Soft Comput (2005) 9: 875–888
DOI 10.1007/s00500-004-0446-8

FOCUS

A. Di Nola · F. Esteva · L. Godo · F. Montagna

Varieties of BL-algebras

Published online: 31 May 2005

Abstract In this paper we overview recent results about the lattice of subvarieties of the variety **BL** of BL-algebras and the equational definition of some families of them.

1 Introduction

It is well-known that the variety **BL** of BL-algebras is the equivalent algebraic semantics for Hájek's basic fuzzy logic BL. Thus subvarieties of **BL** naturally correspond to schematic extensions of BL, i.e., to sets of formulas which include all axioms of BL and which are closed under substitution and under Modus Ponens. Three outstanding subvarieties of **BL** are the variety **G** of Gödel algebras, the variety **Ł** of Wajsberg algebras and the variety **Π** Product algebras, which are the algebraic counterparts of Gödel, Łukasiewicz and Product logics resepctively. Subvarieties of **G**, **Ł** and **Π** are countable and relatively simple to describe [HK], [K], [DNL2], [Pa], and [CT] (see a short summary later in Sect. 3.1). Contrariwise the lattice *Sub*(**BL**) of subvarieties of **BL** seems extremely difficult. For example we will see in Sect. 2 that there are continuum many subvarieties that are generated by the ordinal sum of two totally ordered Wajsberg algebras, see [AM]. Thus the main task of this paper is to describe some of the most important sublattices (or just subposets) of *Sub*(**BL**).

Since one of the reasons for introducing BL was the search of a common fragment of Łukasiewicz Logic, Gödel Logic and Product Logic, one of the most natural sublattices of *Sub*(**BL**) is the lattice of subvarieties of the join of the varieties of Wajsberg algebras, of Gödel algebras and of Product algebras. Such join, which will be called **ŁΠG** in the sequel, is axiomatized in [CEGT], and the lattice of its subvarieties are fully described in [DNEGGS]. The main results about such lattice are sketched in Sect. 3. It turns out that not only **ŁΠG** is a proper subvariety of **BL**, but the lattice of its subvarieties is countable. Thus there are continuum many subvarieties of **BL** which are not subvarieties of **ŁΠG**.

A second reason for introducing BL was the search of the logic of all continuous t-norms and their residuals. In [CEGT] it is shown that BL is in fact complete with respect to the class of the so called *t-norm* BL-algebras, i.e., of all residuated lattices whose monoid operation is a continuous t-norm on [0, 1]. This result suggests the investigation of the subvarieties of **BL** which are generated by a single t-norm algebra. This has been done in [EGM], and the main results will be sketched in Sect. 4. Surprisingly, the poset of such varieties is countable, and any such variety is finitely axiomatizable.

Another fundamental result in the theory of BL-algebras is the decomposition theorem, which says that every linearly ordered BL-algebra is the ordinal sum of a family of linearly ordered Wajsberg hoops, see [AM], [LS] and [MB]. Hence every BL-algebra can be decomposed as a subdirect product of ordinal sums of Wajsberg hoops. This result suggests the investigation of another interesting poset of varieties, namely the poset consisting of all varieties generated by BL chains which are the ordinal sums of finitely many Wajsberg hoops. This problem will be afforded in Sect. 5, on the ground of a number of results contained in [AM]. It turns out that such poset is not countable and is strictly related with the lattice of universal theories of Wajsberg hoops.

Finally, in Sect. 6 we survey the theory of local and perfect BL-algebras, based on results in [DNSEGG], which extends the theory and results already developed for MV-algebras in [ST, T1].

A. Di Nola (✉)
Dipartamento de Matematica e Informatica,
University of Salerno,
Italy
E-mail: adinola@unisa.it

F. Esteva · L. Godo
Artificial Intelligence Research Institut,
CSIC, Spain

F. Montagna
Dipartamento de Matematica,
University of Siena, Italy

2 General properties

In the sequel, given a class $\mathbf{K}$ of algebras of the same type, $\mathbf{I}(\mathbf{K})$, $\mathbf{S}(\mathbf{K})$, $\mathbf{H}(\mathbf{K})$, $\mathbf{P}(\mathbf{K})$ and $\mathbf{P}_u(\mathbf{K})$ denote the classes of isomorphic images, of subalgebras, of homomorphic images, of direct products and of ultraproducts of algebras from $\mathbf{K}$ respectively. We refer to the papers by Cignoli and Torrens and by Montagna in this issue for notions concerning hoops and BL-algebras and for the notion of ordinal sum. Moreover if, for all $j \in J$, $\mathbf{O}_j$ is a combination of $\mathbf{I}$, $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_u$, then for every collection $\{\mathbf{A}_j : j \in J\}$ of linearly ordered Wajsberg hoops, where J has a minimum j_0 and $\mathbf{A}_{j_0}$ is bounded, $\bigoplus_{j\in J} \mathbf{O}_j(\mathbf{A}_j)$ denotes the set of all algebras of the form $\bigoplus_{j\in J} \mathbf{B}_j$, where for $j \in J$, $\mathbf{B}_j \in \mathbf{O}_j(\mathbf{A}_j)$.

The following general properties concerning ordinal sums are proved in [AM].

Proposition 1 *Let $\mathbf{A} = \bigoplus_{i\in I} \mathbf{A}_i$ be a linearly ordered BL-algebra, where all $\mathbf{A}_i$ are linearly ordered hoops, I has a minimum i_0 and $\mathbf{A}_{i_0}$ is bounded. Then $\mathbf{S}(\bigoplus_{i\in I} \mathbf{A}_i) = \bigoplus_{i\in I} \mathbf{S}(\mathbf{A}_i)$, (where of course $\mathbf{S}(\mathbf{A}_{i_0})$ denotes the class of BL-subalgebras of $\mathbf{A}_{i_0}$, and for $i \neq i_0$, $\mathbf{S}(\mathbf{A}_i)$ denotes the class of subhoops of $\mathbf{A}_i$).*

Note that a subhoop of any hoop may be trivial, whereas a BL-subalgebra of a non-trivial Wajsberg algebra is always non-trivial.

Proposition 2 *Let $\mathbf{A} = \bigoplus_{i=0}^{n} \mathbf{A}_i$ be a BL-algebra, where $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are linearly ordered hoops, and $\mathbf{A}_0$ is a linearly ordered BL-algebra. Then*

$$\mathbf{H}(\mathbf{A}) = \mathbf{H}(\mathbf{A}_0) \cup \{\mathbf{A}_0 \oplus \mathbf{H}(\mathbf{A}_1)\} \cup \ldots \cup \{\mathbf{A}_0 \oplus \cdots \oplus \mathbf{A}_{n-1} \oplus \mathbf{H}(\mathbf{A}_n)\}.$$

Proposition 3 *Let $\mathbf{A}_0, \ldots, \mathbf{A}_n$, be as in Proposition 2. Then $\mathbf{ISP}_u(\bigoplus_{i=0}^{n} \mathbf{A}_i) = \bigoplus_{i=1}^{n} \mathbf{ISP}_u(\mathbf{A}_i)$.*

Since the variety of BL-algebras is congruence distributive, Jónsson Lemma (cf. [Bu, Theorem IV.6.8]) holds. Thus if $\mathbf{K}$ is a class of BL-algebras, the subdirectly irreducible members of the variety generated by $\mathbf{K}$ are in $\mathbf{HSP}_u(\mathbf{K})$. Combining this with Propositions 2 and 3, we obtain:

Theorem 1 *Let $\mathbf{A}_0, \ldots, \mathbf{A}_n$ be as in Proposition 2. Then every subdirectly irreducible member of the variety generated by $\bigoplus_{i=1}^{n} \mathbf{A}_i$ is a member of*

$$\mathbf{HSP}_u(\mathbf{A}_1) \cup (\mathbf{ISP}_u(\mathbf{A}_1) \oplus \mathbf{HSP}_u(\mathbf{A}_2)) \cup \ldots \cup (\bigoplus_{i=1}^{n-1} \mathbf{ISP}_u(\mathbf{A}_i) \oplus \mathbf{HSP}_u(\mathbf{A}_n)).$$

We can use the above results to prove that the lattice of subvarieties of $\mathbf{BL}$ is uncountable.

Theorem 2 *Let $\mathbf{BL}(n)$ denote the variety generated by all ordinal sums of at most $n+1$ linearly ordered Wajsberg hoops, the first one bounded. Then:*

1. *The lattice of subvarieties of $\mathbf{BL}(n)$ is countable if and only if $n = 0$;*
2. *For every $n \geq 0$ there are uncountably many subvarieties of $\mathbf{BL}(n+1)$ which are not subvarieties of $\mathbf{BL}(n)$.*

In particular there are uncountably many subvarieties of BL-algebras.

Proof Note that $\mathbf{BL}(0)$ is just the variety of Wajsberg algebras, whose lattice of subvarieties is countable. For $n > 0$ and for any set X of primes, let $\mathbf{W}_X$ be the subalgebra of the standard Wajsberg algebra $[0,1]_L = ([0,1], \otimes, \Rightarrow, 0, 1)$[1] whose universe is the set of all rational numbers of the form n/m where $m, n \in \mathbf{N}$, $m > 0$, $n \leq m$, and every prime which divides m is in X. Let $\mathbf{W}(X, n) = \mathbf{W}_X \oplus \cdots \oplus \mathbf{W}_X$, $n+1$ times, and let $\mathbf{V}(X, n)$ denote the variety generated by $\mathbf{W}(X, n)$. We prove that, for $n, m > 0$, $\mathbf{V}(X, n) \subseteq \mathbf{V}(Y, m)$ iff $n \leq m$ and $X \subseteq Y$.

The right-to-left implication is trivial, because $\mathbf{W}(X, n)$ is a subalgebra of $\mathbf{W}(Y, m)$.

For the other direction, if $n > m$, then by Propositions 2 and 3 every member of $\mathbf{HSP}_u(\mathbf{W}(Y, m))$ has $m+1$ Wajsberg components at most, whereas $\mathbf{W}(X, n)$ is a subdirectly irreducible member of $\mathbf{V}(X, n)$ with $n+1$ components. Hence there is a subdirectly irreducible element in $\mathbf{V}(X, n) \setminus \mathbf{V}(Y, m)$, and the claim follows.

If $X \not\subseteq Y$, then let $p \in X \setminus Y$. As usual, let $(n)x = x \oplus \overset{n}{\cdots} \oplus x$. Then the fact that $1/p$ is not in $\mathbf{W}_Y$ can be expressed by the universal formula $\forall x \sim (\neg x = (p-1)x)$, where $\sim$ denotes negation in classical logic. Hence the above formula is true in $\mathbf{W}_Y$ and not in $\mathbf{W}_X$. It follows that $\mathbf{W}_X \notin \mathbf{ISP}_u(\mathbf{W}_Y)$, and by Theorem 1, $\mathbf{W}(X, n) \notin \mathbf{HSP}_u(\mathbf{W}(Y, m))$, and once again the result follows.

Summing up, when X ranges over all non-empty sets of prime numbers, $\mathbf{V}(X, n+1)$ describes an uncountable set of subvarieties of $\mathbf{BL}(n+1)$ which are not subvarieties of $\mathbf{BL}(n)$. □

Notation. There are several equivalent presentations of BL-algebras. In this paper we assume a BL-algebra $\mathbf{A}$ to be an algebraic structure of the form $(A, \wedge, \vee, *, \rightarrow, 0, 1)$, where $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, *, 1)$ is a commutative monoid with the unit element 1, and $(*, \rightarrow)$ is an adjoint pair satisfying the divisibility condition $x \wedge y = x * (x \rightarrow y)$ and pre-linearity $(x \rightarrow y) \vee (y \rightarrow x) = 1$. The corresponding negation operation is defined as $\neg x = x \rightarrow 0$ and the equivalence operation as $x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$.

3 Subvarieties of BL generated by single-component chains

Taking into account that any (saturated) BL-chain is an ordinal sum (of BL-algebras) of copies of Wajsberg, Gödel and

[1] Recall that for all $x, y \in [0,1]$, $x \otimes y = \max(0, x+y-1)$ and $x \Rightarrow y = \min(1, 1-x+y)$. Further, $\neg x = x \Rightarrow 0 = 1-x$ and $x \oplus y = \neg x \Rightarrow y = \min(1, x+y)$.

Product chains, we will call a *single-component* BL-chain any BL-chain which is either a Gödel, Wajsberg or Product chain (that is, it is not an ordinal sum of more than one component). In [CEGT], an equation denoted $(L\Pi G)$, which is only satisfied for single-component BL-chains, is introduced. The variety defined by the equations of **BL** plus this equation is called **ŁΠG** and is the least variety containing **Ł**, **Π** and **G**. In that paper equational characterizations are also provided for **ŁΠ**, the least variety containing **Ł** and **Π**, for **ΠG**, the least variety containing **Π** and **G**, and for **ŁG**, the least variety containing **Ł** and **G**.

In [DNEGGS] a full description and equational characterizations of all subvarieties generated by a family of single-component BL-chains is given and they are shown to coincide with the subvarieties of **ŁΠG**. The content of this section summarizes the results of that paper. The first subsection is devoted to recall the well-known equational characterization of the subvarieties of Product, Gödel and MV varieties, while the structure of the lattice of subvarieties and the equational characterization of all subvarieties of **ŁΠG** is presented in the second subsection.

3.1 Gödel, Product and Wajsberg subvarieties and their equational characterization

We review here which are the subvarieties of the three basic varieties, i.e. the variety of Gödel algebras **G**, the variety of Product algberas **Π** and the variety of Wajsberg algebras **Ł**. Recall that Gödel algebras are BL-algebras satisfying the equation

$x = x * x,$

Wajsberg algebras[2] are BL-algebras satisfying the equation

$x = \neg\neg x$

and Product algebras are BL-algebras satisfying

$x \wedge \neg x = 0$

$\neg\neg z \to ((x * x \to y * z) \to (x \to y)) = 1$

The case of **Π** is very simple since, as proved by Cignoli and Torrens in [CT], the only proper subvariety of **Π** is the variety of Boolean algebras **B**. It is also known that the lattice (chain in this case) of subvarieties of **G** is characterized by the following results (see e.g., [HK, Go]).

Theorem 3 *(1) The variety* **G** *is the variety generated by any infinite Gödel chain and it is defined from the equations of* **BL** *by adding the equation*

$$x \to x * x = 1 \qquad (G)$$

(2) The set of subvarieties of **G** *is the set* $\{\mathbf{G}_n \mid n \geq 2\}$ *where* $\mathbf{G}_n$ *is the variety generated by the finite Gödel chain with cardinal equal to n.* $\mathbf{G}_n$ *contains all Gödel chains of length at most n and it is equationally defined by the equations of* **G** *plus the equation*

$$\bigvee_{i=1,2,\dots,n} (x_i \to x_{i+1}) = 1 \qquad (G_n)$$

(3) $\mathbf{B} = \mathbf{G}_2 \subset \mathbf{G}_3 \subset \mathbf{G}_4 \subset \dots \mathbf{G}_n \dots \subset \mathbf{G}$.

Although these results are well-known, let us comment some basic facts. First, all infinite Gödel chains satisfy the same equations since each one of them generates the full variety **G**. On the other hand, a Gödel chain with at most n elements satisfies the equation (G_n) while chains with more than n elements do not satisfy it. Finally, notice that $\mathbf{G}_2$ coincides with the two element Boolean algebra **B**.

The rest of the subsection focuses on the subvarieties of Wajsberg algebras, also known after Chang's work as *MV-algebras*. Actually, it is well known that, although they use different languages, they are definitionally equivalent and we will use both names indistinctly. The usual language of MV-algebras uses Lukasiewicz product ($\otimes$) and sum ($\oplus$) as main operators. Lukasiewicz product corresponds to the $*$ operation of the BL-algebra, while the addition is given by $x \oplus y = \neg x \to y$, which in the particular case of MV-algebras it is also equivalent to $\neg(\neg x * \neg y)$. The following results will be expressed in the language of MV-algebras, as found in the literature, and after they will be rewritten in the language of BL-algebras.

In [K], Komori gave a complete description of the lattice of all subvarieties of **Ł**. He proved that all subvarieties are generated by their chains and they are finitely axiomatizable. Indeed, Grigolia in [GR] gave an axiomatization of the varieties generated by each finite MV-chain, and Rodríguez and Torrens in [RT] gave finite axiomatizations for the subvarieties generated by finite families of finite MV-chain, both using Wajsberg axioms. Finally, following Grigolia's work, Di Nola and Lettieri gave in [DNL2] an equational characterization of all varieties of MV-algebras. In the following we describe the subvarieties and their axiomatizations.

First of all, Komori proved that every subvariety of **Ł** has a finite set of generators. Every generator is a finite chain $S_n = \left\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\right\}$ or an infinite chain $S_n^w = \Gamma((\mathbb{Z} \times \mathbb{Z}), (n, 0))$, where Γ is the Mundici functor [CDOM] between MV-algebras and lattice-ordered abelian groups with strong unit. Here $\mathbb{Z}$ is seen as the totally ordered additive group of integers, and $\mathbb{Z}\times\mathbb{Z}$ is the lexicographic product of $\mathbb{Z}$ by itself. Thus, $S_n^w = \left\{(x, y) : x \in \left\{\frac{1}{n}, \dots, \frac{n-1}{n}\right\}, y \in \mathbb{Z}\right\} \cup \{(0, y) : y \in \mathbb{Z}^+\} \cup \{(1, -y) : y \in \mathbb{Z}^+\} \cup \{(0, 0), (1, 0)\}$, where $\mathbb{Z}^+ = 1, 2, \dots$; furthermore notice that $S_n = \Gamma(\mathbb{Z}, n) \subset S_n^w$.

Actually Komori ([K, Theorem 4.11]) proved that if a subvariety **W** of **Ł** is proper, then there exist two sets of integers $I = \{\alpha_1, \dots, \alpha_s\}$ and $J = \{\beta_1, \dots, \beta_t\}$ such that $I \cup J \neq \emptyset$ and $\mathbf{W} = \mathbf{V}(S_{\alpha_1}, \dots, S_{\alpha_s}, S_{\beta_1}^w, \dots, S_{\beta_t}^w)$[3]. Moreover, Di Nola and Lettieri [DNL2] proved that it is possible to assume, if $I \neq \emptyset$, $\alpha_1 < \alpha_2 < \dots < \alpha_s$ such that α_i is not a divisor

[2] In fact, Wajsberg algebras were introduced in [RT] using $\neg$ and $\to$ as primitive operations, but this presentation is definitionally equivalent to our presentation using the operations of BL-algebras.

[3] As usual, $\mathbf{V}(A_1, \dots, A_n)$ denotes the subvariety of **Ł** generated by $A_1, \dots, A_n$.

of α_j whenever $i < j \le s$; if $J \ne \emptyset$, $\beta_1 < \beta_2 < \cdots < \beta_t$ such that β_i is not a divisor of β_j whenever $i < j \le t$; if both $I \ne \emptyset$ and $J \ne \emptyset$, α_i is not a divisor of β_j for every $i = 1, \ldots, s$ and $j = 1, \ldots, t$. Note that if α_i is a divisor of β_j, then $S_{\alpha_i} \subset S_{\beta_j} \subset S^w_{\beta_j}$.

The finite equational characterization of every subvariety of **Ł** provided by [DNL2] is given in the next theorem. For every $i \in \mathbb{Z}^+$, we set

$\delta(i) = \{n \in \mathbb{Z}^+ : n \text{ is divisor of } i\}$.

Furthermore, if J is a nonempty finite subset of $\mathbb{Z}^+$ and i = 2, 3, ..., we put

$\Delta(i, J) = \delta(i) \setminus \cup_{j \in J} \delta(j)$.

In the case that $J = \emptyset$, then we define $\Delta(i, J) = \delta(i)$.

Theorem 4 *[DNL2] Let* **W** *be a proper subvariety of* **Ł**. *Then there exist finite sets I and J of integers of* $\mathbb{Z}^+$, *with* $I \cup J \ne \emptyset$, *such that* $\mathbf{A} \in \mathbf{W}$ *iff* **A** *satisfies the following equations:*

(1) $((n+1)(x^n))^2 = 2(x^{n+1})$, *where* $n = \max\{I \cup J\}$;
(2) $(p(x^{p-1}))^{n+1} = (n+1)(x^p)$, *for every positive integer* $1 < p < n$ *such that p is not a divisor of any* $i \in I \cup J$;
(3) *if* $I \ne \emptyset$, $(n+1)(x^q) = (n+2)(x^q)$, *for every* $q \in \cup_{i \in I} \Delta(i, J)$.

In this theorem, as usual, x^n denotes $x * \overset{n}{\ldots} * x$ and nx denotes $x \oplus \overset{n}{\ldots} \oplus x$.

Example 1 The variety $\mathbf{V}(S^w_4)$ is defined by the following equations:

$(5x^4)^2 = 2x^5$;
$(3x^2)^5 = 5x^3$;

This corresponds to $I = \emptyset$ and $J = \{4\}$. The variety $\mathbf{V}(S^w_3, S^w_4)$ is defined by the single equation:

$(5x^4)^2 = 2x^5$;

This corresponds to $I = \emptyset$ and $J = \{3, 4\}$. Finally, the variety $\mathbf{V}(S_5, S^w_8, S^w_{12})$ is defined by the following equations:

$13x^{12})^2 = 2x^{13}$;
$(7x^6)^{13} = 13x^7$;
$(9x^8)^{13} = 13x^9$;
$(10x^9)^{13} = 13x^{10}$;
$(11x^{10})^{13} = 13x^{11}$;

This corresponds to $I = \{5\}$, $J = \{8, 12\}$ and $\Delta(5, J) = \{5\}$. □

The above equations by Di Nola and Lettieri are given using the operations of MV-algebras. In order to obtain the equations of the subvarieties generated by a single-component BL-chain, we will need to translate them into equations using operations of **BL**. Namely, the equations appearing in (1), (2) and (3) of Theorem 4 can be equivalently written using only $*$ and $\neg$ as follows:

$$[\neg((\neg(x^n))^{n+1})]^2 \leftrightarrow \neg((\neg(x^{n+1}))^2) = 1 \qquad E_1(n)$$
$$[\neg((\neg(x^{p-1}))^p)]^{n+1} \leftrightarrow \neg((\neg(x^p))^{n+1}) = 1 \qquad E_2(n, p)$$
$$\neg((\neg(x^q))^{n+1}) \leftrightarrow \neg((\neg(x^q))^{n+2}) = 1 \qquad E_3(n, q)$$

3.2 Subvarieties of **ŁΠG**

As already mentioned, the least variety containing **Ł**, **Π** and **G** called **ŁΠG** has been equationally characterized in [CEGT] by the equations of **BL** plus the following equation:

$$(x \to x * y) \to [(x \to 0) \vee y \vee ((x \to x * x) \wedge (y \to y * y))] = 1 \quad (L\Pi G)$$

It is easy to check that the only chains contained in **ŁΠG** are the single-component ones, i.e., chains in one of the varieties **Ł**, **Π** and **G**[4]. As a consequence we have that the class of varieties generated by single-component chains coincides with the full lattice of subvarieties of **ŁΠG**. Next theorem gives a full description of the subvarieties of **ŁΠG**.

Theorem 5 *The lattice of the subvarieties of* **ŁΠG** *is the direct product of the lattices of the subvarieties of* **Ł**, **Π** *and* **G**. *Moreover the sublattice of the subvarieties of* **ŁΠ**, **ŁG** *and* **ΠG** *are the direct product of the lattice of the subvarieties of* **Ł** *and* **Π**, *of* **Ł** *and* **G** *and of* **Π** *and* **G** *respectively.*

The following result is crucial in the proof of this theorem: let **V** be a subvariety of **ŁΠG** and let $\mathbf{V}_L$, $\mathbf{V}_\Pi$ and $\mathbf{V}_G$ the subvarieties of **V** containing the algebras of **V** satisfying respectively the equations of **Ł**, **Π** and **G**. Then:

(1) $\mathbf{V}_L$, $\mathbf{V}_\Pi$ and $\mathbf{V}_G$ are the subvarieties obtained as the intersections of **V** with **Ł**, **Π** and **G** respectively, and they determine univocally the variety **V** as subvariety of **ŁΠG**.
(2) For each $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$, subvarieties of **Ł**, **Π** and **G** respectively, there exists a unique subvariety **V** of **ŁΠG** such that $\mathbf{V}_L = \mathbf{V}_1$, $\mathbf{V}_\Pi = \mathbf{V}_2$ and $\mathbf{V}_G = \mathbf{V}_3$.

Therefore, the above theorem also provides a complete description of the lattice of subvarieties of **ŁΠG**, which is depicted in Fig. 1.

Taking into account that all subvarieties of **Ł**, **Π** and **G** are finitely generated, it follows that all subvarieties of **ŁΠG** are finitely generated as well. To obtain the equations of these subvarieties (as subvarieties of **ŁΠG**) we need to combine the equations characterizing subvarieties of **Ł**, of **G** and **Π** that is:

$$\neg\neg x \to x = 1 \qquad (L)$$
$$[\neg((\neg(x^n))^{n+1})]^2 \leftrightarrow \neg((\neg(x^{n+1}))^2) = 1 \qquad E_1(n)$$
$$[\neg((\neg(x^{p-1}))^p)]^{n+1} \leftrightarrow \neg((\neg(x^p))^{n+1}) = 1 \qquad E_2(n, p)$$
$$\neg((\neg(x^q))^{n+1}) \leftrightarrow \neg((\neg(x^q))^{n+2}) = 1 \qquad E_3(n, q)$$
$$x \to (x * x) = 1 \qquad (G)$$
$$\bigvee_{i=1,2,\ldots,n} (x_i \to x_{i+1}) = 1 \qquad (G_n)$$
$$l(x \wedge \neg x) \to 0 = 1 \qquad (\Pi 1)$$
$$(x \to x * x) \to ((x \to 0) \vee x) = 1 \qquad (\Pi 3)$$

[4] For any non-trivial ordinal sum take two elements x, y from different components such that $0 < x < y < 1$ and at least one of them is non idempotent. Then $x \to x * y = x \to x = 1$, but $(x \to 0) \vee y \vee ((x \to x * x) \wedge (y \to y * y)) < 1$ since $x \to 0 < y < 1$, $y < 1$ and either $x * x < x$ or $y * y < y$.

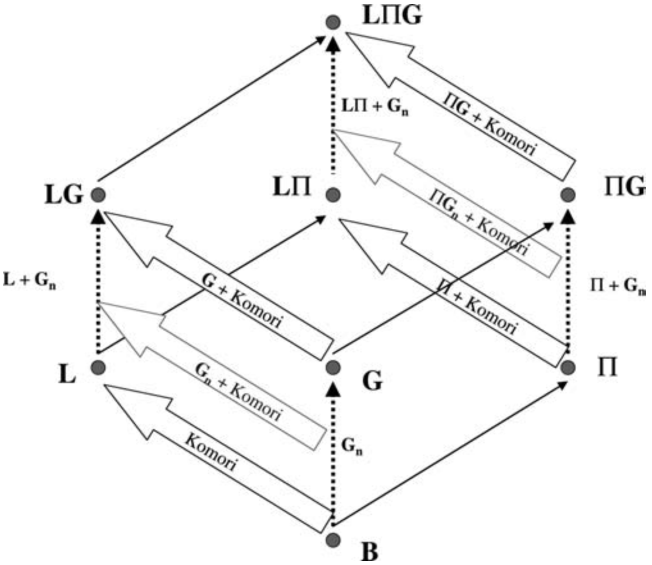

Fig. 1 Graph of subvarieties of **ŁΠG**

Notation : Notice that all the equations written so far have been deliberately written under the form "*term* = 1". In this way it will be easy to combine them and also will allow us to use to a safe notation to refer to the left hand side terms of these equations (the term that is not 1) by just adding the name of a variable(s) to the corresponding label. That is, for instance, $L(x)$ will stand for $\neg\neg x \to x$, $G(x)$ for $x \to (x * x)$, and so forth for $\Pi 1(x)$, $\Pi 3(x)$ and $G_n(x_1, \dots, x_{n+1})$ respectively.

Using this above notation, subvarieties of **ŁΠG** are equationally characterized as follows. Let **V** be a subvariety of **ŁΠG** and let $\mathbf{V}_L$, $\mathbf{V}_\Pi$ and $\mathbf{V}_G$ be as in (1). Then there exists a set of equations $\{E_i^{V_L}(x) = 1 \mid i \in I_1\}$ characterizing $\mathbf{V}_L$ as subvariety of **Ł**; analogously, let $\{E_i^{V_\Pi}(x) = 1 \mid i \in I_2\}$ and $\{E_i^{V_G}(x) = 1 \mid i \in I_3\}$ be the set of equations defining $\mathbf{V}_\Pi$ and $\mathbf{V}_G$ as subvarieties of **Π** and **G** respectively.

Theorem 6 *With the above assumptions,* **V** *is the subvariety of* **ŁΠG** *determined by the equation*

$$\left[L(x) \wedge \left(\bigwedge_{i\in I_1} E_i^{V_L}(x)\right)\right] \vee$$
$$\left[\Pi 1(x) \wedge \Pi 3(x) \wedge \left(\bigwedge_{i\in I_2} E_i^{V_\Pi}(x)\right)\right] \vee$$
$$\left[G(x) \wedge \left(\bigwedge_{i\in I_3} E_i^{V_G}(x)\right)\right] = 1.$$

Of course, in many cases, this is not the simplest equation defining the subvariety. For example, the equational characterization of the varieties **ŁΠ**, **ΠG** and **ŁG** given in [CEGT] is much simpler. Namely, they consist of the equations of **BL** plus the equation

$$x \to (x * y) = \neg x \vee y \qquad (L\Pi)$$

for the variety **ŁΠ**; the equation

$$(x \wedge \neg x) = 0 \qquad (\Pi 1)$$

and $(L\Pi G)$ for the variety **ΠG**; and the equation

$$(\neg\neg x \to x) \vee (x \to x * x) = 1 \qquad (LG)$$

and $(L\Pi G)$ for the variety **ŁG**. In [DNEGGS], the interested reader can find equational characterizations and lattice descriptions for the particular cases of subvarieties of **ŁΠ**, **ŁG** and **ΠG**.

An interesting remark is that all the equations used above in the characterization of the subvarieties of **ŁΠG** except for (G_n), involve only one variable. In particular, the variety **Π**, as extension of **ŁΠG**, can be characterized by the equations (Π1) and (Π3), both using only one variable. This is obviously not true for characterizing **Π** as subvariety of **BL**.

Notice also that the standard completeness results for the logics corresponding to the varieties **BL**, **ŁΠG**, **ŁΠ**, **ŁG** and **ΠG** given in [CEGT] do not extend to arbitrary axiomatic extensions of BL, in particular to arbitrary axiomatic extensions of **ŁΠG**. This is the case for example of $\mathbf{S}_n$ or $\mathbf{G}_n$. In fact there is no t-norm algebra belonging to these varieties so it is impossible that the corresponding logics be standard complete.

4 Varieties generated by t-norm algebras

In this section we study the subvarieties of **BL** generated by a special class of BL-chains, called *regular*, containing the standard BL-chains, that is, the BL-chains defined by a t-norm and its residuum on [0, 1]. Mainly we prove that there exist a special subset of them, called *canonical regular* BL-chains with the property that for each regular BL-chain there exists a canonical regular BL-chain defining the same subvariety. Moreover two different canonical regular BL-chains define different subvarieties. In this section we give also two algorithms. The first one, given two canonical BL-chains, checks whether the subvariety generated by one chain is embeddable into the subvariety generated by the other. The second algorithm, given a canonical BL-chain, finds a finite set of equations defining the variety generated by the chain. From a logical point of view, this means that we provide, for each regular BL-chain, an effective method to find the axiomatic extensions of BL defining a logic complete with respect to the given BL-chain. In particular, in the t-norms setting this means that we provide for each continuous t-norm, an effective method to find the axiomatic extensions of BL defining the logic of the given continuous t-norm. Let **A** be a BL-algebra and let **V**(**A**) denote the variety generated by **A**. In this section we also prove that the set of subvarieties {**V**(**A**) | **A** is a standard BL-chain} is countable while there are uncontably many subvarieties of **BL**. Finally let us mention that most of the proofs are not included but can be found in the paper [EGM] that is the basic reference of this section. Only a few alternative, shorter proofs of some results are included.

4.1 Regular BL-chains and their generated varieties

In the decomposition of a standard BL-chain as an ordinal sum of Wajsberg hoops, there is always a first component, which is necessarily bounded, and any component appearing in the decomposition is (isomorphic to) one of following hoops:

- **2**, the hoop defined on the set of two idempotent elements $\{0, 1\}$, coinciding with the two-element Boolean algebra.
- $\mathcal{L}$, the hoop defined on $[0, 1]$ by Lukasiewicz's t-norm and its residuum, and coinciding with the corresponding standard BL-algebra. (This is obviously a bounded hoop.)
- $\mathcal{C}$, the cancellative hoop defined on the semi-open interval $(0, 1]$ by the product t-norm and its residuum.

A BL-chain **A** is called *regular* if it is the ordinal sum of Wajsberg hoops of the form $\mathcal{L}$, $\mathcal{C}$ and **2**, and **A** has a first component (which is either $\mathcal{L}$ or **2**). The class of regular BL-chains will be denoted by *REG*. Obviously t-norm algebras, those isomorphic to $[0, 1]_\star = ([0, 1], \min, \max, \star, \rightarrow, 0, 1)$ for some continuous t-norm $\star$, are regular but there are regular BL-chains that are not t-norm algebras. Furthermore, the subclass of *REG* consisting of those BL-chains which are *finite* ordinal sums of Wajsberg hoops will be denoted by Fin.

Definition 1 *Let* $\mathbf{A} \in REG$. *Then* $Fin(\mathbf{A})$ *denotes the set of all finite ordinal sums of Wajsberg hoops* $\mathbf{W}_0, \ldots, \mathbf{W}_n$ *such that the following conditions hold:*

- *Each* $\mathbf{W}_i$ *is isomorphic either to* **2**, *or to* $\mathcal{C}$ *or to* $\mathcal{L}$.
- $\mathbf{W}_0$ *is either* **2** *or* $\mathcal{L}$.
- *There are components* $\mathbf{A}_0 < \ldots < \mathbf{A}_n$ *of* **A** *such that* $\mathbf{A}_0$ *is the first component of* **A**, *and for every* i, *if* $\mathbf{W}_i$ *is isomorphic to* $\mathcal{L}$, *then* $\mathbf{A}_i$ *is isomorphic to* $\mathcal{L}$, *if* $\mathbf{W}_i$ *is isomorphic to* $\mathcal{C}$ *then* $\mathbf{A}_i$ *is isomorphic either to* $\mathcal{C}$ *or to* $\mathcal{L}$, *and if* $\mathbf{W}_i$ *is isomorphic to* **2** *then* $\mathbf{A}_i$ *is isomorphic either to* **2** *or to* $\mathcal{L}$.

In the sequel, and to simplify notation, if $\star$ is any continuous t-norm, then we will write $Fin(\star)$ for $Fin([0, 1]_\star)$. The last definition plays an important role, since it turns out that, for any regular BL-chain **A**, the set $Fin(\mathbf{A})$ fully determines the variety generated by **A**.

First steps toward this result are given in the following lemma, which is based on Proposition 3.

Lemma 1 *Let* $\mathbf{A} \in REG$. *Then:*

(1) Every finitely generated subalgebra of **A** *is a subalgebra of at least one algebra in* $Fin(\mathbf{A})$.
(2) Every algebra in $Fin(\mathbf{A})$ *is in* $\mathbf{ISP}_u(\mathbf{A})$.
(3) $\mathbf{ISP}_u(\mathbf{A}) = \mathbf{ISP}_u(Fin(\mathbf{A}))$.

Actually, from (3) of Lemma 1 one can check that $\mathbf{V}(\mathbf{A}) = \mathbf{V}(Fin(\mathbf{A}))$ for any $\mathbf{A} \in REG$. Indeed, the following chain of equalities hold: $\mathbf{V}(\mathbf{A}) = \mathbf{V}(\mathbf{ISP}_u(\mathbf{A})) = \mathbf{V}(\mathbf{ISP}_u(Fin(\mathbf{A})))$ $= \mathbf{V}(Fin(\mathbf{A}))$. But we can prove more than this. In fact we are going to show that for any $\mathbf{A}, \mathbf{B} \in REG$, we have $\mathbf{V}(\mathbf{A}) \subseteq \mathbf{V}(\mathbf{B})$ iff $Fin(\mathbf{A}) \subseteq Fin(\mathbf{B})$.

To prove this characterization, what we actually do is, given a regular BL-chain **A**, to characterise those regular BL-chains in Fin which do not belong to $Fin(\mathbf{A})$. We start by considering the following terms:

$e_{\mathcal{L}}(x) : (x \rightarrow x^2) \vee ((x \rightarrow x^3) \rightarrow x^2)$
$e_{\mathcal{C}}(x) : x \rightarrow x^2$
$e_{\mathbf{2}}(x) : (x \rightarrow x^3) \rightarrow x^2$

where expressions of the form x^m stand for abbreviations of $x \star \cdots \star x$, m times. Notice that:

- the equation $e_{\mathcal{L}}(x) = 1$ is valid in **2** and in cancellative hoops and it is not valid in any MV chain with more than two elements.
- the equation $e_{\mathcal{C}}(x) = 1$ is valid in **2** and it is not valid either in any MV chain with more than two elements or in non-trivial cancellative hoop.
- the equation $e_{\mathbf{2}}(x) = 1$ is valid in any cancellative hoop and it is not valid either in **2** or in any MV chain.

This leads us to define an equation associated to each BL-chain of Fin. Namely, let $\mathbf{A} = \bigoplus_{i=0,n} \mathbf{A}_i \in Fin$, and for each $i = 0, \ldots, n$ let $e_i^{\mathbf{A}}$ be $e_{\mathcal{L}}$ if $\mathbf{A}_i$ is an MV algebra with more than two elements, be $e_{\mathcal{C}}$ if $\mathbf{A}_i$ is a non-trivial cancellative hoop, and be $e_{\mathbf{2}}$ if $\mathbf{A}_i$ is **2**. Then we define the following equation:

$$(e_{\mathbf{A}}) : \left[\left(\bigwedge_{i=0\ldots n-1} ((x_{i+1} \rightarrow x_i) \rightarrow x_i)\right) \star (\neg\neg x_0 \rightarrow x_0) \rightarrow \left(\bigvee_{i=0\ldots n} x_i\right)\right] \vee \left[\bigvee_{i=0\ldots n} \mathrm{e}_i^{\mathbf{A}}(x_i)\right] = 1.$$

By construction, the equation $(e_{\mathbf{A}})$ is not valid in **A** (Hint: take a sequence of values $0 \leq x_0 < x_1 < \cdots < x_n < 1$ such that for each $i = 0, ..., n$, $x_i \in \mathbf{A}_i$. Hence $(\bigwedge_{i=0\ldots n-1}((x_{i+1} \rightarrow x_i) \rightarrow x_i)) \star (\neg\neg x_0 \rightarrow x_0) = 1$, but $\bigvee_{i=0\ldots n} x_i = x_n < 1$ and $e_i^{\mathbf{A}}(x_i)) < 1$ for all $i = 0, ..., n$.) Moreover, it is not difficult to prove the following stronger result.

Lemma 2 *Let* $\mathbf{D} \in REG$, *and let* $\mathbf{A} \in Fin$. *Then the equation* $(e_{\mathbf{A}})$ *is valid in all* $\mathbf{B} \in Fin(\mathbf{D})$ *iff* $\mathbf{A} \notin Fin(\mathbf{D})$.

Theorem 7 *Let* $\mathbf{D}, \mathbf{E} \in REG$. *Then* $\mathbf{V}(\mathbf{D}) \subseteq \mathbf{V}(\mathbf{E})$ *iff* $Fin(\mathbf{D}) \subseteq Fin(\mathbf{E})$. *Thus, in particular,* $\mathbf{V}(\mathbf{D}) = \mathbf{V}(\mathbf{E})$ *iff* $Fin(\mathbf{D}) = Fin(\mathbf{E})$

Proof One direction is easy. If $Fin(\mathbf{D}) \subseteq Fin(\mathbf{E})$ then $\mathbf{V}(\mathbf{D}) = \mathbf{V}(Fin(\mathbf{D})) \subseteq \mathbf{V}(Fin(\mathbf{E})) = \mathbf{V}(\mathbf{E})$. As for the other direction, assume $\mathbf{V}(\mathbf{D}) \subseteq \mathbf{V}(\mathbf{E})$ and $Fin(\mathbf{D}) \not\subseteq Fin(\mathbf{E})$. Then there is $\mathbf{A} \in Fin(\mathbf{D})$ and $\mathbf{A} \notin Fin(\mathbf{E})$. Then, by Lemma 2, the equation $(e_{\mathbf{A}})$ will be valid in $Fin(\mathbf{E})$, hence in $\mathbf{V}(Fin(\mathbf{E})) = \mathbf{V}(\mathbf{E})$, hence in $\mathbf{V}(\mathbf{D})$, hence in $Fin(\mathbf{D})$. Now, again by Lemma 2, $\mathbf{A} \notin Fin(\mathbf{D})$, contradiction. □

By using (3) of Lemma 1, a consequence of this result is that, for any $\mathbf{A}, \mathbf{B} \in REG$, $\mathbf{A} \in \mathbf{V}(\mathbf{B})$ iff $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{B})$. Hence if **A** satisfies all equations valid in **B**, it also satisfies all universal formulas valid in **B**.

4.2 Canonical regular BL-chains and embedding algorithm

Thanks to Theorem 7, in the first part of this section we show that regular BL-chains admit a kind of canonical form, in terms of their generated varieties, as *finite* ordinal sums whose components may be either basic components (i.e. $\mathbf{2}, \mathcal{C}, \mathcal{L}$ components) or complex components consisting in turn of ordinal sums of infinite copies of $\mathbf{2}, \mathcal{C}, \Pi = \mathbf{2} \oplus \mathcal{C}$ and $\mathcal{L}$ components. This canonical representation is used in the second part of the section to develop an algorithm to check when the variety generated by a regular BL-chain $\mathbf{A}$ is included in the variety generated by another regular BL-chain $\mathbf{B}$.

In the sequel, $\mathcal{C}^\infty$ denotes the ordinal sum of ω copies of $\mathcal{C}$, Π^∞ denotes the ordinal sum of ω copies of Π, and $\mathcal{L}^\infty$ denotes the ordinal sum of ω copies of $\mathcal{L}$. (Recall that $\mathcal{G}$ denotes in fact the ordinal sum of continuum many copies of $\mathbf{2}$.)

The proofs of next lemmas, propositions and theorems mainly use the result of Theorem 7 and also that of Lemma 2, that is, they are proved by computing the set of $Fin(\mathbf{A})$ for the different regular BL-chains appearing in them.

Lemma 3 *Let* $\mathbf{B} \in REG$. *Assume* $\mathbf{B} = \mathbf{D} \oplus \mathbf{A} \oplus \mathbf{D}'$, *with* $\mathbf{D}$ *and* $\mathbf{D}'$ *possibly empty. If* $\mathbf{A}$ *has no* $\mathcal{L}$ *component, then:*

(1) If $\mathbf{A}$ *has infinitely many* $\mathbf{2}$ *components and no* $\mathcal{C}$ *component, then* $\mathbf{V}(\mathbf{B}) = \mathbf{V}(\mathbf{D} \oplus \mathcal{G} \oplus \mathbf{D}')$.
(2) If $\mathbf{A}$ *has infinitely many* $\mathcal{C}$ *components and no* $\mathbf{2}$ *component, then* $\mathbf{V}(\mathbf{B}) = \mathbf{V}(\mathbf{D} \oplus \mathcal{C}^\infty \oplus \mathbf{D}')$.
(3) If $\mathbf{A}$ *has infinitely many alternations*[5] *of* $\mathcal{C}$ *and* $\mathbf{2}$, *then* $\mathbf{V}(\mathbf{B}) = \mathbf{V}(\mathbf{D} \oplus \Pi^\infty \oplus \mathbf{D}')$.

If $\mathbf{A}$ *has infinitely many* $\mathcal{L}$ *components, then:*

(4) $\mathbf{V}(\mathbf{B}) = \mathbf{V}(\mathbf{D} \oplus \mathcal{L}^\infty)$.

Moreover in this last case, if the first Wajsberg component of $\mathbf{B}$ *is* $\mathcal{L}$ *then* $\mathbf{V}(\mathbf{B}) = \mathbf{BL}$, *otherwise* $\mathbf{V}(\mathbf{B}) = \mathbf{SBL}$.

Definition 2 *A regular BL-algebra* $\mathbf{H}$ *is said to be* canonical *iff either* $\mathbf{H} = \mathcal{L}^\infty$, *or* $\mathbf{H} = \mathbf{2} \oplus \mathcal{L}^\infty$, *or* $\mathbf{H}$ *is a* finite *ordinal sum of components of the form* $\mathcal{L}, \mathbf{2}, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty$ *and* Π^∞, *where*

(1) each component $\mathcal{G}$ *is* not *preceded and* not *followed by* $\mathbf{2}$ *or by another* $\mathcal{G}$;
(2) each component $\mathcal{C}^\infty$ *is* not *preceded and* not *followed by* $\mathcal{C}$ *or by another* $\mathcal{C}^\infty$.
(3) each component Π^∞ *is* not *preceded and* not *followed by* $\mathbf{2}, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty$ *or by another* Π^∞.

Theorem 8 *For every regular BL-algebra* $\mathbf{H}$ *there is a canonical regular BL-algebra* $\mathbf{K}$ *such that* $\mathbf{V}(\mathbf{H}) = \mathbf{V}(\mathbf{K})$.

The proof is done case by case. For regular chains having infinite number of $\mathcal{L}$ components the result is easy. For the case of regular chains having a finite number of $\mathcal{L}$ components the result is obtained proving first that the number of these components must be the same and then studying the ordinal sums contained in between two adjacent $\mathcal{L}$ components.

Next theorem proves that two different canonical BL-chains generate different varieties.

Theorem 9 *Let* $\mathbf{H} = \bigoplus_{i=0,n} \mathbf{H}_i$ *and* $\mathbf{K} = \bigoplus_{i=0,m} \mathbf{K}_j$ *be two canonical regular BL-chains. Then* $\mathbf{V}(\mathbf{H}) = \mathbf{V}(\mathbf{K})$ *if and only if* $n = m$ *and* $\mathbf{H}_i = \mathbf{K}_i$ *for each* $i = 1, \ldots, n$.

This theorem means that there are as many subvarieties of BL generated by regular BL-chains as canonical regular BL-chains. Since canonical regular BL-chains are finite ordinal sums of components belonging to the set $\{\mathcal{L}^\infty, \mathcal{L}, \mathbf{2}, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty, \Pi^\infty\}$, we obtain as a corollary that the set varieties generated by single regular BL-algebras is *countable*. Since t-norm algebras are regular algebras, the set of varieties generated by single t-norm algebras is obviously countable as well[6].

The results about canonical regular BL-chains can be particularized to t-norm (standard) BL-chains, the (regular) chains defined on $[0, 1]$ by a continuous t-norm and its residuum. Actually, due to Lemma 3, if we restrict ourselves to t-norm algebras, it turns out that their corresponding canonical regular algebras are indeed t-norm algebras (i.e., ordinal sums of $\mathcal{G}$, Π and $\mathcal{L}$ components) with the only exception of those t-norm algebras that generate the whole subvariety **SBL** (those with infinitely-many $\mathcal{L}$ components but not starting with $\mathcal{L}$): $\mathbf{2} \oplus \mathcal{L}^\infty$ is not a t-norm algebra, so it is replaced by $\Pi \oplus \mathcal{L}^\infty$, that also generates the variety of SBL-algebras. So, we introduce the following definition.

Definition 3 *A BL-chain* $\mathbf{A}$ *is said to be a* canonical t-norm algebra *iff either* $\mathbf{A} = \mathcal{L}^\infty$, *or* $\mathbf{A} = \Pi \oplus \mathcal{L}^\infty$, *or* $\mathbf{A}$ *is a* finite *ordinal sum of components of the form* $\mathcal{L}$, Π, $\mathcal{G}$ *and* Π^∞, *where each component* $\mathcal{G}$ *is* not *preceded and* not *followed by another* $\mathcal{G}$, *and each component* Π^∞ *is* not *preceded and* not *followed by* $\mathcal{G}$, *or by* Π *or by another* Π^∞.

As an example, it is easy to check that $\mathbf{2} \oplus \Pi^\infty \oplus \mathcal{G} \oplus \mathcal{C}^\infty$ is not a canonical regular algebra, while $\mathcal{L} \oplus \mathcal{C} \oplus \mathcal{L} \oplus \Pi^\infty$ is a canonical regular algebra but not a canonical t-norm algebra. Finally, $\mathcal{G} \oplus \mathcal{L} \oplus \Pi^\infty$ is indeed a canonical t-norm algebra.

If $\star$ is a continuous t-norm, let us say that $\star$ is *canonical* if the corresponding BL-chain $[0, 1]_\star$ is canonical in the above sense. Then for t-norm algebras, Theorem 8 becomes as follows.

Theorem 10 *For every continuous* t*-norm* $\star$ *there is a canonical continuous* t*-norm* $\circ$ *such that* $\mathbf{V}([0, 1]_\star) = \mathbf{V}([0, 1]_\circ)$.

From these results we can define an easy algorithm to decide, given two canonical BL-chains $\mathbf{A}$ and $\mathbf{B}$, whether one has $\mathbf{A} \in \mathbf{V}(\mathbf{B})$, or equivalently whether $\mathbf{V}(\mathbf{A}) \subseteq \mathbf{V}(\mathbf{B})$. Note that this occurs just when $\mathbf{A}$ can be embedded into an ultrapower of $\mathbf{B}$, hence in this case $\mathbf{A}$ satisfies not only all

[5] We say that $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$, with $\mathbf{A}_i \in \{\mathcal{C}, \mathbf{2}\}$ for all i, has infinitely many alternations of $\mathcal{C}$ and $\mathbf{2}$ if for every $n \in \mathbb{N}$ there are $i_0 < i_1 < \cdots < i_n \in I$ such that for $j = 0, \ldots, n-1$, if $\mathbf{A}_{i_j} = \mathcal{C}$ then $\mathbf{A}_{i_{j+1}} = \mathbf{2}$, and if $\mathbf{A}_{i_j} = \mathbf{2}$, then $\mathbf{A}_{i_{j+1}} = \mathcal{C}$.

[6] Indeed, this result for t-norm algebras was already known, actually in [Ha] it is shown that every variety generated by a t-norm BL-algebra is Co-NP complete, and there are only countably many Co-NP complete sets.

identities valid in $\mathbf{B}$, but also all universal formulas which are valid there. So the algorithm really checks if $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{B})$. Thus our algorithm is called *embedding algorithm*. In the following we write $\mathbf{D} \preceq \mathbf{E}$ for $\mathbf{D} \in \mathbf{ISP}_u(\mathbf{E})$, for $\mathbf{D}$ and $\mathbf{E}$ considered as Wajsberg hoops. The following facts about the embedding relation $\preceq$ are the basis of the algorithm:

1. $\mathcal{C} \preceq \mathcal{L}$ and $\mathbf{2} \preceq \mathcal{L}$.
2. If $\mathbf{E}$ is any finite ordinal sum of linearly ordered hoops from the set $\{\Pi^\infty, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty, \mathbf{2}\}$, the first one bounded, then $\mathcal{L} \not\preceq \mathbf{E}$.
3. If $\mathbf{E}$ is any finite ordinal sum of $\mathcal{G}$, $\mathcal{C}$, $\mathcal{C}^\infty$ and $\mathbf{2}$, then $\mathbf{E} \preceq \Pi^\infty$.
4. If $\mathbf{E}$ is any finite ordinal sum of components isomorphic to $\mathbf{2}$, then $\mathbf{E} \preceq \mathcal{G}$.
5. If $\mathbf{E}$ is any finite ordinal sum of components isomorphic to $\mathcal{C}$, then $\mathbf{E} \preceq \mathcal{C}^\infty$.
6. If $\mathbf{E}$ is any finite ordinal sum of $\mathcal{C}$, $\mathcal{C}^\infty$, $\mathbf{2}$, $\mathcal{L}$ and $\mathcal{G}$, then $\Pi^\infty \not\preceq \mathbf{E}$.
7. If $\mathbf{E}$ is any finite ordinal sum of $\mathbf{2}$, $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{C}^\infty$, then $\mathcal{G} \not\preceq \mathbf{E}$.
8. If $\mathbf{E}$ is any finite ordinal sum of $\mathbf{2}$, $\mathcal{L}$, $\mathcal{C}$ and $\mathcal{G}$, then $\mathcal{C}^\infty \not\preceq \mathbf{E}$.
9. If $\mathbf{E}$ is any ordinal sum of two or more linearly ordered hoops, then $\mathbf{E} \not\preceq \mathcal{L}$, $\mathbf{E} \not\preceq \mathcal{C}$ and $\mathbf{E} \not\preceq \mathbf{2}$.

Using this and Proposition 3, a linear algorithm can be devised to check, given two canonical regular BL-chains $\mathbf{A}$ and $\mathbf{B}$, whether $\mathbf{A} \in \mathbf{V}(\mathbf{B})$. The main steps are the following, for full details consult [EGM].

Embedding algorithm.

- If $\mathbf{B} = \mathcal{L}^\infty$, then $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{B})$.
- If $\mathbf{B} = \mathbf{2} \oplus \mathcal{L}^\infty$, then $\mathbf{A} \in \mathbf{ISP}_u(\mathbf{B})$ iff $\mathbf{A}$ is of the form $\mathbf{2} \oplus \mathbf{H}$ (i.e., if the first component of $\mathbf{A}$ is not $\mathcal{L}$).
- If $\mathbf{A} \in \{\mathcal{L}^\infty, \mathcal{G} \oplus \mathcal{L}^\infty\}$ and $\mathbf{B} \notin \{\mathcal{L}^\infty, \mathcal{G} \oplus \mathcal{L}^\infty\}$, then $\mathbf{A} \notin \mathbf{ISP}_u(\mathbf{B})$.
- It remains to consider the case where $\mathbf{A} = \bigoplus_{i=0}^n \mathbf{A}_i$ and $\mathbf{B} = \bigoplus_{j=0}^m \mathbf{B}_j$, where for all $i \leq n$ and for all $j \leq m$, $\mathbf{A}_i, \mathbf{B}_j \in \{\mathbf{2}, \mathcal{L}, \mathcal{G}, \mathcal{C}, \mathcal{C}^\infty, \Pi^\infty\}$. In this case what we do is to succesively check for each index $i = 1, \dots, n$ whether there exists the minimum index $j_i \leq m$ such that $\bigoplus_{k=0}^i \mathbf{A}_i \preceq \bigoplus_{j=0}^{j_i} \mathbf{B}_j$ but $\bigoplus_{k=0}^i \mathbf{A}_i \not\preceq \bigoplus_{j=0}^{j_i-1} \mathbf{B}_j$. If for a first index i the corresponding index j_i does not exist it means that $\mathbf{A} \notin \mathbf{ISP}_u(\mathbf{B})$ since one has $\bigoplus_{k=0}^i \mathbf{A}_i \not\preceq \mathbf{B}$, hence $\mathbf{A} \notin \mathbf{V}(\mathbf{B})$. Otherwise, if for each i we can find such a j_i, then we can assure that $\mathbf{A} \in \mathbf{V}(\mathbf{B})$.

  Two remarks are in order here:

  Remark 1: recall that for $i = 0$, $\preceq$ is meant as embedding of BL-algebras, hence if $\mathbf{A}_0 = \mathcal{L} \neq \mathbf{B}_0$ then directly $\mathbf{A} \notin \mathbf{V}(\mathbf{B})$.

  Remark 2: assume we have found the index j_i for $i = 0, \dots, k$ and now we have to look for j_{k+1}. If $\mathbf{A}_k \oplus \mathbf{A}_{k+1} \preceq \mathbf{B}_{j_k}$ still holds then $j_{k+1} = j_k$. Otherwise, $j_{k+1} = \min\{j \mid j > j_k, \mathbf{A}_{i+1} \preceq \mathbf{B}_j\}$.

Notice that, in particular, if both $\mathbf{A}$ and $\mathbf{B}$ contain finitely-many $\mathcal{L}$ components, then a necessary condition for having $\mathbf{A} \in \mathbf{V}(\mathbf{B})$ is that $\mathbf{B}$ must have at least as many $\mathcal{L}$ components as $\mathbf{A}$ has.

Example 2 Consider the canonical regular algebras $\mathbf{A} = \mathbf{2} \oplus \mathbf{2} \oplus \mathcal{L} \oplus \mathbf{2} \oplus \mathcal{C}^\infty \oplus \mathcal{G}$ $(n = 5)$ and $\mathbf{B} = \Pi^\infty \oplus \mathcal{L} \oplus \Pi^\infty$ $(m = 2)$, and let us check whether $\mathbf{A} \in \mathbf{V}(\mathbf{B})$ using the above algorithm. It turns out that $j_0 = j_1 = 0$, $j_2 = 1$ and $j_3 = j_4 = j_5 = 2$. Hence $\mathbf{A} \in \mathbf{V}(\mathbf{B})$.

Now consider the algebras $\mathbf{A} = \mathbf{2} \oplus \mathcal{L} \oplus \mathcal{C} \oplus \mathbf{2} (n = 3)$ and $\mathbf{B} = \Pi^\infty \oplus \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{C}^\infty (m = 3)$ and let us run the algorithm again. In this case, $j_0 = 0$, $j_1 = 1$, $j_2 = 3$ but j_3 is undefined since $\mathbf{2} \oplus \mathcal{L} \oplus \mathcal{C} \preceq \Pi^\infty \oplus \mathcal{L} \oplus \mathcal{G} \oplus \mathcal{C}^\infty$ but $\mathbf{2} \not\preceq \mathcal{C}^\infty$. □

4.3 Axiomatization of varieties generated by a regular BL-chain

In the rest of the section, given $\mathbf{A} \in REG$, we denote by $\mathbf{A}^\perp$ the set $Fin \setminus Fin(\mathbf{A})$. Moreover, given $\mathbf{M} \subseteq REG$ we denote by $Min(\mathbf{M})$ the set of minimal elements of $\mathbf{M}$ with respect to the embedding relation $\preceq$ (i.e., $\mathbf{B} \preceq \mathbf{D}$ iff $\mathbf{B} \in \mathbf{ISP}_u(\mathbf{D})$). In particular, the sets $Min(\mathbf{A}^\perp)$, for $\mathbf{A} \in REG$, will play a major role in the rest of this section.

The relation $\preceq$ can be extended to classes of algebras: if $\mathbf{M}$ and $\mathbf{M}'$ are two classes of algebras, we shall write $\mathbf{M} \preceq \mathbf{M}'$ iff for all $\mathbf{D} \in \mathbf{M}'$ there exists $\mathbf{B} \in \mathbf{M}$ such that $\mathbf{B} \preceq \mathbf{D}$. For instance, for any $\mathbf{A} \in REG$, we have $Min(\mathbf{A}^\perp) \preceq \mathbf{A}^\perp$.

To present the first main result we need first a previous lemma.

Lemma 4 *Let* $\mathbf{M}, \mathbf{M}' \subseteq Fin$ *such that* $\mathbf{M} \preceq \mathbf{M}'$. *Then, every algebra which satisfies the set of equations* $\{(e_\mathbf{D}) \mid \mathbf{D} \in \mathbf{M}\}$ *will also satisfy the equations* $\{(e_\mathbf{E}) \mid \mathbf{E} \in \mathbf{M}'\}$.

Theorem 11 *Let* $\mathbf{A}$ *be a regular BL-chain. Then:*

(1) $\mathbf{V}(\mathbf{A})$ *is axiomatized by* $AX(\mathbf{A}) = \{e_\mathbf{B} : \mathbf{B} \in \mathbf{A}^\perp\}$.
(2) $\mathbf{V}(\mathbf{A})$ *is axiomatized by* $AX_0(\mathbf{A}) = \{e_\mathbf{B} : \mathbf{B} \in Min(\mathbf{A}^\perp)\}$.

Proof We prove (1), (2) is an easy consequence of (1) and Lemma 4.

If $\mathbf{D} \in \mathbf{A}^\perp$, then by Lemma 2, $e_\mathbf{D}$ is valid in (every element of) $Fin(\mathbf{A})$, hence it is valid in every element of $\mathbf{V}(Fin(\mathbf{A})) = \mathbf{V}(\mathbf{A})$. It follows that every member of $\mathbf{V}(\mathbf{A})$ satisfies $AX(\mathbf{A})$. As for the other direction, assume $\mathbf{B}$ is a BL-chain that satisfies $AX(\mathbf{A})$. We have to prove that $\mathbf{B} \in \mathbf{V}(\mathbf{A})$. Assume $B \notin \mathbf{V}(\mathbf{A})$. By Theorem 7, it follows that $Fin(\mathbf{B}) \not\subseteq Fin(\mathbf{A})$. Hence there exists a regular $\mathbf{D} \in Fin(\mathbf{B})$ and $\mathbf{D} \notin Fin(\mathbf{A})$. Then, on the one hand, since $\mathbf{D} \in Fin(\mathbf{B})$, by Lemma 2 it follows that the equation $e_\mathbf{D}$ is not valid in $Fin(\mathbf{B})$. On the other hand, since $\mathbf{D} \notin Fin(\mathbf{A})$, we have $e_\mathbf{D} \in AX(\mathbf{A})$. Therefore, by hypothesis, $\mathbf{B}$ satisfies $e_\mathbf{D}$, and therefore $e_\mathbf{D}$ is valid in $Fin \cap \mathbf{ISP}_u(B) = Fin(\mathbf{B})$. Contradiction. □

It only remains to show that for any regular BL-chain $\mathbf{A}$ (thus for any t-norm algebra too), $\mathbf{V}(\mathbf{A})$ is *finitely axiomatizable*. Due to the previuos theorem, it is enough to show that, for any canonical regular BL-chain $\mathbf{A}$, $Min(\mathbf{A}^{\perp})$ is always finite. This is shown in full detail in [EGM] in a case by case basis. Actually, what is described is a general procedure to find all the elements of $Min(\mathbf{A}^{\perp})$, showing that in any case there are only finitely many. Thus, we do not only derive that $\mathbf{V}(\mathbf{A})$ is finitely axiomatizable but also, and very important, we can obtain the equations axiomatizing $\mathbf{V}(\mathbf{A})$ as a BL-extension.

The procedure of finding $Min(\mathbf{A}^{\perp})$ can be thought as expanding a tree, where the root node is the empty ordinal sum, and each node is an ordinal sum that corresponds to a possible expansion (with a $\mathbf{2}$, $\mathcal{C}$ or $\mathcal{L}$ component) of its parent. The basic idea of the tree building procedure is that nodes are successively expanded until it is checked they do not belong to $Fin(\mathbf{A})$. From them, only those which are minimal are kept. During the expansion process, nodes can be *closed* (i.e., their branches are pruned) as soon as they are checked to embedd another currently open node. The algorithm described below makes use of the following notion of degree of maximal embeddability of one algebra into another one.

Definition 4 *Let* $\mathbf{A}$ *be a canonical regular BL-chain, with* $\mathbf{A} = \mathbf{A}_1 \oplus ... \oplus \mathbf{A}_n$*, where* $\mathbf{A}_i \in \{\mathbf{2}, \mathcal{C}, \mathcal{G}, \mathcal{C}^{\infty}, \Pi^{\infty}, \mathcal{L}\}$*, for* $i = 1, \ldots, n$*. Let* $\mathbf{B} \in Fin$*. Then the degree of "maximal embeddability" of* $\mathbf{B}$ *in* $\mathbf{A}$ *is defined as follows:*

$$g(\mathbf{B} \hookrightarrow \mathbf{A}) = \begin{cases} k, & \text{if } \mathbf{B} \in Fin(\mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_k) \text{ and} \\ & \quad \mathbf{B} \notin Fin(\mathbf{A}_1 \oplus \cdots \oplus \mathbf{A}_{k-1}) \\ n+1, & \text{if } \mathbf{B} \notin Fin(\mathbf{A}) \end{cases}$$

where obviously $k \leq n$.

In the following description we use $\mathcal{U}_{\emptyset}$ to denote the empty ordinal sum, and by convention we take $g(\mathcal{U}_{\emptyset} \hookrightarrow \mathbf{A}) = 0$ for any $\mathbf{A}$.

procedure $find\_Min^{\perp}(\mathbf{A})$
% input: $\mathbf{A} = \mathbf{A}_1 \oplus ... \oplus \mathbf{A}_n$, canonical regular BL-chain,
% – where $\mathbf{A}_1 \in \{\Pi, \mathcal{L}, \mathbf{2}, \mathcal{G}, \Pi^{\infty}\}$ and
% $\mathbf{A}_i \in \{\mathcal{C}, \Pi, \mathcal{L}, \mathbf{2}, \mathcal{G}, \mathcal{C}^{\infty}, \Pi^{\infty}\}$ for $i > 1$ –
% output: $minimal\_list$
% – list in which minimal elements of $\mathbf{A}^{\perp}$ are stored –
% auxiliary list: $open\_list$
% – list containing nodes to be expanded –
$n = length(\mathbf{A})$; $open\_list = [\mathcal{U}_{\emptyset}]$; $minimal\_list = [\,]$;
do while $open\_list \neq [\,]$
$\mathcal{U} = first(open\_list)$; $k = g(\mathcal{U} \hookrightarrow \mathbf{A})$;
if $k = 0$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{L}\}$;
if $0 < k < n$ then
$expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}, \mathcal{U} \oplus \mathcal{L}\}$;
if $k = n$ then $expanded\_nodes = \{\mathcal{U} \oplus \mathbf{2}, \mathcal{U} \oplus \mathcal{C}\}$;
for all $\mathcal{U}' \in expanded\_nodes$ do
$l = g(\mathcal{U}' \hookrightarrow \mathbf{A})$;
if $l = 1$ or ($1 < l \leq n$ and and $\mathbf{A}_l \neq \Pi^{\infty}$) then
$open\_list = update(open\_list, \mathcal{U}')$;
if $l = n + 1$ then
$minimal\_list = update(minimal\_list, \mathcal{U}')$;
end for
$open\_list = remove(open\_list, \mathcal{U})$;
end do
end procedure
function $update(list, \mathcal{U})$
% inputs: $list$ to be possibly updated with node $\mathcal{U}$
% output: $list$ after being updated
for all $\mathbf{W} \in list$ do
if $g(\mathbf{W} \hookrightarrow \mathbf{A}) = g(\mathcal{U} \hookrightarrow \mathbf{A})$ then do
if $\mathbf{W} \preceq \mathcal{U}$ then return $list$;
if $\mathcal{U} \prec \mathbf{W}$ then $list = remove(list, \mathbf{W})$;
end do
end for
$list = append(list, \mathcal{U})$; return $list$;
end function

As an example, let us consider a continuous t-norm $\star$ isomorphic to $\mathcal{G} \oplus \mathcal{L} \oplus \Pi^{\infty} \oplus \mathcal{L}$. The above procedures yield the expanded tree of Fig. 2, where one can get $Min(\star^{\perp}) = \{\mathcal{L}, \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{L} \oplus \mathbf{2}, \mathbf{2} \oplus \mathcal{C} \oplus \mathcal{L} \oplus \mathcal{C}\}$.

5 Varieties generated by ordinal sums of finitely many Wajsberg hoops

In this section we move from varieties generated by regular BL-chains, i.e., ordinal sums whose Wajsberg components can be either $\mathcal{L}$, $\mathbf{2}$ and $\mathcal{C}$, to varieties generated by BL-chains which are ordinal sums of finitely many (arbitrary) Wajsberg hoops.

We start from the following observation. Let $\mathbf{A} = \bigoplus_{i=0}^{n} \mathbf{A}_i$ be a linearly ordered BL-algebra, where $\mathbf{A}_0$ is a linearly ordered Wajsberg algebra and for $i > 0$, $\mathbf{A}_i$ is a linearly ordered Wajsberg hoop. Let $\mathbf{V}(\mathbf{A})$ be the variety generated by $\mathbf{A}$. Recall that by Theorem 1 the subdirectly irreducible members of $\mathbf{V}(\mathbf{A})$ are in

$$\mathbf{HSP}_u(\mathbf{A}_0) \cup (\mathbf{ISP}_u(\mathbf{A}_0) \oplus \mathbf{HSP}_u(\mathbf{A}_1)) \cup \cdots \cup (\bigoplus_{i=0}^{n-1} \mathbf{ISP}_u(\mathbf{A}_i) \oplus \mathbf{HSP}_u(\mathbf{A}_n)).$$

Since the behavior of the operators $\mathbf{ISP}_u$ and $\mathbf{HSP}_u$ on linearly ordered Wajsberg algebras and Wajsberg hoops are well-known, the above result will allow us to describe the containment relation on varieties of BL-algebras generated by finite ordinal sums of linearly ordered Wajsberg hoops. We recall some known definitions and results.

The *radical* of a Wajsberg algebra $\mathbf{A}$, $Rad(\mathbf{A})$, is the intersection of all maximal filters of $\mathbf{A}$. It is easily shown that the radical of a Wajsberg algebra is a cancellative subhoop of $\mathbf{A}$. We say that the *rank* of $\mathbf{A}$ is n if $\mathbf{A}/Rad(\mathbf{A})$ is isomorphic to the Wajsberg chain of $n+1$ elements, denoted by $\mathbf{Wa}_n$. If $\mathbf{A}/Rad(\mathbf{A})$ is infinite, then we say that $\mathbf{A}$ *has infinite rank*. For every Wajsberg algebra $\mathbf{A}$ of finite rank n, let $d(\mathbf{A})$ denote the greatest k such that $\mathbf{Wa}_k$ is embeddable into $\mathbf{A}$. (That $d(\mathbf{A})$ is finite when $rank(\mathbf{A})$ is finite follows from [Gi]). Finally, for $k > 0$, we say that $\frac{1}{k}$ *is in* $\mathbf{A}$ iff $\mathbf{Wa}_k$ is embeddable in $\mathbf{A}$. We say that *every rational in* $\mathbf{A}$ *is in* $\mathbf{B}$ if for every $k > 0$, if $\frac{1}{k}$ is in $\mathbf{A}$, then it is also in $\mathbf{B}$.

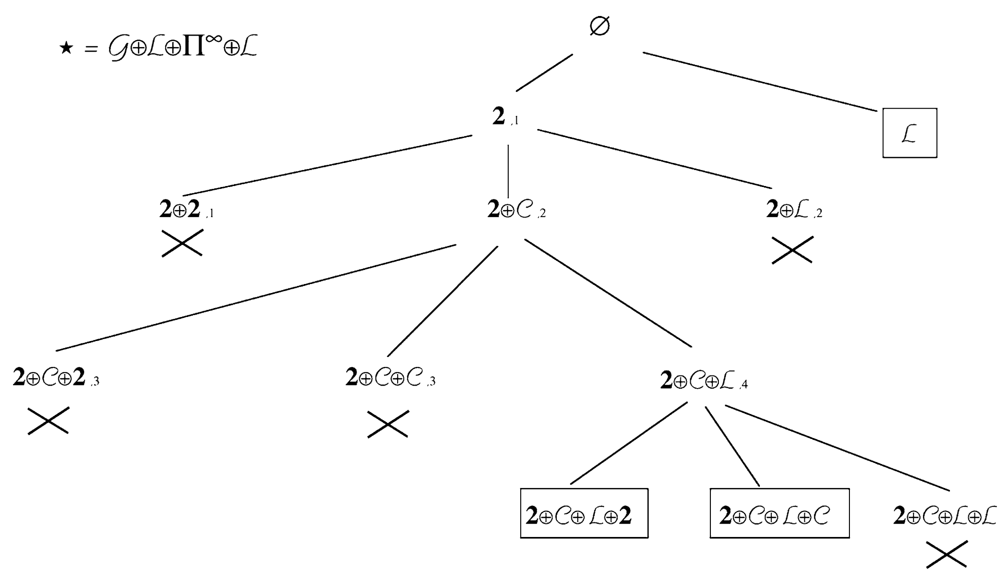

Fig. 2 Expanded tree for $\star = \mathcal{G} \oplus \mathcal{L} \oplus \Pi^\infty \oplus \mathcal{L}$

We define two relations $\lhd$ and $\preceq$ on Wajsberg algebras and on Wajsberg hoops as follows:

(1) If $\mathbf{D}$ is a non-trivial linearly ordered cancellative hoop, then $\mathbf{C} \preceq \mathbf{D}$ iff $\mathbf{C} \lhd \mathbf{D}$ iff $\mathbf{C}$ is a cancellative hoop.
(2) If $\mathbf{D}$ has infinite rank, then $\mathbf{C} \lhd \mathbf{D}$ always, and $\mathbf{C} \preceq \mathbf{D}$ iff $\mathbf{C}$ is either a cancellative hoop, or every rational in $\mathbf{C}$ is in $\mathbf{D}$. (see [Gi]).
(3) If $\mathbf{D}$ is infinite and has finite rank m, then $\mathbf{C} \lhd \mathbf{D}$ iff either $\mathbf{C}$ is a cancellative hoop or $rank(\mathbf{C})$ divides m. Moreover, $\mathbf{C} \preceq \mathbf{D}$ iff either $\mathbf{C}$ is cancellative or $rank(\mathbf{C})$ divides $rank(\mathbf{D})$ and $d(\mathbf{C})$ divides $d(\mathbf{D})$.
(4) If $\mathbf{D}$ is finite, say $\mathbf{D} = \mathbf{Wa}_n$, then $\mathbf{C} \lhd \mathbf{D}$ iff $\mathbf{C} \preceq \mathbf{D}$ iff $\mathbf{C} = \mathbf{Wa}_m$ for some m which divides n.

We recall the following results, which they are either proved in, or follow easily from [F], [GMT] and [Gi], and which are also recalled (with a different notation) in [AM].

Proposition 4 *Let $\mathbf{C}$, $\mathbf{D}$ be either linearly ordered Wajsberg hoops or linearly ordered Wajsberg algebras. Then $\mathbf{C} \lhd \mathbf{D}$ iff $\mathbf{C} \in \mathbf{HSP}_u(\mathbf{D})$, and $\mathbf{C} \preceq \mathbf{D}$ iff $\mathbf{C} \in \mathbf{ISP}_u(\mathbf{D})$.*

As a consequence, we obtain:

Theorem 12 . *Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$, $\mathbf{B}_1, \ldots, \mathbf{B}_m$ be linearly ordered (non-trivial) Wajsberg hoops, let $\mathbf{A}_0$, $\mathbf{B}_0$ be linearly ordered (non-trivial) Wajsberg algebras, let $\mathbf{A} = \bigoplus_{i=0}^n \mathbf{A}_i$, $\mathbf{B} = \bigoplus_{j=0}^m \mathbf{B}_j$, and let $\mathbf{V}(\mathbf{A})$ and $\mathbf{V}(\mathbf{B})$ be the varieties generated by $\mathbf{A}$ and $\mathbf{B}$ respectively. Then $\mathbf{V}(\mathbf{B}) \subseteq \mathbf{V}(\mathbf{A})$ iff $m \leq n$, and there are $0 = i_0 < i_1 < \ldots < i_m \leq n$ such that for $j < m$, $\mathbf{B}_j \preceq \mathbf{A}_{i_j}$, and $\mathbf{B}_m \lhd \mathbf{A}_{i_m}$.*

Proof If the condition of Theorem 12 holds, then

$$\mathbf{B} \in \bigoplus_{j=0}^{m-1} \mathbf{ISP}_u(\mathbf{A}_{i_j}) \oplus \mathbf{HSP}_u(\mathbf{A}_{i_m})$$

$$\subseteq \mathbf{HSP}_u(\bigoplus_{j=0}^{m} \mathbf{A}_{i_j}) \subseteq \mathbf{HSP}_u(\mathbf{A}),$$

therefore $\mathbf{V}(\mathbf{B}) \subseteq \mathbf{V}(\mathbf{A})$. Conversely, if $\mathbf{V}(\mathbf{B}) \subseteq \mathbf{V}(\mathbf{A})$, then every subdirectly irreducible member of $\mathbf{V}(\mathbf{B})$ is a subdirectly irreducible member of $\mathbf{V}(\mathbf{A})$, therefore it is in

$$\mathbf{HSP}_u(\mathbf{A}_0) \cup \bigcup_{k=1}^{n} (\bigoplus_{i=0}^{k-1} \mathbf{ISP}_u(\mathbf{A}_i) \oplus \mathbf{HSP}_u(\mathbf{A}_k)).$$

Now consider the BL-algebra $\mathbf{D} = \bigoplus_{j=0}^m \mathbf{D}_j$ where for $j < m$, $\mathbf{D}_j = \mathbf{B}_j$, and in addition:

(1) If $\mathbf{B}_m$ has infinite rank, then $\mathbf{D}_m$ is the (hoop reduct of the) Wajsberg algebra $[0, 1]_L$ on [0, 1].
(2) If $\mathbf{B}_m$ is finite, then $\mathbf{D}_m = \mathbf{B}_m$.
(3) If $\mathbf{B}_m$ is cancellative, then $\mathbf{D}_m$ is the cancellative hoop (0, 1] equipped with product and product residuation.
(4) If $\mathbf{B}_m$ is the reduct of an infinite Wajsberg algebra with finite rank k, then $\mathbf{D}_m$ is the Chang algebra with rank k generated by $\frac{1}{k}$ and a positive infinitesimal ε.

Note that $\mathbf{D}_m$ is simple in cases (i), (ii) and (iii), and in case (iv) it is subdirectly irreducible, with minimum non-trivial filter generated by $1-\varepsilon$. Since the minimum non-trivial filter of $\mathbf{D}_m$ is also the minimum non-trivial filter of $\mathbf{D}$, it follows that $\mathbf{D}$ is a subdirectly irreducible member of $\mathbf{V}(\mathbf{B})$. Hence $\mathbf{D} \in \mathbf{V}(\mathbf{A})$, therefore $\mathbf{D} \in \mathbf{HSP}_u(\mathbf{A}_0) \cup \bigcup_{i=1}^n (\bigoplus_{j=0}^{i-1} \mathbf{ISP}_u(\mathbf{A}_j) \oplus \mathbf{HSP}_u(\mathbf{A}_i))$. Hence there are $i_0 = 0 < \ldots < i_m \leq n$ such that $\mathbf{B}_m \lhd \mathbf{A}_{i_m}$ and for $j < m$, $\mathbf{B}_j \preceq \mathbf{A}_{i_j}$ as desired. □

6 The variety generated by perfect BL-algebras

The starting point of the present section is the observation that in every BL-algebra $\mathbf{A}$, one can consider the largest MV-algebra $\mathbf{MV}(\mathbf{A})$ which is a subalgebra of $\mathbf{A}$ [ST]. Indeed, a BL-algebra $\mathbf{A}$ is an MV-algebra iff $\neg\neg x = x$ for all $x \in A$, where $\neg x = x \to 0$. Recall that an operation of addition is defined in an MV-algebra $\mathbf{A}$ by setting $x \oplus y = \neg(\neg x * \neg y)$ for all $x, y \in A$.

From [ST] we know that in any BL-algebra $\mathbf{A} = (A, \wedge, \vee, *, \rightarrow, 0, 1)$, the greatest MV-subalgebra of $\mathbf{A}$ is given by

$$\mathbf{MV}(\mathbf{A}) = (MV(A), \wedge, \vee, *_{MV}, \oplus_{MV}, 0, 1),$$

where $MV(A) = \{\neg x : x \in A\}$, via the following operations [ST]:

$$\neg x *_{MV} \neg y = \neg x * \neg y$$

$$\neg x \oplus_{MV} \neg y = \neg\neg x \rightarrow \neg y = \neg(x * y) = \neg(\neg\neg x * \neg\neg y).$$

Moreover, in [CT2] it is shown that the mapping $\neg\neg : A \rightarrow MV(A)$ is a morphism of BL-algebras, i.e., it preserves the BL-algebra operations.

We shall explore the role of $\mathbf{MV}(\mathbf{A})$ in the class of BL-algebras and its properties. What we expect is that giving conditions on $\mathbf{MV}(\mathbf{A})$, we get informations on $\mathbf{A}$, or by considering classes of MV-algebras we can induce the consideration of interesting classes of BL-algebras. This is what we do in the rest of this section, relating the properties (local or perfect) of a BL-algebra $\mathbf{A}$ with the properties (local or perfect) of the corresponding MV-algebra $\mathbf{MV}(\mathbf{A})$, and studying these types of BL-algebras and the varieties generated by them.

6.1 General results

Let $\mathbf{A}$ be a BL-algebra, $\mathbf{MV}(\mathbf{A})$ its greatest MV-subalgebra, $Filt(\mathbf{A})$ the set of all proper filters of $\mathbf{A}$ and $Id(\mathbf{MV}(\mathbf{A}))$ the set of all proper ideals of $\mathbf{MV}(\mathbf{A})$[7]. There is an interesting interplay between $Filt(\mathbf{A})$ and $Id(\mathbf{MV}(\mathbf{A}))$. Indeed, we can map any element of $Filt(\mathbf{A})$ into an element of $Id(\mathbf{MV}(\mathbf{A}))$ and viceversa. To show this, assume D to be a proper filter of a BL-algebra $\mathbf{A}$ and P to be an ideal of $\mathbf{MV}(\mathbf{A})$. Then we define:

$$\neg D = \{\neg x \in MV(A) : x \in D\},$$

$$P^{\neg} = \{x \in L : \neg x \in P\},$$

$$Rad(\mathbf{A}) = \bigcap\{F : F \in Filt(\mathbf{A}), F\ maximal\},^{8}$$

$$M(\mathbf{MV}(\mathbf{A})) = \bigcap\{I : I \in Id(\mathbf{MV}(\mathbf{A})), I\ maximal\}$$

$$S(\mathbf{A}) = \{x \in A : \neg\neg x \geq \neg x\}.$$

With these definitions, it easy to define a mapping from $Filt(A)$ to $Id(MV(A))$ and viceversa. Indeed we have the following propoerties (see [DNSEGG] for proofs):

Proposition 5 *Let* $\mathbf{A}$ *be a BL-algebra,* $D \in Filt(\mathbf{A})$ *and* $P \in Id(\mathbf{MV}(\mathbf{A}))$. *Then the following statements hold:*

(1) $\neg D \in Id(\mathbf{MV}(\mathbf{A}))$;
(2) $(\neg D)^{\neg} \supseteq D$;
(3) *If* D *is maximal then* $\neg D$ *is maximal and* $(\neg D)^{\neg} = D$;
(4) $P^{\neg} \in Filt(\mathbf{A})$;
(5) $\neg(P^{\neg}) = P$;
(6) *If* P *is maximal then* $P^{\neg}$ *is maximal;*
(7) $(M(\mathbf{MV}(\mathbf{A})))^{\neg} = Rad(\mathbf{A})$;
(8) $Rad(\mathbf{A}) \subseteq S(\mathbf{A})$;

Regarding the radical of BL-algebras, the following important properties are proved in [DNSEGG].

Proposition 6 *For any BL-algebra* $\mathbf{A}$ *the following conditions hold:*

(1) $Rad(\mathbf{A}) = \{x \in A : \neg(x^n) < x \text{ for all } n\}$;
(2) for all $x \in A$, $\neg\neg x \rightarrow x \in Rad(\mathbf{A})$.

Finally, recalling that an MV-algebra $\mathbf{A}$ is semisimple iff the intersection of its maximal ideals is $\{0\}$, we have the following statement.

Proposition 7 *Let* $\mathbf{A}$ *be a* BL*-algebra. Then* $\mathbf{A}$ *is a semisimple* MV*-algebra iff* $Rad(\mathbf{A}) = \{1\}$.

6.2 Local BL-algebras

In the structure of certain types of algebraic systems one attempts to classify the "atoms" of the theory, i.e., those algebras from which the others are in somehow composed. One usually tries first to classify the simple structures, those with no proper homomorphic images. At next level of complexity one studies the indecomposable ones, i.e. those algebras that cannot be written as a non-trivial direct sum.

In the theory of MV-algebras the simple structures are, up to isomorphism, subalgebras of the standard MV-algebra [0, 1]. Such algebras are also called *locally finite*. Not every MV-subalgebra, however, can be built up from the simple algebras. On the other hand, every MV-algebra can be obtained subdirectly from products of MV-chains. It turns out there are also non-linearly ordered indecomposable MV-algebras, which we call *local* MV-algebras. Local MV-algebras are just those MV-algebras having exactly one maximal ideal (or equivalently, one maximal filter). Refinements of the notion of local MV-algebra are given by *perfect* MV-algebras and *singular* MV-algebras. We recall that an MV-algebra $\mathbf{A}$ is said to be singular if there are $x, y \in A$ of finite order[9] such that $x * y \in M(\mathbf{A}) \setminus \{0\}$. $\mathbf{A}$ is said to be perfect if and only if for every element $a \in A$ exactly one of a and $\neg a$ is of finite order. It happens that a perfect MV-algebra $\mathbf{A}$ is a local MV-algebra which is generated by its unique maximal ideal I. Actually we have that $A = I \cup \neg I$, where $\neg I = \{\neg x \in A \mid x \in I\}$.

Unfortunately the class of perfect MV-algebras does not form a variety, in fact it is not closed under direct products. However, it remains an interesting class, because the class

[7] Notice that in MV-algebras we can indistinctly deal with filters or ideals since they are dual, but this is not the case in BL-algebras, since the negation is not necessarily involutive. In the latter case, the suitable theory is filter theory, which is in good correspondence with congruences.

[8] Note that in MV-algebras literature $Rad(A)$ usually refers to the intersection of maximal ideals.

[9] The order of x is the smallest natural n for which $nx = x \oplus \overset{n}{\ldots} \oplus x = 1$. If it exists we write $MV\text{-ord}(x) = n$, otherwise we write $MV\text{-ord}(x) = \infty$.

of perfect MV-algebras, as a full subcategory of the category of all MV-algebras, is equivalent to the category of abelian lattice ordered groups [DNL1]. The variety $\mathbf{V}(Perf)$ generated by all perfect MV-algebras can be equationally characterized by the equation $(2x)^2 = 2(x^2)$. Every element from the variety $\mathbf{V}(Perf)$ can be subdirectly represented by perfect MV-chains. We conclude this short report on local and perfect MV-algebras by recalling that local MV-algebras can be classified into three pairwise disjoint subclasses. Indeed, any local MV-algebra is either perfect, or locally finite or singular.

Bearing in mind the above described results for MV-algebras, we try to parallel as much as possible the above theory to BL-algebras.

In a BL-algebra $\mathbf{A}$, the *order* of an element $x \in A$ is defined as the smallest natural n such that $x^n = x * \overset{n}{\cdots} * x = 0$. In this case we put $\mathrm{ord}(x) = n$, otherwise $\mathrm{ord}(x) = \infty$ if no such n exists. By convention, we put $x^0 = 1$ for any $x \neq 0$. We point out that although this notion of BL-order does not coincide in general with the notion of MV-order given before, it is easy to check that in any MV-algebra it holds that $MV\text{-}\mathrm{ord}(x) = \mathrm{ord}(\neg x)$.

In [ST] the authors extend the notion of local MV-algberas to BL-algebras by defining a BL-algebra to be *local* if it has exactly one maximal filter, and they prove that a BL-algebra $\mathbf{A}$ is local iff, for all $x \in A$, $\mathrm{ord}(x) < \infty$ or $\mathrm{ord}(\neg x) < \infty$. Then, like for MV-algebras, the class of local BL-algebras can be partitioned in three pairwise disjoint subclasses. Strictly speaking a local BL-algebra $\mathbf{A}$ is said to be

(1) *perfect* iff, for all $x \in A$, $\mathrm{ord}(x) < \infty$ iff $\mathrm{ord}(\neg x) = \infty$;
(2) *locally finite* iff, for all $x \in A - \{1\}$, $\mathrm{ord}(x) < \infty$;
(3) *peculiar* iff there exist $x, y \in A - \{0, 1\}$, such that $\mathrm{ord}(x) = \infty$, $\mathrm{ord}(y) < \infty$, $\mathrm{ord}(\neg y) < \infty$.

Indeed, a locally finite BL-algebra is a locally finite MV-algebra ([T1], Theorem 1). Moreover, taking in account that $\mathrm{ord}(\neg x) = MV\text{-}\mathrm{ord}(x)$ and $\mathrm{ord}(x) = \mathrm{ord}(\neg\neg x) = MV\text{-}\mathrm{ord}(\neg x)$ for all $x \in A$, we get:

Theorem 13 *A local BL-algebra* $\mathbf{A}$ *is perfect iff* $\mathbf{MV}(\mathbf{A})$ *is perfect.*

Theorem 14 *Let* $\mathbf{A}$ *be a local BL-algebra such that* $\mathbf{A} \neq \mathbf{MV}(\mathbf{A})$. *Then* $\mathbf{A}$ *is peculiar iff* $\mathbf{MV}(\mathbf{A}) \neq \{0, 1\}$ *is either singular or locally finite.*

We illustrate Theorem 14 with the following examples:

Example 3 Let $\star : [0, 1]^2 \to [0, 1]$ be the continuous t-norm defined as

$$x \star y = \begin{cases} \max\{x + y - 1/2, 0\} & \text{if } x, y \in [0, 1/2], \\ x \wedge y & \text{otherwise.} \end{cases}$$

$\star$ is the ordinal sum of the t-norm of Łukasiewicz and min with respect to the intervals $[0, 1/2]$ and $[1/2, 1]$. Its residuum "$\to$" is defined as

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ \min\{1/2, 1/2 - x + y\} & \text{if } y < x \leq 1/2, \\ y & \text{otherwise.} \end{cases}$$

By [H2], $\mathbf{L} = ([0, 1], \wedge, \vee, \star, \to, 0, 1)$ is a t-norm algebra, hence a BL-chain, which is actually peculiar. Further, $\mathbf{MV}(\mathbf{L}) = [0, 1/2) \cup \{1\}$, which is locally finite. □

Now we give an example of a peculiar BL-algebra $\mathbf{L}$ with $\mathbf{MV}(\mathbf{L})$ being singular.

Example 4 Let $\mathbb{N}$ and $\mathbb{R}$ be the set of integers and real numbers, respectively. Let F be an ultrafilter of subsets of $\mathbb{N}$ containing the cofinite subsets of $\mathbb{N}$ and $\mathbb{R}^* = \mathbb{R}^{\mathbb{N}}/F$ be the ultrapower determined from F. Denote by $\mathbf{L}_1 = [0, 1]^*$ the unit interval of $\mathbb{R}^*$, structured as an MV-algebra with the same operations which define the Łukasiewicz algebra over $[0, 1]$. $\mathbf{L}_1$ is a singular MV-algebra [BDNL]. Now, let $\mathbf{L}_2$ be the Chang's MV-algebra [Ch] and let $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$ be the ordinal sum of $\mathbf{L}_1$ with $\mathbf{L}_2$. Then $\mathbf{L}$ is peculiar and $\mathbf{MV}(\mathbf{L}) = L_1 - \{1_1\} \cup \{1\}$ is singular because is MV-isomorphic to $\mathbf{L}_1$. □

6.3 The variety generated by perfect BL-algebras

The class of perfect BL-algebras, although closed by homomorphic images and subalgebras, is not a variety. We can consider the subvariety of BL-algebras generated by all perfect BL-algebras. Let us denote such a variety by $\mathbb{P}_0$. Then we have:

Theorem 15 *Let Each* $\mathbf{A} \in \mathbb{P}_0$ *is a subdirect product of perfect BL-chains.*

Using this result and the fact that any BL-chain satifies the equation

$$\neg\Big(\big(\neg(x^2)\big)^2\Big) = \Big(\neg\big((\neg x)^2\big)\Big)^2 \qquad (p_0)$$

if and only if the chain is perfect, we obtain the following equational characterization of $\mathbb{P}_0$

Theorem 16 *The variety* $\mathbb{P}_0$*, generated by perfect BL-algebras, is the subvraiety of* $\mathbf{BL}$ *defined by the equation* (p_0).

Now we examine the generators of the variety $\mathbb{P}_0$.

By ([H2], Theorem 3), any BL-chain $\mathbf{A}$ can be isomorphically embedded into a saturated BL-chain $\mathbf{A}^\infty$.

Theorem 17 *If* $\mathbf{A}$ *is a perfect BL-chain, so is* $\mathbf{A}^\infty$.

By Theorem 15, the variety $\mathbb{P}_0$ has an infinite number of generators which are the perfect BL-chains. By Theorem 17, we can choose perfect saturated BL-chains as generators of $\mathbb{P}_0$, since a non-saturated BL-chain is a subalgebra of the corresponding saturated BL-chain.

On the other hand, according to [T2], we define a BL-algebra $\mathbf{A}$ to be *bipartite* (resp. *strongly bipartite*) if $A = M \cup \neg M$ for some (resp. every) maximal filter M. Next theorem characterizes these types of algebras and their relations to the correspondign largest MV-subalgebras.

Theorem 18 *A BL-algebra* $\mathbf{A}$ *is bipartite for a maximal filter* M *if and only if* $S(\mathbf{A}) \subseteq M$. *Moreover,* $\mathbf{A}$ *is strongly bipartite if and only any of the following three conditions hold:*

(1) $S(\mathbf{A}) = Rad(\mathbf{A})$
(2) $S(\mathbf{A})$ *is a proper filter of* $\mathbf{A}$
(3) $\mathbf{A}/Rad(\mathbf{A})$ *is a Boolean algebra*

Theorem 19 *Let* $\mathbf{A}$ *be a* BL*-algebra. Then* $\mathbf{A}$ *is (resp. strongly) bipartite iff* $\mathbf{MV}(\mathbf{A})$ *is (resp. strongly) bipartite.*

It turns out that the class of strongly bipartite BL-algebras form a variety, indeed it can be checked that any linearly ordered BL-algebra is strongly bipartite if and only it satisfies the following set of equations:

$$[\neg\neg(x^n) \wedge \neg x] \vee [\neg\neg x \wedge (\neg x)^n] = \neg x \wedge \neg\neg x, \quad \text{for all } n,$$

or equivalently, by this other set of equations

$$[\neg(x^n) \vee x] \wedge [\neg((\neg x)^n) \vee \neg x] = x \vee \neg x, \quad \text{for all } n.$$

The interesting thing is that the variety of strongly bipartite algebras coincides with the variety $\mathbb{P}_0$ generated by the class of perfect algebras.

Theorem 20 *A* BL*-algebra* $\mathbf{A}$ *is strongly bipartite if and only if* $\mathbf{A} \in \mathbb{P}_0$.

Finally we could ask ourselves about the class of bipartite BL-algebras. This class is not a variety, it is only closed by direct products and subalgebras. Actually, if a direct product contains a component which is bipartite, the product itself is bipartite. From this, it is easy to prove the following last result.

Theorem 21 *The variety generated by bipartite BL-algebras is the full variety* **BL**.

Acknowledgements Francesc Esteva and Lluís Godo acknowledge partial support of the Spanish CICYT project MULOG, TIN2004-07933-C03-01.

References

[AFM] Aglianó P, Ferreirim IMA, Montagna F Basic hoops: an algebriac study of continuous t-norms. Studia Logica (to appear)

[AM] Aglianó P, Montagna F (2003) Varieties of BL-algebras I: general properties. J Pure Appl Algebra 181:105–129

[BDNL] Belluce LP, Di Nola A, Lettieri A (1993) Local MV-algebras. Rend Circ Mat Palermo 42:347–361

[BF] Blok WJ, Ferreirim IMA (2000) On the structure of hoops. Algebra Universalis 43:233–257

[Bu] Burris S, Sankappanavar HP (1981) A course in Universal Algebra, Graduate texts in Mathematics, Springer, Berlin Heidelberg New York

[MB] Busaniche M (2005) Decomposition of BL-chains. To appear in Algebra Universalis Vol. 52, Number 4, 519–525

[Ch] Chang CC (1958) Algebraic analysis of many-valued logic. Trans Am Math Soc 88:467–490

[CEGT] Cignoli R, Esteva F, Godo L, Torrens A (2000) Basic fuzzy logic is the logic of continuous t-norms and their residua. Soft Computing 4:106–112

[CDOM] Cignoli R, D'Ottaviano IML, Mundici D (2000) Algebraic Foundations of Many-valued Reasoning, Kluwer, Doredrecht

[CT] Cignoli R, Torrens A (2000) An algebraic analysis of product logic. Mult Val Logic 5:45–65

[CT2] Cignoli R, Torrens A (2003) Hájek basic fuzzy logic and Łukasiewicz infinite-valued logic. Arch Math Logic 42:361–370

[DNL1] Di Nola A, Lettieri A (1994) Perfect MV-algebras are categorically equivalent to abelian ℓ-groups. Studia Logica 53:417–432

[DNL2] Di Nola A, Lettieri A (1999) Equational characterization of all varities of MV algebras. J Algebra 22:463–474

[DNEGGS] Di Nola A, Esteva F, Garcia P, Godo L, Sessa S (2002) Subvarieties of BL-algebras generated by single-component chains. Arch Math Logic 41:673–685

[DNSEGG] Di Nola A, Sessa S, Esteva F, Godo L, Garcia P (2002) The variety generated by perfect BL-algebras: an algebraic approach in a fuzzy logic setting. Ann Math Artif Intell 35:197–214

[EGM] Esteva F, Godo L, Montagna F (2004) Equational Characterization of the Subvarieties of BL Generated by T-Norm Algebras Studia logica 76, 161–200

[F] Ferreirim IMA (1992) On varieties and quasi varieties of hoops and their reducts. PhD Thesis, University of Illinois

[Gi] Gispert J (2002) Universal classes of MV-chains with applications to many valued logics. Math Logic Q 48:581–601

[GMT] Gispert J, Mundici D, Torrens A (1999) Ultraproducts of **Z** with an Application to Many-Valued Logics. J Algebra 219:214–233

[Go] Gottwald S (2001) A Treatise on Many-valued Logics. Studies in logic and computation, Research Studies Press, Baldock

[GR] Grigolia RS (1977) Algebraic Analysis of Lukasiewicz-Tarski's n-valued logical systems. In: Wojcicki AR, Malinowski G (eds) Selected papers on Lukasiewicz sentential calculus. Ossolineum, Wroclaw, pp 81–92

[H1] Hájek P (1998) Metamathematics of fuzzy logic. In: Trends in logic-studia logica library, vol 4. Kluwer, Dordercht/Boston/London

[H2] Hájek P (1998) Basic fuzzy logic and BL-algebras. Soft Computing 2:124–128

[Ha] Haniková Z (2002) A note on propositional tautologies of individual continuous t-norms vol 12. Neural Netw World (5) 453–460

[HK] Hecht T, Katrinak T (1972) Equational classes of relative Stone algebras. Notre Dame J Formal Logic 13:248–254

[K] Komori Y (1981) Super- Łukasiewicz implicational logics. Nagoya Math J 84:1119–133

[LS] Laskowski MC, Shashoua YV (2002) A classification of BL-algebras. Fuzzy Sets Syst 131:271–282

[MS] Mostert PS, Shields AL (1957) On the structure of semigroups on a compact manifold with boundary. Ann Math 65:117–143

[Mu] Mundici D (1986) Interpretations of AFC*-algebras in Łukasiewicz sentential calculus. J Funct Anal 65:15–63

[Pa] Panti G (1999) Varieties of MV algebras. J Appl Non-Classical Logic 9:141–157

[RT] Rodríguez AJ, Torrens A (1994) Wajsberg algebras and Post algebras. Studia Logica 53:1–19

[ST] Turunen E, Sessa S (2001) Local BL-algebras. Int J Multiple Valued Logic 6:229–249

[T1] Turunen E (1999) BL-algebras and fuzzy logic. Mathware and Soft Comput 1:49–61

[T2] Turunen E (2001) Boolean deductive systems of BL-algebras. Arch Math Logic 40:467–473