoted $C_1$ and $C_2$. When we glue them together at their common boundary points, we get the graph $T_1 \bigcup_{x_i,x_j} T_2$, which gets assigned the factorization $C_1 \otimes C_2$. The tensor product is with respect to $\mathbb{Q}[x_i, x_j]$.

We see that this way any graph that is built up from $\Gamma^0$, $\Gamma^1$ and loose arcs gets assigned a $w$-factorization with

$$w \quad = \quad \sum_i (-1)^{s_i} x_i^{n+1}$$

where the summation is over all boundary points $x_i$ and $s_i$ is 0 or 1 depending on the orientation of $x_i$. Such a factorization will consist of tensor products of

$C(\Gamma^0)$, $C(\Gamma^1)$ and $L^i_j$. Since a resolution is indeed built up from these elementary graphs and does not have any boundary points it gets assigned a 0-factorization. A 0-factorization is in fact a chain-complex, but to prevent confusion, we will still refer to it as a 0-factorization. Such a 0-factorization has only two homology groups $H^0$ and $H^1$ since it only has two chain spaces.

There are several ways to construct the same resolution from elementary graphs, however the resulting factorizations will all be isomorphic up to homotopy. So every resolution gets assigned a unique homology.

**Lemma 15** *0-factorizations constructed in this way corresponding to the same resolution are quasi-isomorphic.*

We will not prove this.

Notice that this homology is doubly graded. It has its degree in $(\mathbb{Z}, \mathbb{Z}_2)$. The first degree comes from the grading of $R$ and the second degree is the homological degree of the 0-factorization.

**Lemma 16** *If a resolution has $k$ components and $m := k + 1(mod\ 2)$ then $H^m = \{0\}$. ($m$ denotes the $\mathbb{Z}_2$ grading)*

We will not prove this.

This means that, although the resolution gets assigned a doubly graded 0-factorization, we can ignore the $\mathbb{Z}_2$ gradation.

**Definition 45** *A morphism $f : M \to N$ of matrix factorizations is a pair of maps $(f^0, f^1)$ $f^i : M^i \to N^i$ such that they commute with the chain maps $d^i$.*

The morphism $\chi_0 :\ C(\Gamma_0) \to C(\Gamma_1)$ is given by:

$$\chi_0 \quad = \quad (U_0, U_1)$$

$$U_0 = \begin{pmatrix} x_4 - x_2 & 0 \\ a_1 & 1 \end{pmatrix} \quad , \quad U_1 = \begin{pmatrix} x_4 & -x_2 \\ -1 & 1 \end{pmatrix}$$

with:

$$a_1 \quad = \quad -u_2 + \frac{u_1 + x_1 u_2 - \pi_{23}}{x_1 - x_4}$$

The morphism $\chi_1 :\ C(\Gamma_1) \to C(\Gamma_0)$ is given by:

$$\chi_1 \quad = \quad (V_0, V_1)$$

$$V_0 = \begin{pmatrix} 1 & 0 \\ -a_1 & x_4 - x_2 \end{pmatrix} \quad , \quad V_1 = \begin{pmatrix} 1 & x_2 \\ 1 & x_4 \end{pmatrix}$$

To assign a chain complex to a link we once again form a cube and define maps on the edges of this cube. However since the resolutions can be defined as tensor products of the elementary graphs we now only have to look at morphisms between these elementary graphs and the cube can be formed by taking the tensor product of these morphisms. That is: suppose we have a diagram $D$, then to every positive crossing we will assign the complex:

$$0 \to C(\Gamma^0)\{1-n\} \stackrel{\chi_0}{\to} C(\Gamma^1)\{-n\} \to 0$$

to every negative crossing we assign the complex:

$$0 \to C(\Gamma^1)\{n\} \stackrel{\chi_1}{\to} C(\Gamma^0)\{n-1\} \to 0$$

And then we take the tensor product over all these complexes. It follows from the definition of the tensor product of factorizations that this results in a chain complex equal to the one we would have obtained if we would have used the cube construction.

These factorizations have finite rank as $R$-modules. However the graded dimension of $R$ as a vector space over $\mathbb{Q}$ is an infinite series:

$$\mathrm{qdim}(R) = \sum_{i=0}^{\infty} kq^{2i}$$

To keep things finite we define the graded dimension of a factorization $M$ as the following 2-variable polynomial:

$$\mathrm{gdim}(M) := \sum_{j\in\mathbb{Z}, i\in\{0,1\}} \dim(H^{i,j}(M))q^j s^i$$

**Lemma 17** $\mathrm{gdim}(C(\Gamma))$ *satisfies the skein relations in figure 8.*

We will not entirely prove this, but we will outline how this might be proven. We could first try to prove this statement only for graphs that look exactly like in the above picture. However, these pictures represent all resolutions they are part of. That is, the relations should not only hold for exactly these graphs, but for all resolutions that contain them. So the next step is to prove:

$$A \cong B \quad \Rightarrow \quad A \otimes X \cong B \otimes X$$

for matrix factorizations $A$, $B$ and $X$, where $A \cong B$ means that they have the same graded dimension. If this statement holds then this implies that for any two resolutions that are related as in the skein relations, the lemma holds.

**Lemma 18** *Suppose we have a resolution $\alpha$ with 0-factorizaton $C(\Gamma)$, then* $\mathrm{gdim}(C(\Gamma)) = p_\alpha$. *(Where $p_\alpha$ stands for the polynomial that is constructed from the skein relations in section 9.1.)*

Proof: this follows from the previous lemma if the skein relations indeed determine a unique polynomial $p_\alpha$. $\square$

Now we want that the graded Euler-characteristic of the entire chain-complex is the homfly-polynomial, which is an alternating sum of the polynomials $p_\alpha$, which is an alternating sum of the graded dimensions corresponding to the resolutions. If our vectorspaces were finite dimensional this would indeed be equal to the euler characteristic. However, we are working with infinite dimensional vector spaces. So the question is:

*How do we know that the alternating sum of the graded dimensions of the chain spaces in equal to the Euler-characteristic?* Finally we need to show that the resulting homology is invariant under Reidemeister moves.

At this point it might not be entirely clear why we are using matrix factorizations. However, we can see there is some advantage in it. For instance for matrix factorizations we can make a distinction between internal tensor product and external product. In Khovanov homology every resolution consists simply of a finite amount of unknots. So every resolution is described by a positive integer $k$. This makes it easy to assign a vector space. We simply used $V^{\otimes k}$. For Khovanov-Rozanski homology things are a lot more complicated since, as we can see from figure 7, the resolutions are much more complex (although they can be simplified using the relations of figure 8). Therefore we first assign matrix factorizations to the elementary graphs $\Gamma^0$ and $\Gamma^1$. We can build up any resolution from these graphs. To do this we sometimes need to glue boundary points together and sometimes we just need disjoint union. This respectively corresponds to internal and external tensor product of the corresponding matrix factorizations.

# 10 Appendix A: Homology and Euler Characteristic

Chain complexes and homology arise in many branches of mathematics. In this thesis for example we use them as a means of categorification. But they originated from algebraic topology. Here we will give a quick overview to better understand these concepts.

## 10.1 Euler's Formula

A very easy explanation of what Euler Characteristic actually is, is given by John Baez. He explains this as follows: Suppose we have two islands. If we would build a bridge between them this bridge would make these two islands effectively into one.

So a bridge mathematically behaves like a 'negative island'. But what if we would build another bridge between these two islands?

If we still consider a bridge as a negative island this would mean that we had zero islands now. This seems strange at first. However, suppose we would then build a giant deck between the two bridges joining them together sort of like bridge between the two bridges.

This should then behave like a negative bridge, which is the negative of the negative of an island. This indeed works out, because building a bridge between the two bridges can be imagined as if we were making our land into one big island again. This is consistent with the rule that we count every island as 1, every bridge as -1 and every bridge between bridges as 1 again. Then two islands with two bridges and a bridge between the bridges counts as 1. While we could interpret this whole configuration just as well as one big island, which would also count as 1.

We could also look at this as if the two bridges between the two islands create a sort of lake in between them. A lake counts as a negative island. Two islands with two bridges can be seen as one island with a lake in the middle, so this counts as 0. And building a bridge between the two bridges is like filling up the lake with land, effectively removing the lake. This lead Euler to define the *Euler characteristic:*

**Definition 46** *The Euler characteristic is defined as:*

$$\chi = V - E + F$$

*Where $V$ stands for the number of vertices (islands) of a space, $E$ the number of edges (bridges) and $F$ the number of faces (bridges between bridges).*

It can be used as a topological invariant for two-dimensional surfaces. Euler states for instance that for any simple polyhedron this number is 2. If we attach $g$ handles to a sphere then the Euler characteristic is:

$$\chi = 2 - 2g$$

If we remove a point from a surface then the Euler characteristic decreases by one, since this is the same as removing a face while the number of vertices and edges remains the same. So for any 2-dimensional orientable manifold with $k$ punctures we have:

$$\chi = 2 - 2g - k$$

Notice that the Euler characteristic can be easily generalized to higher dimensions.

## 10.2 Singular Homology

**Definition 47** *Let $\{e_0, e_1, e_2, ...\}$ be the standard basis of a real vector space. Then the standard p-simplex is $\Delta_p = \{x = \sum_{i=0}^{p} \lambda_i e_i | \sum \lambda_i = 1, 0 \leq \lambda_i \leq 1\}$*

So $\Delta_0$ is a point, $\Delta_1$ is a line, $\Delta_2$ is a filled triangle and $\Delta_3$ is a filled tetrahedron.

**Definition 48** *The $i^{th}$ face map $F_i^p$ is the linear map $\Delta_{p-1} \to \Delta_p$ which maps the first $i$ basis vectors to themselves and maps the last $p - i$ basis vectors $e_n$ to $e_{n+1}$. In other words: it maps the standard $p-1$-simplex to one of the faces of the standard p-simplex.*

**Definition 49** *If $X$ is a topological space then a map $\sigma_p : \Delta_p \to X$ is called a singular p-simplex. The singular p-chain group $\Delta_p(X)$ is the free abelian group based on the singular p-simplices.*

This means a $p$-chain is a formal sum of $p$-simplices. Notice that a singular 1-simplex is simply a path.

**Definition 50** *If $\sigma$ is a singular p-simplex then the $i^{th}$ face of $\sigma$ is $\sigma^{(i)} = \sigma \circ F_i^p$. The boundary of $\sigma$ is $\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}$. If $c = \sum_\sigma n_\sigma \sigma$ is a p-chain then we define $\partial_p c = \sum_\sigma n_\sigma \partial_p \sigma$.*

This means $\partial_p$ is a homomorphism $\Delta_p(X) \to \Delta_{p-1}(X)$ such that $\partial_p \partial_{p+1} = 0$. We put $\Delta_p(X) = \{0\}$ for $p < 0$ and $\partial_p = 0$ for $p \leq 0$. The sequence of groups $\Delta_i(X)$ and homomorphisms $\partial_i : \Delta_i(X) \to \Delta_{i-1}(X)$ is called the singular chain complex of $X$.

Chains in the kernel of $\partial_p$ are called cycles and chains in the image of $\partial_{p+1}$ are called boundaries. So we have:

$$im\ \partial_{p+1} \subset \ker\ \partial_p \subset \Delta_p(X).$$

Here $im\ \partial_{p+1}$ is a subgroup of $\ker\ \partial_p$ which is in turn a subgroup of $\Delta_p(X)$ so that the following definition makes sense:

**Definition 51** *The $p^{th}$ singular homology group of a space $X$ is:*

$$H_p(x) = \ker\ \partial_p / im\ \partial_{p+1}$$

Two chains $c_1$ and $c_2$ are said to be homologous if $c_1 - c_2 = \partial d$ for some $p+1$ chain $d$ in other words: $c_1$ and $c_2$ are in the same homology class. Homology groups are invariant under homeomorphism so they form topological invariants.

If $f$ and $g$ are paths in $X$ such that $f(1) = g(0)$ then we denote by $f * g$ the concatenation of the two paths.

**Lemma 19** *The 1-chain $f * g - f - g$ is a boundary.*

Proof: this can be seen if we define a map from the standard 2-simplex to $X$ such that it is $f$, $g$ and $f * g$ on its respective faces so that the boundary of the standard simplex is is mapped to $f * g - f - g$. This means that the 1-chain $f + g$ is homologous to the 1-simplex $f * g$. $\square$

### 10.2.1 Example 1

Say we have two arcwise connected spaces $X$ and $Y$. The singular 0-chain group consists of formal sums of points in $X$ and $Y$. Any 0-chain is by definition a cycle. A 0-chain $x_1 - x_2$ is a boundary if and only if $x_1$ and $x_2$ are in the same arc-component because then there is a path from $x_1$ to $x_2$ and then $x_1 - x_2$ is the boundary of the 1-simplex corresponding to this path. So if we have a 0-chain $\sigma = \sum n_i x_i$ with integers $n_i$ and points $x_i \in X$ then its homology class is determined by the number: $\epsilon = \sum n_i$. That is:

$$H_0(X) \cong \mathbb{Z}$$

and similarly:

$$H_0(X \sqcup Y) \cong \mathbb{Z}^2$$

This means that $H_0$ 'measures' the number of arc-components. The number of arc-components is the rank of the group $H_0$.

### 10.2.2 Example 2

Say we have the real two dimensional plane with the origin left out: $\mathbb{R}^2\backslash\{0\}$. $c_1$ and $c_2$ are two 1-chains, such that their images are (homologous to) paths in the plane. If both paths are closed and wind around the origin exactly once then $c_1 - c_2$ is a boundary. Notice that the fact that they are closed paths means they are cycles, since $\partial c_1$ is $f(1) - f(0) = 0$ because $f(0) = f(1)$ where $f$ denotes the path corresponding to $c_1$. Also, if $c_1$ and $c_2$ both do not enclose the origin, they are homologous.

So $\Delta_1(X)$ is the group of formal sums of paths in $X$, ker $\partial_1$ is the group of closed paths in $X$ and $im\ \partial_2$ is the group of closed paths that form boundaries of 2-dimensional area's. $H_1(X)$ is the group of homology-equivalence classes of closed paths.

A contractible path is homologous to the constant path in any point. Also, if $c$ is any 1-chain and $x$ is the constant path through the point $x$ then we see that $x + c$ is homologous to $c$ because $x$ is a boundary (notice that we can view $x$ as a 1-chain which is the boundary of the constant 2-simplex in the point $x$). This means that any constant path, or any closed path that does not enclose the origin is a representative for the identity-element of $H_1$. $H_1$ is then generated by one element: namely the equivalence class of paths that wind around the origin once, so $H_1 \cong \mathbb{Z}$.

### 10.2.3 Example 3

Say we have now the 2-dimensional plane with two punctures: $\mathbb{R}^2\backslash\{x_1, x_2\}$. Now $H_1$ is generated by two equivalence classes: paths that enclose $x_1$, and paths that enclose $x_2$. So we have: $H_1 \cong \mathbb{Z}^2$. We could say that the first homology group measures the number of punctures in the plane.

### 10.2.4 Example 4

Let's extend this to $\mathbb{R}^3$. We embed $\mathbb{R}^2\backslash\{0\}$ into $\mathbb{R}^3\backslash\{0\}$. We see that our paths encircling the origin are now no longer nontrivial. However, if we do not only take away the origin, but the entire $z$-axis (a 'one-dimensional puncture') everything is equivalent to our second example. So in a 3-dimensional space we can say that $H_1$ measures the number of '1-dimensional' punctures and in general it measures in an $n$-dimensional space the number of $n-2$-dimensional punctures.

### 10.2.5 Example 5

Again let's take a look at $X = \mathbb{R}^3\backslash\{0\}$. A 2-cycle is a sphere embedded in $X$. Such a sphere is a boundary if and only if it does not enclose the origin. Hence $H_2$ measures the number of points removed from $\mathbb{R}^3$. And in general $H_2$ measures the number of $n-3$-dimensional punctures of $\mathbb{R}^n$. Even more general: $H_i$ measures the number of $n - i - 1$-dimensional punctures.

## 10.3 CW-Complexes

**Definition 52** *Let $K^{(0)}$ be a discrete set of points. We call these points the 0-cells. If $K^{(n-1)}$ has been defined then we define $K^{(n)}$ as follows: let $\{f_{\partial\sigma}\}$ be a collection of maps $f_{\partial\sigma} : S^{n-1} \to K^{(n-1)}$ where $\sigma$ is an index ranging over some set. Let $Y$ be the disjoint union of copies $D^n_\sigma$ of $D^n$ and let $B$ be the corresponding union of the boundaries $S^{n-1}_\sigma$ then these maps $f_{\partial\sigma}$ form together a map $f : B \to K^{(n-1)}$. Then we define:*

$$K^{(n)} = K^{(n-1)} \cup_f Y$$

*Which means the union of $K^{(n-1)}$ with $Y$ but with $x$ and $f(x)$ identified for all $x \in B$. The map $f_{\partial\sigma}$ is called the attaching map for the cell $\sigma$.*

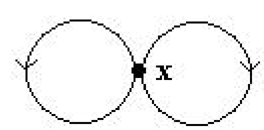
*If $K^{(n)}$ has been defined for all positive integers then we define $K = \cup K^{(n)}$.*

We call $K$ a CW-complex. $K$ has the weak topology. This means that a set is open if and only if its intersection with every $K^{(n)}$ is open in $K^{(n)}$. $K^{(n)}$ is called the $n$-skeleton of $K$.

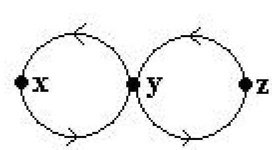
So a CW-complex is in fact something like a 'step by step recipe' for building a topological space form elementary bricks.

### 10.3.1 Example: figure Eight

The easiest way to define a CW-structure on the 'figure eight' is to start with a single point $K^{(0)} = \{x\}$ and then attach two 1-cells, which are two copies of $I = [0, 1]$. We refer to them as $I_1$ and $I_2$. Then the two attaching maps $f_{\partial I_1}$ and $f_{\partial I_1}$ are maps from $\{0, 1\}$ to $\{x\}$ so there is only one way of attaching these 1-cells.




But we could also define a more complicated CW-structure on the figure eight. For instance we could start with three points: $K^{(0)} = \{x, y, z\}$ and with four 1-cells. This means we would have four copies of $I = [0, 1]$ which we refer to as $I_1, I_2, I_3$ and $I_4$. Then for instance $f_{\partial I_1}$ maps 0 to $x$ and 1 to $y$. $f_{\partial I_2}(0) = y$, $f_{\partial I_2}(1) = z$, $f_{\partial I_3}(0) = z$, $f_{\partial I_3}(1) = y$, $f_{\partial I_4}(0) = y$ and $f_{\partial I_3}(1) = x$.



## 10.4 Homology of a CW-Complex

Notice that the image of a singular $p$-cycle in the $n$-skeleton of $K$ is always homologically trivial if $p > n$.

**Lemma 20** *any $n$-simplex in $K$ is homologically equivalent to an $n$-simplex whose image is in $K^{(n)}$.*

Proof: see for instance [4] $\square$.

This means that, in order to calculate $H_n$ of $K$, we can restrict ourselves to $K^{(n)}$. In fact one can prove that $H_n$ is determined by the degrees of the attaching maps.

**Definition 53** *Suppose $f$ is a map from $S^n$ to $S^n$ and $f_*$ is the induced homomorphism on the singular homology groups $H_n(S^n) \cong \mathbb{Z}$. Then the degree* $\deg(f)$ *of $f$ is defined by: $f_*(a) = \deg(f)a$ for all $a \in H_n(S^n)$*

If we for instance consider $S^1$ as the unit circle in the complex plane and $f(z) = z^k$ then $\deg(f) = k$.

**Lemma 21** *The singular homology groups $H_*(K)$ of a CW-complex $K$ are isomorphic to the homology of the chain complex $C_*(K)$ where the chain groups are the $n$-cells and the boundary operator $\partial_n : C_n \to C_{n-1}$ is given by:*

$$\partial_n \sigma = \sum_\tau [\tau : \sigma]\tau$$

*where $\tau$ runs over all $n-1$ cells and $[\tau : \sigma]$ is the degree of the attaching map $f_{\partial\sigma}$.*

This means in particular that the Euler characteristic of the singular chain complex of $K$ is equal to the Euler characteristic of $C_*(K)$. However we can see that this can also be calculated as the alternating sum of the ranks of the chain groups: $\chi = \sum(-1)^i C_i$ but this simply the number of 0-cells minus the number of 1-cells plus the number of 2-cells minus... etc. In other words, this is the number of vertices minus the number of edges plus the number of faces minus... etc. So this is in fact the same as the Euler characteristic we defined in section 10.1!

Since singular homology is defined in terms of maps from $\Delta_p$ *to* $X$ a continuous map $f : X \to Y$ induces a homomorphism $\Delta_p(X) \to \Delta_p(Y)$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \sigma_p \uparrow & & \\ \Delta_p & & \end{array}$$

The simplex $\tau_p : f \circ \sigma_p$ is a simplex in $\Delta_p(Y)$. So composition with $f$ indeed induces a homomorphism from $\Delta_p(X)$ to $\Delta_p(Y)$. Moreover, it induces a homomorphism $f_* : H_p(X) \to H_p(Y)$. This makes $H_*$ into a functor.

The reason for us to talk about singular homology is that it makes very well clear how abstract notions such as homology and Euler characteristic can be used to 'throw away' redundant information. They leave us with purely topological information. Also it shows how homology is a huge refinement of the Euler characteristic. This is essential in Khovanov's theory.

## 10.5 Homology in general

In general we call any sequence of maps between abelian groups $\partial_i : C_i \to C_{i-1}$ such that $\partial_{i-1} \circ \partial_i = 0$ a chain complex. The homology groups $H_i$ are defined as ker $\partial_i / im\ \partial_{i+1}$ The Euler characteristic of the complex is then defined as the alternating sum of the ranks of the homology groups:

$$\chi := \sum (-1)^i rank(H_i)$$

The rank of $H_i$ is called the $i^{th}$ betti number. Notice however that if the chain spaces are finite dimensional vector spaces this sum is exactly the same as the alternating sum of the dimensions of the chain spaces $C_i$ themselves:

$$\chi = \sum (-1)^i \dim(C_i)$$

**Definition 54** *The number $i$ is called the height of the chain space $C_i$.*

If the groups $H^i$ do not define a functor, but rather a contravariant functor, we call them co-homology groups. An example is for instance deRahm cohomology. deRahm cohomology is a contravariant fuctor because it is defined in terms of maps *from* $X$ to a tensor-bundle ($p$-forms).

$$\begin{array}{ccc} & & \wedge^p Y \\ & & \uparrow \alpha \\ X & \xrightarrow{f} & Y \end{array}$$

(To be precise: $\alpha$ is a section of a tensor-bundle) So composition with $f$ makes a $p$-form $\alpha$ on $Y$ into a $p$-form $\beta = \alpha \circ f$ on $X$. For us the distinction between homology and cohomology is not really important so we'll stick with the term homology, although Khovanov himself uses the term cohomology.

# 11 Appendix B: Hopf Algebras and Quantum Groups

## 11.1 Hopf-Algebras

**Definition 55** *An algebra $(A, m, \eta)$ is a vector space $A$ over a field $k$ together with a linear map $m : A \otimes A \to A$ called the multiplication and a linear map $\eta : k \to A$ called the unit.*

**Definition 56** *The algebra is called commutative when $m(a \otimes b) = m(b \otimes a)$ for every $a, b \in A$.*

If $A$ has an algebra structure then it naturally induces an algebra structure on $A \otimes A$. The multiplication $\tilde{m} : (A \otimes A) \otimes (A \otimes A) \to A \otimes A$ is given by:

$$\tilde{m}((a \otimes b) \otimes (c \otimes d)) = m(a \otimes c) \otimes m(b \otimes d)$$

and the unit is given by:

$$\tilde{\eta}(1) = 1 \otimes 1$$

**Definition 57** *A co-algebra $(C, \Delta, \epsilon)$ is a vector space $C$ over a field $k$ together with a co-associative linear map $\Delta : V \to V \otimes V$ called the co-multiplication and a linear map $\epsilon : V \to k$ called the co-unit.*

**Definition 58** *A co-algebra $(C, \Delta, \epsilon)$ is called co-commutative if $\tau \circ \Delta = \Delta$, where $\tau$ denotes the flip map: $\tau(a \otimes b) = (b \otimes a)$, for all $a$ and $b$ in $C$.*

**Definition 59** *A bi-algebra $(B, m, \eta, \Delta, \epsilon)$ is a vector space $B$ over a field $k$ together with a linear maps $m, \eta, \Delta, \epsilon$ such that $(B, m, \eta)$ is an algebra, $(B, \Delta, \epsilon)$ is a co-algebra and $\Delta$ and $\epsilon$ are algebra morphisms.*

**Definition 60** *A bi-algebra morphism is a linear map between bi-algebras that preserves the bi-algebra structure.*

Suppose $A$ and $B$ are two algebras and $\phi$ and $\psi$ are two algebra morphisms from $A$ to $B$. then we define the convolution product $\phi * \psi$, with $x \in A$, as follows:

**Definition 61** $\phi * \psi(x) = m((\phi \otimes \psi)(\Delta(x)))$

**Definition 62** *An antipode is a bi-algebra endomorphism $S$ such that $S * \mathrm{Id} = \mathrm{Id} * S = \eta \circ \epsilon$.*

**Definition 63** *A Hopf-algebra is a bialgebra with an antipode.*

The Hopf-algebra structure is designed after the structure that arises in the space of polynomials on a Lie-group. Suppose we have a matrix Lie-group $G$. Then we can interpret the entries of the matrices that make up $G$ as coordinates on $G$. Take for instance the group $GL_2(k)$ consisting of all $2\times 2$ matrices with $k$-valued entries and non-zero determinant. We denote the entries of these matrices by $a, b, c, d$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then we define the coordinate ring $O(GL_2(k))$ as the ring of all polynomials in the variables $a, b, c$ and $d$.

**Lemma 22** *The coordinate ring $O(G)$ of a group $G$ has a hopf-algebra structure.*

The multiplication $m$ of $O(G)$ is simply defined by pointwise multiplication: $(f \cdot g)(x) = f(x) \cdot g(x)$. The unit is given by: $\eta(1) = \mathbf{1}$ with $1 \in k$ and $\mathbf{1}$ is the function in $O(G)$ defined by $\mathbf{1}(x) = 1$ for all $x \in G$.

Now we use the fact that there is a multiplication defined on $G$ to define a co-multiplication on $O(G)$. The group multiplication is a map from $G \times G$ to $G$. This means that if we have a function $f$ on $G$ then we can define a function $\tilde{f}$ on $G \times G$ as follows: $\tilde{f}(x, y) = f(xy)$. So the multiplication on G naturally induces a map $\Delta$ from $O(G)$ to $O(G \times G)$ sending $f$ to $\tilde{f}$.

**Lemma 23** *$O(G\times G)$ is isomorphic to $O(G)\otimes O(G)$ (if $G$ is an affine algebraic Lie Group, but we will not go into this).*

Proof: this follows from the fact that we can write an element of $O(G \times G)$ as a polynomial in the coordinates of $G\times G$ which can then be written as a product of two polynomials in two distinct sets of variables which are both coordinates of $G$. $\square$

This map $\Delta$ is a co-multiplication. The co-unit $\epsilon$ is simply defined by evaluating the function $f$ in the unit element of $G$:

$$\epsilon(f) := f(e)$$

We still have to check if the co-multiplication and the co-unit are indeed algebra morphisms.

**Lemma 24** *The maps $\Delta$ and $\epsilon$ as defined here are algebra morphisms.*

Proof:

$$\epsilon(f \cdot g) = (f \cdot g)(e) = f(e) \cdot g(e) = \epsilon(f) \cdot \epsilon(g)$$

$$\Delta(f{\cdot}g)(x, y) = f{\cdot}g(x{\cdot}y) = f(x{\cdot}y){\cdot}g(x{\cdot}y) = \Delta(f)(x, y){\cdot}\Delta(g)(x, y) = \Delta(f){\cdot}\Delta(g)(x, y) \quad \square$$

The antipode $S$ is defined by:

$$S(f)(x) := f(x^{-1})$$

We see that we can always obtain a Hopf-algebra from a Lie-group. Such a Hopf-algebra is always commutative, but in general it is not co-commutative. Let's see what this all means for $SL_2(k)$ for instance. $SL_2(k)$ consists of all matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

With $ad - bc = 1$. This means that $O(SL_2(k))$ is defined as the ring of all polynomials in the variables $a, b, c$ and $d$. Divided out by the ideal generated by $ad - bc - 1$.

$$O(SL_2(k)) := k[a, b, c, d]/(ad - bc - 1)$$

The algebra structure of this ring automatically corresponds to the algebra structure defined above for general coordinate rings. For a polynomial $f$ we find the co-unit $\epsilon(f)$ by evaluating $f$ at the identity element $e$ of $SL_2(k)$. This means evaluating $f$ at $a = d = 1$ and $b = c = 0$.

For the co-multiplication we have:

$$\Delta(a) = a \otimes a + b \otimes c$$

$$\Delta(b) = a \otimes b + c \otimes d$$

$$\Delta(c) = c \otimes a + d \otimes c$$

$$\Delta(d) = c \otimes b + d \otimes d$$

Since $\Delta$ is an algebra morphisms it suffices to know how it acts on these four generators of the algebra. These relations follow directly from the above defined co-multiplication for general groups.

## 11.2 From Hopf-algebras to Knots

**Definition 64** *The category of tangles is a category in which the objects are finite ordered sets and the morphisms are tangles. For instance: a (4,2)-tangle is a morphism from a set of 4 elements to a set of 2 elements.*

**Definition 65** *The category of oriented tangles consists of objects that are finite sequences of $+$ and $-$ signs. If $a$ is sequence of $k$ signs and $b$ a sequence of $l$ signs then a morphism from $a$ to $b$ is an oriented tangle of type $(k, l)$ for which the bottom $k$ points are marked according to the source object and the top $l$ points are marked according to the target object.*

In order to describe knots we apply a functor from this category to the category of representations of a certain Hopf-algebra. This category is a strict tensor category since if $V$ is a representation, then the co-multiplication induces an action on $V \otimes V$ or on any $n$-fold tensor product. Also, if $V$ is a representation then the antipode induces an action on the dual space $V^*$. The co-unit makes the ground field $k$ into a representation. The morphisms in this category are linear maps that preserve the hopf-algebra representation structure.

For instance the object $(+)$ is mapped under this functor to a representation $V$ and the object $(-)$ is mapped to its dual representation $V^*$. A sequence of $m$ signs is mapped to an $m$-fold tensorproduct of representations $V$ and $V^*$ such that every $+$ is mapped to $V$ and every $-$ is mapped to $V^*$. For instance:

$$(+,+,-,-,+) \quad \Rightarrow \quad V \otimes V \otimes V^* \otimes V^* \otimes V$$

The empty sequence is then mapped to the ground field $k$.

An $(m, n)$-tangle is then mapped to a morphism from an $m$-fold tensor product of representations to an $n$-fold tensor product of reps. Since a knot is a $(0, 0)$-tangle it is mapped to a $k$-linear morphism from $k$ to $k$, which is itself an element of $k$.

We want these maps to satisfy the same relations as the tangles do, which means specifically that they should satisfy the three Reidemeister moves. So we need an $R$-matrix.

### 11.3 Quantisation

It turns out that, if we want such an $R$-matrix, the Hopf-Algebra should be both non-commutative and non-cocommutative. We have allready seen an example of a Hopf-algebra that is commutative but non-cocommutative. Namely, the space of functions on a Lie-group. The solution to this is to redefine the multiplication such that it is no longer commutative. This must be done in such a way that it keeps its Hopf-algebra structure. This new object is then called a *Quantum Group*.

**Lemma 25** *The dual space of a Hopf Algebra is also a Hopf Algebra*

Suppose we have the Hopf algebra $(A, m, \eta, \Delta, \epsilon, S)$ then we have the dual Hopf algebra $(A^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$.

Here the dual $\Delta^*$ of the co-multiplication defines a multiplication:

$$(\Delta^*(f \otimes g))(x)) = f \otimes g(\Delta(x))$$

The dual $\epsilon^*$ of the co-unit defines a unit:

$$(\epsilon^*(1))(x) = \epsilon(x)$$

$m^*$ defines a co-multiplication:

$$(m^*(f))(x \otimes y) = f(m(x \otimes y))$$

$\eta^*$ defines a co-unit:

$$\eta^*(f) = f(1)$$

and the dual $S^*$ of the antipode defines another antipode:

$$(S^*(f))(x) = f(S(x))$$

**Lemma 26** *Let $L$ be the Lie-Algebra of $G$. Then the universal enveloping algebra $U(L)$ can be considered as the dual Hopf-algebra of $O(G)$.*

Although the coordinate ring $O(G)$ of $G$ is a hopf algebra that is easier to define, one usually prefers to work with its dual $U(L)$. That is because we know from semi-simple theory how to construct representations of $U(L)$. When we have quantised $O(G)$ we define $U_q(L)$ to be the dual of $O(G)_q$.

## 11.4 $U_q(sl(2))$

The standard example of a quantum group is the quantisation of the Lie-algebra $sl(2)$ consisting of all $2 \times 2$ matrices with zero trace. $sl(2)$ is 3-dimensional and its generators are usually denoted by $H$, $E$ and $F$. They satisfy the following relations:

$$[H, E] = 2E$$

$$[H, F] = -2F$$

$$[E, F] = H$$

However we are interested in its quantised enveloping algebra, which satisfies 'quantised' versions of these relations:

**Definition 66** *The quantized universal evneloping algebra $U_q(sl(2))$ is defined as the algebra generated by the four variables $E$,$F$,$K$,$K^{-1}$ with the relations:*

$$KK^{-1} = K^{-1}K = 1 \tag{63}$$

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F \tag{64}$$

*and*

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \tag{65}$$

With the following relations $U_q(sl(2))$ becomes a Hopf-algebra:

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = \epsilon(K^{-1}) = 1,$$

and

$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K.$$

From equations (63), (64) and (65) we can determine its 'highest weight representations'. We will do this explicitly for a complex 2-dimensional representation. This representation is usually denoted by $V_1$ (in general an $n+1$-dimensional representation of $U_q(sl(2))$ is denoted by $V_n$). It is spanned by two vectors which we'll call $v_0$ and $v_1$. We define $v_0$ to be an eigenvector of $K$ (since $\mathbb{C}$ is algebraically closed $K$ must have at least one eigenvector), so without any loss of generality we can define:

$$Kv_0 = \lambda v_0, \quad Ev_0 = 0, \quad Fv_0 = v_1$$

From this and (64) we deduce:

$$Kv_1 = KFv_0 = q^{-2}FKv_0 = q^{-2}\lambda Fv_0 = q^{-2}\lambda v_1$$

Also we have:

$$Ev_1 = EFv_0 = [E,F]v_0 + FEv_0 = [E,F]v_0 = \frac{K-K^{-1}}{q-q^{-1}}v_0 = \frac{\lambda-\lambda^{-1}}{q-q^{-1}}v_0$$

We know that $Fv_1$ should be zero, otherwise we'd have a third eigenvalue for $K$ which is impossible in a 2-dimensional space. So:

$$\begin{aligned}
Fv_1 &= 0 && (66)\\
&= EFv_1 = [E,F]v_1 + FEv_1 = \frac{K-K^{-1}}{q-q^{-1}}v_1 + F\frac{\lambda-\lambda^{-1}}{q-q^{-1}}v_0 && (67)\\
&= \frac{q^{-2}\lambda - q^2\lambda^{-1}}{q-q^{-1}}v_1 + \frac{\lambda-\lambda^{-1}}{q-q^{-1}}v_1 && (68)\\
&\Rightarrow\ q^{-2}\lambda - q^2\lambda^{-1} + \lambda - \lambda^{-1} = 0 && (69)\\
&\Rightarrow\ \lambda = q && (70)
\end{aligned}$$

All put together we see that $V_1$ is determined by:

$$Kv_0 = qv_0, \quad Ev_0 = 0, \quad Fv_0 = v_1$$

$$Kv_1 = q^{-1}v_1, \quad Ev_1 = v_0, \quad Fv_1 = 0$$

The fact that $U_q(sl(2))$ is a Hopf-algebra induces automatically a representation on the tensor product $V_1 \otimes V_1$. The action of an element $x$ of $U_q(sl(2))$ on an element $v \otimes w$ of the tensor product is defined through the co-product:

$$x \cdot v \otimes w \quad = \quad \Delta(x) \cdot v \otimes w$$

So we have for instance:

$$\begin{aligned}
K \cdot v_0 \otimes v_0 &= K \otimes K \cdot v_0 \otimes v_0 = Kv_0 \otimes Kv_0 = q^2 v_0 \otimes v_0\\
F \cdot v_0 \otimes v_0 &= (K^{-1} \otimes F + F \otimes 1) \cdot v_0 \otimes v_0 = q^{-1}v_0 \otimes v_1 + v_1 \otimes v_0\\
E \cdot v_0 \otimes v_0 &= (1 \otimes E + E \otimes K) \cdot v_0 \otimes v_0 = 0
\end{aligned}$$

In the same way we calculate:

$$K \cdot v_0 \otimes v_1 = v_0 \otimes v_1$$

$$F \cdot v_0 \otimes v_1 = v_1 \otimes v_1$$

$$E \cdot v_0 \otimes v_1 = v_0 \otimes v_0$$

$$K \cdot v_1 \otimes v_0 = v_1 \otimes v_0$$

$$F \cdot v_1 \otimes v_0 = q v_1 \otimes v_1$$

$$E \cdot v_1 \otimes v_0 = q v_0 \otimes v_0$$

$$K \cdot v_1 \otimes v_1 = q^{-2} v_1 \otimes v_1$$

$$F \cdot v_1 \otimes v_1 = 0$$

$$E \cdot v_1 \otimes v_1 = v_1 \otimes v_0 + q^{-1} v_0 \otimes v_1$$

If we define: $w_0 := v_0 \otimes v_0, \quad w_1 := q^{-1} v_0 \otimes v_1 + v_1 \otimes v_0, \quad w_2 := v_1 \otimes v_1$ and $t := v_0 \otimes v_1 - q^{-1} v_1 \otimes v_0$ then we can summarize this as:

$$K w_0 = q^2 w_0, \quad F w_0 = w_1, \quad E w_0 = 0$$

$$K w_1 = w_1, \quad F w_1 = (q + q^{-1}) w_2, \quad E w_1 = (q + q^{-1}) w_0$$

$$K w_2 = q^{-2} w_2, \quad F w_2 = 0, \quad E w_2 = w_1$$

and:

$$K t = 1 - q^{-1} t, \quad F t = 0, \quad E t = 0$$

Now since we want to describe knots by morphisms between representations of Hopf-algebras (in this case $U_q(sl(2))$) we need an isomorphism of $V_1 \otimes V_1$ that is linear over $U_q(sl(2))$ and that satisfies the Yang-Baxter equation. Let $\phi$ be such an isomorphism. We have:

$$\phi(K w_0) = K \phi(w_0)$$

but we also have:

$$\phi(K w_0) = \phi(q^2 w_0) = q^2 \phi(w_0)$$

from which follows:

$$K \phi(w_0) = q^2 \phi(w_0)$$

We see that $\phi(w_0)$ is an eigenvalue of $K$ with eigenvalue $q^2$, which means it must be a multiple of $w_0$. Therefore we conclude:

$$\phi(w_0) = \alpha w_0$$

for some complex number $\alpha$. In the same way we can derive $\phi(w_1) = \alpha' w_1$ for some complex number $\alpha'$. Furthermore we have:

$$\phi(w_1) = \phi(Fw_0) = F\phi(w_0)$$

so we have:

$$\alpha' w_1 = F\alpha w_0 = \alpha F w_0 = \alpha w_1 \quad \Rightarrow \quad \alpha' = \alpha$$

This means:

$$\phi(w_i) = \alpha w_i \text{ for } i = 0, 1, 2$$

Also we can derive:

$$\phi(t) = \beta t$$

for some complex number $\beta$.

We now want to write $\phi$ as a matrix with respect to the basis $\{v_0 \otimes v_0, \quad v_0 \otimes v_1, \quad v_1 \otimes v_0, \quad v_1 \otimes v_1\}$:

$$\begin{array}{lcl}
\phi(v_0 \otimes v_0) & = & \phi(w_0) \quad = \quad \alpha v_0 \otimes v_0 \\
\phi(v_0 \otimes v_1) & = & \phi(\dfrac{w_1 + qt}{q + q^{-1}}) \quad = \quad \dfrac{\phi(w_1) + q\phi(t)}{q + q^{-1}} \quad = \quad \dfrac{\alpha w_1 + q\beta t}{q + q^{-1}} \\
 & = & \dfrac{\alpha(q^{-1} v_0 \otimes v_1 + v_1 \otimes v_0) + \beta(q v_0 \otimes v_1 - v_1 \otimes v_0)}{q + q^{-1}} \\
 & = & \dfrac{\alpha q^{-1} + \beta q}{q + q^{-1}} v_0 \otimes v_1 \quad + \quad \dfrac{\alpha - \beta}{q + q^{-1}} v_1 \otimes v_0 \\
\phi(v_1 \otimes v_0) & = & \dfrac{\alpha - \beta}{q + q^{-1}} v_0 \otimes v_1 \quad + \quad \dfrac{\alpha q + \beta q^{-1}}{q + q^{-1}} v_1 \otimes v_0 \\
\phi(v_1 \otimes v_1) & = & \phi(w_2) \quad = \quad \alpha v_1 \otimes v_1
\end{array}$$

We can now write $\phi$ in matrix form:

$$\phi = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & x & y & 0 \\ 0 & y & z & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$$

With $x = \frac{\alpha q^{-1} + \beta q}{q + q^{-1}}$, $y = \frac{\alpha - \beta}{q + q^{-1}}$ and $z = \frac{\alpha q + \beta q^{-1}}{q + q^{-1}}$. After tedious computations (see [8]) one can show that this is an $R$-matrix if $x = 0$, or $y = 0$ or $z = 0$. Let's suppose that $x = 0$. Then we have $y = \alpha q^{-1}$ and $z = (1 + q^{-2})\alpha$. From which follows:

$$\phi = \alpha q^{-1} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Notice that if $\alpha = q^{-1}$ then we have:

$$q^2\phi - q^{-2}\phi^{-1} = (q - q^{-1})\mathrm{Id} \tag{71}$$

The fact that $\phi$ is an $R$-matrix means that it satisfies the Reidemeister moves $RII$ and $RIII$. However, we want a quantity that also satisfies $RI$. Therefore we need a so called *'enhanced R-matrix'* which is a pair $(\phi, \mu)$ with $\phi$ an $R$-matrix and $\mu$ an automorphism of $V_1$ satisfying certain equations. According to Kassel [8] we have such an enhanced $R$-matrix if indeed $\alpha = q^{-1}$ holds.

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