On Implicative Closure Operators in Approximate Reasoning*

R. O. Rodríguez[1], F. Esteva[2], P. Garcia[2] and L. Godo[2]

[1] Dpto. de Computación, Fac. Ciencias Exactas y Naturales
Universidad de Buenos Aires, 1428 Buenos Aires, Argentina
ricardo@dc.uba.ar

[2] Institut d'Investigació en Intel·ligència Artificial, CSIC
Campus Univ. Autònoma Barcelona, 08193 Bellaterra, Spain
{*esteva,pere,godo*}*@iiia.csic.es*

Abstract

This paper introduces a new class of fuzzy closure operators called implicative closure operators, which generalize some notions of fuzzy closure operators already introduced by different authors. We show that implicative closure operators capture some usual consequence relations used in Approximate Reasoning, like Chakraborty's graded consequence relation, Castro et al.'s fuzzy consequence relation, similarity-based consequence operators introduced by Dubois et al. and Gerla's canonical extension of classical closure operators. We also study the relation of the implicative closure operators to other existing fuzzy inference operators as the Natural Inference Operators defined by Boixader and Jacas and the fuzzy operators defined by Biacino, Gerla and Ying.

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1 Introduction

Many works have been devoted to extend the notions of closure operators, closure systems and consequence relations from two valued logic to many

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valued logic. Concerning closure operators, one of the first works was done by Michálek in [25] in the framework of Fuzzy Topological Spaces. Nevertheless the best well-known approach to many-valued closure operators is due to Pavelka [27]. He defines such operators (in the standard sense of Tarski) as mappings from fuzzy sets of formulas to fuzzy sets of formulas. But before going into details let us introduce some notation conventions we shall use throughout this paper.

In the following we shall denote by $\mathcal{L}$ a propositional language, by $\mathcal{P}(\mathcal{L})$ its power set and by $\mathcal{F}(\mathcal{L})$ the set of L-fuzzy subsets of $\mathcal{L}$, where $(L, \wedge, \vee, \leq, 0, 1)$ is a complete distributive lattice. Propositions of $\mathcal{L}$ will be denoted by lower case letters p,q, ..., and fuzzy sets of propositions by upper case letters A,B, etc. For each $A \in \mathcal{F}(\mathcal{L})$ and each $p \in \mathcal{L}$, $A(p) \in L$ will stand for the membership degree of p to A. Moreover, the lattice structure of L induces a related lattice structure on $\mathcal{F}(\mathcal{L})$, $(\mathcal{F}(\mathcal{L}), \cap, \cup, \subseteq, \bar{0}, \bar{1})$, which is complete and distributive as well, where $\cap, \cup$ are the pointwise extensions of the lattice operations $\wedge$ and $\vee$ to $\mathcal{F}(\mathcal{L})$, i.e.

$$\begin{array}{rcl} (A \cap B)(p) & = & A(p) \wedge B(p), \text{ for all } p \in \mathcal{L} \\ (A \cup B)(p) & = & A(p) \vee B(p), \text{ for all } p \in \mathcal{L} \ , \end{array}$$

and where the lattice (subsethood) ordering and top and bottom elements are defined respectively by

$$\begin{array}{ccl} A \subseteq B & \text{iff} & A(p) \leq B(p) \text{ for all } p \in \mathcal{L} \\ \bar{0}(p) = 0 & \text{and} & \bar{1}(p) = 1, \text{ for all } p \in \mathcal{L} \ . \end{array}$$

For any $k \in L$, we shall also denote by $\bar{k}$ the constant fuzzy set defined by $\bar{k}(p) = k$ for all $p \in \mathcal{L}$.

Now we are ready to follow our introduction with Pavelka's definition of fuzzy closure operator.

Definition 1 (Fuzzy closure operator [27]) *A fuzzy closure operator on the language $\mathcal{L}$ is a mapping $\tilde{C} : \mathcal{F}(\mathcal{L}) \mapsto \mathcal{F}(\mathcal{L})$ fulfilling, for all $A, B \in \mathcal{F}(\mathcal{L})$, the following properties:*

$\tilde{C}1$*)* fuzzy inclusion*: $A \subseteq \tilde{C}(A)$*

$\tilde{C}2$*)* fuzzy monotony*: if $A \subseteq B$ then $\tilde{C}(A) \subseteq \tilde{C}(B)$*

$\tilde{C}3$*)* fuzzy idempotence*: $\tilde{C}(\tilde{C}(A)) \subseteq \tilde{C}(A)$.*

On the other hand, Chakraborty extends in [8] the concept of consequence relation by defining *graded consequence relations* as fuzzy relations between crisp sets of formulas and formulas. To do so, in [9] he assumes to have a t-norm-like[1] operation $\otimes$ in L such that $(L, \otimes, 1, \leq, \Rightarrow)$ is a complete residuated lattice.

Definition 2 (Graded consequence relation [8]) *Let* $\otimes$ *be a t-norm operation on* L. *A fuzzy relation* $g_c : \mathcal{P}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ *is called a graded consequence relation if, for every* $A, B \in \mathcal{P}(\mathcal{L})$ *and* $p, q \in \mathcal{L}$, g_c *fulfills:*

$g_c1)$ fuzzy reflexivity*:* $g_c(A, p) = 1$ *for all* $p \in A$

$g_c2)$ fuzzy monotony*: if* $B \subseteq A$ *then* $g_c(B, p) \leq g_c(A, p)$

$g_c3)$ fuzzy cut*:* $[\inf_{q \in B} g_c(A, q)] \otimes g_c(A \cup B, p) \leq g_c(A, p)$*.*[2]

In [17] Gerla examines the links between fuzzy closure operators and graded consequence relations. Castro et al. point out in [7] that several methods of approximate reasoning used in Artificial Intelligence, such as Polya's models of plausible reasoning [28] or Nilsson's probabilistic logic [26], are not covered by the formalism of graded consequence relations, and they introduce a new concept of consequence relations, called *fuzzy consequence relations* which, unlike Chakraborty's graded consequence relation, apply over fuzzy sets of formulas.

Definition 3 (Fuzzy consequence relation [7]) *A fuzzy relation* $f_c : \mathcal{F}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ *is called a fuzzy consequence relation if the following three properties hold for every* $A, B \in \mathcal{F}(\mathcal{L})$ *and* $p, q \in \mathcal{L}$*:*

$f_c1)$ fuzzy reflexivity*:* $A(p) \leq f_c(A, p)$

$f_c2)$ fuzzy monotony*: If* $B \subseteq A$ *then* $f_c(B, p) \leq f_c(A, p)$

$f_c3)$ fuzzy cut*: if for all* p, $B(p) \leq f_c(A, p)$, *then for all* q, $f_c(A \cup B, q) \leq f_c(A, q)$

However, it is worth noticing that fuzzy consequence relations as defined above, when restricted over crisp sets of formulas, become only a particular class of graded consequence relations. Namely, regarding the two versions of the fuzzy cut properties, (g_c3) and (f_c3), it holds that for $A, B \in \mathcal{P}(\mathcal{L})$, if $B(p) \leq f_c(A, p)$ for all $p \in \mathcal{L}$, it is clear that $\inf_{q \in B} f_c(A, q) = 1$.

[1] It can be called in this way because it satisfies all the main properties of t-norms, i.e. commutativity, associativity, monotony, 1 is neutral and 0 is a null element.

[2] by the residuation this axiom is equivalent to $[\inf_{q \in B} g_c(A, q)] \leq g_c(A \cup B, p) \Rightarrow g_c(A, p)$

To conclude this brief overview, let us point out the in the classical setting there are wellknown relationships of interdefinability among closure operators, consequence relations and closure systems. In the general framework, fuzzy closure operators and fuzzy consequence relations are related in a analogous way, as proved in [7]:

- if $\tilde{C}$ is a fuzzy closure operator then f_c, defined as $f_c(A, p) = \tilde{C}(A)(p)$, is a fuzzy consequence relation.
- if f_c is a fuzzy consequence relation then $\tilde{C}$, defined as $\tilde{C}(A) = f_c(A, \cdot)$, is a fuzzy closure operator.

Therefore, via these relationships, the fuzzy idempotence property ($\tilde{C}3$) for closure operators and the fuzzy cut property (f_c3) for consequence relations become equivalent.

Moreover, also as in the classical setting, a fuzzy closure operator $\tilde{C}$ on $\mathcal{F}(\mathcal{L})$ defines its corresponding closure system $\mathcal{C}$ as the set of closed fuzzy sets, i.e. $\mathcal{C} = \{T \in \mathcal{F}(\mathcal{L}) \mid \tilde{C}(T) = T\}$. Conversely, if $\mathcal{F}(\mathcal{L})$ is a complete lattice, then every complete inf-semilattice $\mathcal{C} \subseteq \mathcal{F}(\mathcal{L})$ containing the maximum defines a closure operator $\tilde{C}$ whose closure system is $\mathcal{C}$ [29]. And such a closure operator is defined by $\tilde{C}(A) = \bigcap\{T \in \mathcal{C} \mid A \subseteq T\}$.

In this paper we aim at bridging the gap between Chakraborty's graded and Castro et al.'s fuzzy consequence relations by introducing a new class of closure operators, that we call *implicative closure operators*, whose associated consequence relations generalize at the same time the former consequence relations[3]. In some more detail, after this introduction, in Section 2 we define and characterize implicative closure operators as well as their associated closure systems. In Section 3, we related implicative closure operators to other kinds of fuzzy (inference) operators defined by fuzzy relations, in particular to those defined by preorders and similarity relations. Finally, in Section 4, we show that Implicative Closure operators also capture some well-known approximate entailments, like the approximate and proximity entailments introduced by Dubois et al. [11], the canonical extension of a classical closure operator defined by Gerla [17], Boixader and Jacas' natural inference operators [5], and the fuzzy operators defined by Biacino, Gerla and Ying [4]. But before going into details, we introduce below some background that will be needed in the rest of the paper.

For our task, we need to consider enriched lattice structures as algebras of truth values for fuzzy sets of formulas. Namely, we need to expand

[3] A very close notion to implicative closure operators has been independently introduced by Belohlavek in [1, 2] after our conference papers were published.

lattices $(L, \wedge, \vee, 0, 1)$ to complete BL-algebras[4], i.e. algebraic structures $(L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ where $(L, \wedge, \vee, 0, 1)$ is a complete distributive lattice, $(L, \otimes, 1)$ is a commutative monoid and $(\otimes, \Rightarrow)$ is a residuated pair, i.e. it verifies the residuation condition for all $x, y, z \in L$

$$x \otimes y \leq z \text{ if and only if } \; x \leq y \Rightarrow z,$$

and which further fulfills the following two conditions for all $x, y, z \in L$,

$$x \wedge y = x \otimes (x \Rightarrow y) \quad \text{and} \quad (x \Rightarrow y) \vee (y \Rightarrow x) = 1$$

The operation $\Rightarrow$ is usually called *residuum* of $\otimes$. Main examples of BL-algebras are the ones defined over the real unit interval [0, 1]. In a such a case the operation $\otimes$ is a continuous t-norm and $\Rightarrow$ is its corresponding residuum. The importance of BL-algebras on [0,1] is that they generate the whole variety of BL-algebras [10]. Point-wise extensions of these operations to fuzzy sets of formulas in $\mathcal{F}(\mathcal{L})$ are defined analogously as previously done for the lattice operations $\wedge$ and $\vee$. Throughout this paper, for each $\alpha \in L$, $\alpha \otimes A$ will denote the fuzzy set in $\mathcal{F}(\mathcal{L})$ defined by $(\alpha \otimes A)(p) = \alpha \otimes A(p)$ for all $p \in \mathcal{L}$ and the same for the residuated implication.

On the other hand and in the context of a BL-algebra $(L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$, we shall also make use of the degree of inclusion between two fuzzy sets of formulas to defined as

$$[A \sqsubseteq_{\otimes} B] = \inf_{p \in \mathcal{L}} A(p) \Rightarrow B(p).$$

Notice that, since it holds $x \Rightarrow y = 1$ iff $x \leq y$ for all $x, y \in L$, we have that $[A \sqsubseteq_{\otimes} B] = 1$ iff $A \subseteq B$.

Actually, using the notation of closure operators and the notion of degree of inclusion, the relationship between graded consequence and fuzzy consequence relations become self-evident. As already mentioned, the former is defined only over classical sets while the latter is defined over fuzzy sets, but both yield a fuzzy set of formulas as output. But, having this difference in mind, the two first conditions of both operators, i.e. g_c1, f_c1 and g_c2, f_c2, become syntactically the same than $\tilde{C}1$ and $\tilde{C}2$ respectively,

1) *fuzzy inclusion*: $A \subseteq \tilde{C}(A)$

2) *fuzzy monotony*: If $B \subseteq A$ then $\tilde{C}(B) \subseteq \tilde{C}(A)$

[4] BL-algebras are introduced by Hájek in [19] as the algebraic counterpart of the so-called Basic Fuzzy Logic, which is the logic of continuous t-norms and which is briefly recalled at the end of section 2.

while the third one, the fuzzy cut, become very close one to another:

$g_c3)$ *fuzzy cut*: $([B \sqsubseteq \tilde{C}(A)] \otimes \tilde{C}(A \cup B)) \subseteq \tilde{C}(A)$,
where $[B \sqsubseteq \tilde{C}(A)] = \inf_{q \in B} \tilde{C}(A)(q)$ (recall that B is a classical set).

$f_c3)$ *fuzzy cut*: if $B \subseteq \tilde{C}(A)$ then $\tilde{C}(A \cup B) \subseteq \tilde{C}(A)$

2 Implicative closure operators

In this section we introduce implicative closure operators as a generalization of Chakraborty's graded consequence relations over fuzzy sets of formulas. The adjective *implicative* is due to the fact we generalize the Fuzzy Cut property (g_c3) by means of the above defined degree of inclusion, which in turn depends on the implication operation $\Rightarrow$ of the BL-algebra $\mathbf{L} = (L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ over which fuzzy sets of formulas are defined. Unless stated otherwise, for the rest of the section we shall assume $\mathcal{F}(\mathcal{L})$ be defined over a given (and fixed) BL-algebra $\mathbf{L}$ is fixed.

Definition 4 (Implicative closure operators) *A mapping $\tilde{C} : \mathcal{F}(\mathcal{L}) \longmapsto \mathcal{F}(\mathcal{L})$ is called an implicative closure operator if, for every $A, B \in \mathcal{F}(\mathcal{L})$, $\tilde{C}$ fulfills:*

$\tilde{C}1)$ fuzzy inclusion*:* $A \subseteq \tilde{C}(A)$

$\tilde{C}2)$ fuzzy monotony*: If $B \subseteq A$ then $\tilde{C}(B) \subseteq \tilde{C}(A)$*

$\tilde{C}'3)$ fuzzy cut[5]*:* $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \otimes \tilde{C}(A \cup B) \subseteq \tilde{C}(A)$. .

The corresponding implicative consequence relation, denoted by I_c, and defined as $I_c(A, p) = \tilde{C}(A)(p)$. The translation of the properties of Implicative closure operators to Implicative consequence relation read as follows:

$I_c1)$ *fuzzy reflexivity*: $A(p) \leq I_c(A, p)$

$I_c2)$ *fuzzy monotony*: If $B \subseteq A$ then $I_c(B, p) \leq I_c(A, p)$

$I_c3)$ *fuzzy cut*: $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \otimes I_c(A \cup B, p) \leq I_c(A, p)$.

[5]Due to the residuation property, this axiom could also be presented as $\tilde{C}(A \cup B) \subseteq [B \sqsubseteq_{\otimes} \tilde{C}(A)] \Rightarrow \tilde{C}(A)$.

Now, it is easy to check that the restriction of implicative consequence relations over classical sets of formulas are exactly Chakraborty's graded consequence relations, since if B is a crisp set, $[B \sqsubseteq_{\otimes} \tilde{C}(A)] = \inf_{p \in B} I_c(A, p)$. On the other hand, fuzzy consequence relations are implicative as well, since property I_c3) clearly implies f_c3). Therefore, implicative consequence relations generalize both graded and fuzzy consequence relations.

Implicative closure relations admit also representation theorem which is a generalization of the one given by Chakraborty in [8] for graded consequence relations.

Theorem 1 *A fuzzy relation $I_c : \mathcal{F}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ is an implicative consequence relation if and only if there exists a family of fuzzy sets $\{T_i\}_{i \in I}$ such that $I_c(A, p) = \inf_{i \in I}([A \sqsubseteq_{\otimes} T_i] \Rightarrow T_i(p))$.*

Proof: First, assume a family $\{T_i\}_{i \in I}$ is given and let us prove that I_c as defined above is an implicative consequence relation. Properties I_c1 I_c2 are easy to prove. In order to prove I_c3 we shall make use of the following general properties, where A, B, C, D are arbitrary fuzzy sets:

(i) $(A \Rightarrow C) \cap (B \Rightarrow C) = ((A \cup B) \Rightarrow C)$

(ii) $A \cap B = A \otimes (A \Rightarrow B)$

(iii) if $(A \otimes B) \Rightarrow C \supseteq D$ then $A \Rightarrow C \supseteq B \otimes D$

(iv) $(A \Rightarrow (B \Rightarrow C) = B \Rightarrow (A \Rightarrow C)$

Then, by (i), we have $(A(q) \Rightarrow T_i(q)) \wedge (B(q) \Rightarrow T_i(q)) = (A \cup B)(q) \Rightarrow T_i(q)$. Taking infima with respect to q at both sides and using (ii) we have

$$A^i \otimes (A^i \Rightarrow B^i) = (A \cup B)^i$$

where, for the sake of a simpler notation, we use X^i standing for $[X \sqsubseteq_{\otimes} T_i]$. Therefore,

$$(A^i \otimes (A^i \Rightarrow B^i)) \Rightarrow T_i(p) = (A \cup B)^i \Rightarrow T_i(p)$$

also holds, and taking into account (iii) we get

$$A^i \Rightarrow T_i(p) \geq (A^i \Rightarrow B^i) \otimes ((A \cup B)^i \Rightarrow T_i(p)) \ ,$$

and taking infima over the subscript i in both sides we have

$$I_c(A, p) \geq \inf_{i \in I}\{A^i \Rightarrow B^i\} \otimes I_c(A \cup B, p) \ .$$

Finally let us show that $\inf_{i \in I}\{A^i \Rightarrow B^i\} = \inf_{q \in \mathcal{L}}\{B(q) \Rightarrow I_c(A, q)\}$. Namely, applying (iv) we have the following equalities: $\inf_q\{B(q) \Rightarrow$

$I_c(A,q)\} = \inf_q\{B(q) \Rightarrow \inf_i\{A^i \Rightarrow T_i(q)\}\} = \inf_{i,q}\{A^i \Rightarrow (B(q) \Rightarrow T_i(q))\} = \inf_i\{A^i \Rightarrow \inf_q\{B(q) \Rightarrow T_i(q)\}\} = \inf_i\{A^i \Rightarrow B^i\}$. This ends the proof of property I_c3.

Now suppose a relation I_c fulfilling the three properties is given. Take the family of fuzzy sets of formulas $\{T_D\}_{D\in\mathcal{F}(\mathcal{L})}$ with $T_D(p) = I_c(D,p)$. For any $A \in \mathcal{F}(\mathcal{L})$ it is clear that $\inf_{D\in\mathcal{F}(\mathcal{L})}\{\inf_{q\in\mathcal{L}}\{A(q) \Rightarrow T_D(q)\} \Rightarrow T_D(p)\} \leq \inf_{q\in\mathcal{L}}\{A(q) \Rightarrow I_c(A,q)\} \Rightarrow I_c(A,p) = 1 \Rightarrow I_c(A,p) = I_c(A,p)$, since I_c fulfills fuzzy reflexivity. On the other hand, since I_c satisfies fuzzy cut, we have

$$I_c(D,p) \geq I_c(A\cup D,p) \otimes \inf_{q\in\mathcal{L}}\{A(q) \Rightarrow I_c(D,q)\}\ ,$$

hence

$$I_c(A\cup D,p) \leq \inf_{q\in\mathcal{L}}\{A(q) \Rightarrow I_c(D,q)\} \Rightarrow I_c(D,p)\ .$$

Taking into account that I_c fulfills fuzzy monotony we have $I_c(A,p) \leq I_c(A\cup D,p) \leq \inf_{q\in\mathcal{L}}\{A(q) \Rightarrow I_c(D,q)\} \Rightarrow I_c(D,p)$. Thus

$$I_c(A,p) \leq \inf_{D\in\mathcal{F}}\{\inf_{q\in\mathcal{L}}\{A(q) \Rightarrow I_c(D,q)\} \Rightarrow I_c(D,p)\}$$

and the theorem is proved. □

A family of fuzzy sets $\{T_i\}_{i\in I}$ defining an implicative consequence relation I_c in the sense of the above theorem will be called a set of *generators* of I_c. In terms of closure operators, the above theorem says that $\tilde{C}$ is an implicative closure operator iff there exist a family of fuzzy sets $\{T_i\}_{i\in I}$ such that $\tilde{C}(A) = \bigcap_i\{[A \sqsubseteq_\otimes T_i] \Rightarrow T_i\}$. Observe that, for each generator T_i, we have $\tilde{C}(T_i) = T_i$, i.e. the generators are closed sets with respect to the closure operator they generate.

Let us prove another interesting property of implicative closure operators.

Proposition 1 *Any implicative closure operator $\tilde{C}$ satisfies the next additional property, for each $k \in L$ and $A \in \mathcal{F}(\mathcal{L})$:*

$\tilde{C}4)$ $\tilde{C}(A \otimes \bar{k}) \supseteq \tilde{C}(A) \otimes \bar{k}$.

Proof: Given an implicative closure operator $\tilde{C}$, by the representation theorem 1, there exists a family of fuzzy sets $\{T_i\}_{i\in I}$ such that $\tilde{C}(A) = \bigcap_i\{A^i \Rightarrow T_i\}$, where $A^i = [A \sqsubseteq_\otimes T_i]$. In particular, using the same notation, $\tilde{C}(A\otimes\bar{k}) = \bigcap_i((A\otimes\bar{k})^i \Rightarrow T_i)$. One can easily check that $(A\otimes\bar{k})^i = k \Rightarrow A^i$, hence $\tilde{C}(A\otimes\bar{k}) = \bigcap_i((k \Rightarrow A^i) \Rightarrow T_i)$. Now, taking into account that the inequality $x \otimes (y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow z$ holds for any $x, y, z \in L$, it is clear that $\tilde{C}(A\otimes\bar{k}) \supseteq \bigcap_i(k \otimes (A^i \Rightarrow T_i)) = k \otimes \bigcap_i(A^i \Rightarrow T_i) = k \otimes \tilde{C}(A)$. □

This last property provides us with the clue for another characterization of implicative closure operators.

Theorem 2 *$\tilde{C}$ is an Implicative Closure Operator if and only if it satisfies $\tilde{C}1, \tilde{C}2$, $\tilde{C}3$ and $\tilde{C}4$.*

Proof: From Proposition 1 we only need to prove that if $\tilde{C}$ is an operator satisfying the conditions of the theorem, then it satisfies the property $\tilde{C}'3$. Observe that $(B \Rightarrow D) \subseteq (B \cup A \Rightarrow D \cup A)$. In particular, letting $D = \tilde{C}(A)$ and taking into account that $\tilde{C}(A) \cup A = \tilde{C}(A)$, we have $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \subseteq [(B \cup A) \sqsubseteq_{\otimes} \tilde{C}(A)] \subseteq (B \cup A) \Rightarrow \tilde{C}(A)$, or equivalently, $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \otimes (B \cup A) \subseteq \tilde{C}(A)$. Then applying $\tilde{C}$ to both sides, and using monotony, idempotence and $\tilde{C}4$, we finally obtain $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \otimes \tilde{C}(B \cup A) \subseteq \tilde{C}(A)$, which is the property $C'3$. □

Property $\tilde{C}4$ shows the behaviour of an implicative closure operator with respect to uniform modifications of fuzzy sets of formulas when multiplying (using the monoidal operation $\otimes$) them by a truth-constant. However, this property does not provide an intuitive idea about the kind of operators that are characterized by such a property. In order to offer a better interpretation, we introduce the following more intuitive property:

$\tilde{C}2_{\otimes}$) $\otimes$-monotony: $[A \sqsubseteq_{\otimes} B] \leq [\tilde{C}(A) \sqsubseteq_{\otimes} \tilde{C}(B)]$

This property amounts to the preservation by the closure operator of not only the usual inclusion but also the inclusion degree. Moreover, if we define a degree of equality between fuzzy sets of formulas by a double inclusion schema as

$$A \simeq_{\otimes} B = ([A \sqsubseteq_{\otimes} B]) \otimes ([B \sqsubseteq_{\otimes} A]),$$

the above property $\tilde{C}2_{\otimes}$ leads to this other property

$$A \simeq_{\otimes} B \ \leq \ \tilde{C}(A) \simeq_{\otimes} \tilde{C}(B),$$

which establishes that two sets of consequences are at least as equal (in the above sense) as the original sets. What is interesting is that implicative closure operators can be also characterized in an equivalent way by using this new property instead of the $\tilde{C}4$ property.

Theorem 3 *Let $\tilde{C} : \mathcal{F}(\mathcal{L}) \longmapsto \mathcal{F}(\mathcal{L})$ be a mapping that satisfies fuzzy inclusion and fuzzy idempotence. Then the following two properties are equivalent:*

1. $[B \sqsubseteq_\otimes \tilde{C}(A)] \leq [\tilde{C}(A \cup B) \sqsubseteq_\otimes \tilde{C}(A)]$

2. $[B \sqsubseteq_\otimes A] \leq [\tilde{C}(B) \sqsubseteq_\otimes \tilde{C}(A)]$

Proof:

1 $\Rightarrow$ *2*) Because the graded inclusion is non-decreasing in the second argument and non-increasing in the first, it is obvious that the following properties are verified: $([B \sqsubseteq_\otimes A]) \leq ([B \sqsubseteq_\otimes \tilde{C}(A)])$ and $([\tilde{C}(A \cup B) \sqsubseteq_\otimes \tilde{C}(A)]) \leq ([\tilde{C}(B) \sqsubseteq_\otimes \tilde{C}(A)])$. Therefore, by using transitivity over Property *1*, we obtain the desired inequality.

2 $\Rightarrow$ *1*) Since in any BL-algebra it holds that $x \Rightarrow y \leq (x \vee z) \Rightarrow (y \vee z)$, we have that $[B \sqsubseteq_\otimes \tilde{C}(A)]$ is always smaller than $[(B \cup A) \sqsubseteq_\otimes (\tilde{C}(A) \cup A)]$, and since $\tilde{C}$ satisfies fuzzy inclusion $\tilde{C}(A) \cup A = \tilde{C}(A)$, the last inclusion degree is equal to $[(B \cup A) \sqsubseteq_\otimes \tilde{C}(A)]$. Then, by applying Property *2*, we have $[(B \cup A) \sqsubseteq_\otimes \tilde{C}(A)] \leq [\tilde{C}(B \cup A) \sqsubseteq_\otimes \tilde{C}(\tilde{C}(A))]$. Finally, using the idempotence property for $\tilde{C}$, one gets

$$[B \sqsubseteq_\otimes \tilde{C}(A)] \leq [(B \cup A) \sqsubseteq_\otimes \tilde{C}(A)] \leq [\tilde{C}(B \cup A) \sqsubseteq_\otimes \tilde{C}(A)],$$

and the proof is completed.

$\square$

Corollary 1 *A mapping $\tilde{C} : \mathcal{F}(\mathcal{L}) \longmapsto \mathcal{F}(\mathcal{L})$ satisfies $\tilde{C}1$, $\tilde{C}3$ and $\tilde{C}2_\otimes$ if and only if it is an implicative closure operator.*

Proof: It is enough to remark that property *1* from last theorem is actually an equivalent formulation of the fuzzy cut property in Definition 4, and therefore a closure operator is implicative if and only if it satisfies fuzzy inclusion, fuzzy idempotence and Property *1*. Also, notice that property $\tilde{C}2$ becomes now an easy consequence of $\tilde{C}2_\otimes$. $\square$

It is worth mentioning that closure operators under this last formulation, and some generalizations of them, have been recently and independently studied by Bělohlávek in [1, 2].

Finally, the following theorem characterizes fuzzy closure systems corresponding to implicative closure operators, that we shall accordingly call *implicative closure systems*.

Theorem 4 *A fuzzy closure system $\mathcal{C}_\Rightarrow$ in $\mathcal{F}(\mathcal{L})$ is implicative if and only if, for any $T \in \mathcal{C}_\Rightarrow$ and for any $k \in L$, $\bar{k} \Rightarrow T \in \mathcal{C}_\Rightarrow$.*

Proof: For one direction assume $\mathcal{C}_{\Rightarrow}$ is an implicative closure system. Then there exists an implicative closure operator $\mathcal{C}$ such that

$$\mathcal{C}_{\Rightarrow} = \{T \in \mathcal{F}(\mathcal{L}) \mid \tilde{C}(T) = T\} \ .$$

Now for each $T \in \mathcal{C}_{\Rightarrow}$ and $k \in L$ we have $T \supseteq \bar{k} \otimes (\bar{k} \Rightarrow T)$. Using monotony and $\tilde{C}4$ we have

$$T = \tilde{C}(T) \supseteq \tilde{C}(\bar{k} \otimes (\bar{k} \Rightarrow T)) \supseteq \bar{k} \otimes \tilde{C}(\bar{k} \Rightarrow T),$$

and by the residuation property, we have $\tilde{C}(\bar{k} \Rightarrow T) \subseteq \bar{k} \Rightarrow T$. Finally using inclusion property we have $\tilde{C}(\bar{k} \Rightarrow T) = \bar{k} \Rightarrow T$. Thus, $\bar{k} \Rightarrow T \in \mathcal{C}_{\Rightarrow}$.

As for the other direction, assume now $\mathcal{C}_{\Rightarrow}$ is a subset of $\mathcal{F}(\mathcal{L})$ satisfying that, for all $T \in \mathcal{F}(\mathcal{L})$ and $k \in L$, if $T \in \mathcal{C}_{\Rightarrow}$ then $\bar{k} \Rightarrow T \in \mathcal{C}_{\Rightarrow}$. Since $\mathcal{C}_{\Rightarrow}$ is a fuzzy closure system $\tilde{C}(A) = \bigcap\{T \in \mathcal{C}_{\Rightarrow} \mid A \subseteq T\}$ is a fuzzy closure operator. Hence, only need to prove that $\tilde{C}$ satisfies property $\tilde{C}4$. But, for any $k \in L$, by the assumed property, $\tilde{C}(A) \subseteq \bigcap\{\bar{k} \Rightarrow T \in \mathcal{C}_{\Rightarrow} \mid A \subseteq \bar{k} \Rightarrow T\}$, and this is equivalent to $\tilde{C}(A) \subseteq \bar{k} \Rightarrow \bigcap\{T \in \mathcal{C}_{\Rightarrow} \mid A \otimes \bar{k} \subseteq T\} = \bar{k} \Rightarrow \tilde{C}(A \otimes \bar{k})$, and by the residuation property we obtain $\tilde{C}(A) \otimes \bar{k} \subseteq \tilde{C}(A \otimes \bar{k})$. This ends the proof. □

We conclude this section by showing that the consequence relations associated to classical propositional logic and to Gödel infinitely-valued propositional logic are implicative, while the consequence relations defined by Product and Lukasiewicz infinitely-valued logics are not.

i) The case of classical logic

Proposition 2 *The logical consequence relation $\models$ of classical propositional logic is implicative.*

Proof: Let $\mathcal{L}$ be a propositional language built from a countable set of propositional variables and classical logic connectives. Let $\Omega = \{w_i\}_{i \in I}$ be the set of maximally consistent sets of formulas and write

$$w_i(p) = \begin{cases} 1 & \text{if } p \in w_i \\ 0 & \text{otherwise} \end{cases}$$

for any formula $p \in \mathcal{L}$. It is well known that Ω can be identified with the set of classical interpretations for $\mathcal{L}$. Following the representation theorem of implicative consequence relations we can take Ω as a set of generators and define for each crisp set of formulas Γ the following implicative consequence relation

$$I_c(\Gamma, q) = \inf_i \{[\Gamma \sqsubseteq w_i] \Rightarrow w_i(q)\}.$$

Taking into account that Γ and w_i are classical sets, then $[\Gamma \sqsubseteq w_i]$ is the classical inclusion, i.e.,

$$[\Gamma \sqsubseteq w_i] = \inf_{p \in \mathcal{L}} [\Gamma(p) \Rightarrow w_i(p)] = \begin{cases} 1 & \text{if } \Gamma \subseteq w_i \\ 0 & \text{otherwise} \end{cases}$$

Thus, we have

$$I_c(\Gamma, q) = \begin{cases} \inf\{w_i(q) \mid \Gamma \subseteq w_i\}, & \text{if } \Gamma \text{ is consistent} \\ 1, & \text{if } \Gamma \text{ is not consistent} \end{cases}$$

Hence, in the case Γ is consistent, $I_c(\Gamma, q) = 1$ if and only if any interpretation that satisfies all formulas of Γ also satisfies q. Thus, in general, we have

$$I_c(\Gamma, q) = \begin{cases} 1 & \text{if } \Gamma \models q \\ 0 & \text{otherwise} \end{cases}$$

$\square$

ii) The case of t-norm based residuated many-valued logics

In the case of t-norm based residuated many-valued (fuzzy) logics (see [19]) we will study the three basic logics: Gödel, Product and Lukasiewicz, corresponding to the three basic t-norms: minimum, product and Lukasiewicz respectively. The core of all t-norm based propositional calculi is the logic BL (for Basic Fuzzy Logic) introduced by Hájek in [19]. The language of BL is built from a countable set of propositional variables, a conjunction &, an implication $\rightarrow$ and the truth constant $\overline{0}$. Definable connectives are:

$$\begin{array}{rcl} \varphi \wedge \psi & \text{is} & \varphi \& (\varphi \rightarrow \psi) \\ \varphi \vee \psi & \text{is} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \neg\varphi & \text{is} & \varphi \rightarrow \overline{0} \end{array}$$

Truth functions for & are continuous t-norms and for $\rightarrow$ their corresponding residua. Axioms of BL are the following:

(A1)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2)	$(\varphi \& \psi) \to \varphi$
(A3)	$(\varphi \& \psi) \to (\psi \& \varphi)$
(A4)	$(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$
(A5a)	$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi))$
(A5b)	$((\varphi \& \psi) \to \chi)) \to (\varphi \to (\psi \to \chi))$
(A6)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7)	$\overline{0} \to \varphi$

The only Inference Rule of BL is Modus Ponens. This logic is indeed the logic of continuous t-norms in the sense that a formula is provable in BL iff it is a 1-tautology under each interpretation on [0,1] and under each continuous t-norm and its residuum [10].

The three main many-valued logics cited before can be obtained as axiomatic extensions of BL (see [19]).

1. Lukasiewicz logic L is the extension of BL with the double negation axiom

   $(\neg\neg) \quad \neg\neg\varphi \to \varphi$

2. Gödel logic G is the extension of BL with the contraction axiom

   $(G) \quad \varphi \to \varphi \,\&\, \varphi$

3. Product logic is the extension of BL with the axioms

   $(\Pi 1) \quad \neg\neg\psi \to [((\varphi \& \psi) \to (\chi \& \psi)) \to (\varphi \to \chi)]$

   $(\Pi 2) \quad \varphi \wedge \neg\varphi \to 0$

These three logics are complete with respect to interpretations over the BL-algebras on [0,1] defined by Lukasiewicz, Minimum and Product t-norms respectively.

Implicative consequence relations deal with fuzzy sets of formulas, so we need to interpret them in these logics. In fact, the natural way of interpreting fuzzy set of formulas is to introduce truth-constants into the language, a truth constant $\bar{\alpha}$ for each rational $\alpha \in [0,1]$ if we want to keep the language countable. Then, a fuzzy set of formulas A on $\mathcal{L}$ can be interpreted as the set of formulas $A^{\circ} = \{\overline{\alpha_i} \to \varphi_i \mid \alpha_i = A(\varphi_i), \varphi_i \in \mathcal{L}\}$, since a formula $\bar{\alpha} \to \varphi$ is 1-true iff the truth-value of φ is greater or equal α.

The extension of the above logics with rational truth constants are the so-called Rational Gödel (RG), Rational Product (RΠ) and Rational Pavelka (RL) logics respectively, and they need to introduce two additional axioms for purposes of book-keeping of truth-constants,

$$\begin{array}{ll} (1) & \overline{\alpha} \& \overline{\beta} \equiv \overline{\alpha \otimes \beta}, \\ (2) & \overline{\alpha} \rightarrow \overline{\beta} \equiv \overline{\alpha \Rightarrow_{\otimes} \beta}, \end{array}$$

for all rationals $\alpha, \beta \in [0,1]$, where $\otimes$ and $\Rightarrow_{\otimes}$ are the corresponding t-norm (minimum, product and Lukasiewicz respectively) and its residuum. Completeness results for these logics can be found in [19] for the Rational Pavelka logic and in [14] for the Rational Gödel and Rational Product logics. Just remark that in the case of rational Product logic one has to add the following infinitary rule of inference: from $\varphi \rightarrow \overline{\alpha}$ for each rational $\alpha > 0$, derive $\varphi \rightarrow \overline{0}$.

Taking into account the above interpretation of fuzzy sets of formulas, the natural way to define the consequence operator over fuzzy sets of formulas in each of these logics is the following: for each fuzzy set A on $\mathcal{L}$, $C(A)$ is again a fuzzy set on $\mathcal{L}$ defined by

$$C_{\star}(A)(\varphi) = \sup\{\alpha \mid A^{\circ} \vdash_{\star} \overline{\alpha} \rightarrow \varphi\}$$

for $\star$ denoting RG, RΠ or RL. Remark that, to be consistent with the frame, fuzzy sets of formulas have to take values on the rationals of [0, 1], so we consider in the rest of this section $\mathcal{F}(\mathcal{L}) = \{A : \mathcal{L} \longmapsto [0,1] \cap \mathbf{Q}\}$.

Proposition 3
(i) The consequence operator of Rational Gödel logic C_{RG} is implicative.
(ii) The consequence operators of Rational Product and Pavelka logics, $C_{R\Pi}$ and C_{RL} respectively, are not implicative.

Proof: (i) We have to prove that $C_{RG}(A \otimes \bar{k}) \supseteq C_{RG}(A) \otimes \bar{k}$ for $\otimes = \min$. Let $\mathcal{L} = \{\phi_i\}_{i \in I}$, and let $\alpha_i = A(\phi_i)$ for each $i \in I$. According to our interpretation, the fuzzy set $\bar{k} \otimes A$ corresponds to the set of formulas $(\bar{k} \otimes A)^{\circ} = \{\overline{k \otimes \alpha_i} \rightarrow \phi_i\}_{i \in I}$. Then it will be enough to prove that

$$\{\overline{k \otimes \alpha_i} \rightarrow \phi_i\}_{i \in I} \vdash \overline{k \otimes \beta} \rightarrow \varphi \qquad (*)$$

assuming that $\{\overline{\alpha_i} \rightarrow \phi_i\}_{i \in I} \vdash \overline{\beta} \rightarrow \varphi$. But, due to the book-keeping axioms, proving $(*)$ is equivalent to prove

$$\{\overline{k} \& \overline{\alpha_i} \rightarrow \phi_i\}_{i \in I} \vdash \overline{k} \& \overline{\beta} \rightarrow \varphi \qquad (**)$$

Moreover, in G one has that $\psi \& \chi \rightarrow \varphi$ is provably equivalent to $\psi \rightarrow (\chi \rightarrow \varphi)$, and since the deduction theorem is valid in Gödel logic (see [19]), we have that proving $(**)$ is still equivalent to prove

$$\{\overline{k} \to (\overline{\alpha_i} \to \phi_i)\}_{i \in I}, \overline{k} \vdash \overline{\beta} \to \varphi \qquad (***)$$

Now, by modus ponens, we have that for each $i \in I$,

$$\{\overline{k} \to (\overline{\alpha_i} \to \phi_i), \overline{k}\} \vdash \overline{\alpha_i} \to \phi_i$$

that is, $(***)$ holds true if $\{\overline{\alpha_i} \to \phi_i\}_{i \in I} \vdash \overline{\beta} \to \varphi$, but this was just the hypothesis.

(ii) Suppose that $\otimes$ is Lukasiewicz or Product t-norm. Then, by modus ponens and the book-keeping axioms, we have that the inference

$$\{(\overline{\alpha} \to (p \to q), \overline{\beta} \to p\}) \vdash \overline{\alpha \otimes \beta} \to q$$

is valid in both RΠ and RL, and thus,

$$\{(\overline{\alpha \otimes k} \to (p \to q), \overline{\beta \otimes k} \to p\}) \vdash \overline{\alpha \otimes k \otimes \beta \otimes k} \to q$$

is valid as well. But if $C_{R\Pi}$ and $C_{R\text{Ł}}$ were implicative then

$$\{(\overline{\alpha \otimes k} \to (p \to q), \overline{\beta \otimes k} \to p\}) \vdash \overline{\alpha \otimes \beta \otimes k} \to q$$

should also be valid. Therefore, we should have

$$\alpha \otimes \beta \otimes k \leq \alpha \otimes k \otimes \beta \otimes k$$

for all rationals α, β and k. In particular, this would imply (taking $\alpha = \beta = 1$) that for all rational k, $k \otimes k = k$, which is only true for $\otimes = \min$. □

3 Implicative closure operators and closure operators defined by a fuzzy relation

Different authors have studied the so-called *fuzzy operators* defined by fuzzy relations (see [15]), specially those defined by preorders ([6] and [12]) and by fuzzy similarity relations (see for instance [15] and [24]). We recall the definition and basic properties of a fuzzy operator defined by a fuzzy relation (see [22]). We continue assuming to work with fuzzy sets of formulas over a BL-algebra $\mathbf{L} = (L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$.

Definition 5 *Given an L-fuzzy relation $R: \mathcal{L} \times \mathcal{L} \longmapsto L$ on the language $\mathcal{L}$, the associated* fuzzy operator $\tilde{C}_R$ *over $\mathcal{F}(\mathcal{L})$ is defined by:*

$$\tilde{C}_R(A)(q) = \bigvee_{p \in \mathcal{L}} \{A(p) \otimes R(p,q)\}$$

for all $A \in \mathcal{F}(\mathcal{L})$.

In other words, the image $\tilde{C}_R(A)$ of a fuzzy set A by the fuzzy operator $\tilde{C}_R$ is the $\vee$-$\otimes$ composition of A with R.

Definition 6 *A fuzzy operator $\tilde{C}_R$ is called* upper fuzzy operator *if it satisfies fuzzy inclusion, i.e. if $A \subseteq \tilde{C}_R(A)$ for all $A \in \mathcal{F}(\mathcal{L})$. Upper fuzzy operators which are fuzzy closure operators (in the sense of Definition 1) will be called* upper closure operators.

Proposition 4 *(Cf. [6, 12]) Let $\tilde{C}_R$ be a fuzzy operator. Then:*

- *$\tilde{C}_R$ is an upper fuzzy operator iff R is reflexive.*
- *$\tilde{C}_R$ is an upper closure operator iff R is reflexive and $\otimes$-transitive*[6]*, i.e. iff R is a fuzzy preorder.*

Next we list some interesting properties of upper fuzzy operators..

Proposition 5 *(Cf. [15]) Any upper fuzzy operator $\tilde{C}_R$ satisfies the following properties*[7]*:*

C1: $A \subseteq \tilde{C}_R(A)$

C2: $\tilde{C}_R(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \tilde{C}_R(A_i)$

C4: $\tilde{C}_R(\bar{k}) = \bar{k}$

C5$^{\otimes}$*:* $\tilde{C}_R(\{p\} \otimes \bar{k}) = \tilde{C}_R(\{p\}) \otimes \bar{k}$

C6$^{\otimes}$*:* $\tilde{C}_R(A \otimes \bar{k}) = \tilde{C}_R(A) \otimes \bar{k}$,

where $\{p\}$ denotes a crisp singleton, i.e. $\{p\}(p) = 1$ and $\{p\}(q) = 0$ for $q \neq p$.

Two other properties considered in [15], namely

C3: $\tilde{C}_R \circ \tilde{C}_R = \tilde{C}_R$

C7: $\tilde{C}_R(\{p\})(q) = \tilde{C}_R(\{q\})(p)$,

do not hold in general for upper fuzzy operators. Actually, it is easy to show the following proposition.

[6] R is said to be $\otimes$-transitive if it satisfies $R(p,q) \otimes R(q,r) \leq R(p,r)$ for all $p,q,r \in \mathcal{L}$.
[7] We use the same labelling of properties as in [15].

Proposition 6 *Let $\tilde{C}_R$ be an upper fuzzy operator. Then:*

- *$\tilde{C}_R$ satisfies C3 iff R is reflexive and $\otimes$-transitive (iff R is a fuzzy preorder).*
- *$\tilde{C}_R$ satisfies C7 iff R is symmetric.*

In [23] it is proved that if R is a $\otimes$-similarity, the operator $\tilde{C}_R$ satisfies the properties *C1*, *C2*, *C3*, *C4*, *C5*$^{\otimes}$, *C6*$^{\otimes}$ and *C7*. We have also seen that operators $\tilde{C}_R$ are fuzzy closure operators iff R is a fuzzy preorder, as it was stated in [12]. Moreover, it is obvious from C5$^{\otimes}$ and Theorem 2, that all upper closure operators (defined by a fuzzy preorder) are implicative. The converse is not true in general since $\tilde{C}4$ does not imply C5$^{\otimes}$. On the other hand, an interesting fact is that any implicative closure operator satisfies a property called *coherence* in [6].

Proposition 7 *If $\tilde{C}$ is an implicative closure operator, then $\tilde{C}$ fulfills the following property:*

$\tilde{C}5$) coherence*: $\tilde{C}(A)(q) \geq \tilde{C}(\{p\})(q) \otimes A(p)$, for each fuzzy set A on $\mathcal{L}$ and each $p, q \in \mathcal{L}$.*

Proof: The proof is easy by observing that any fuzzy set A can be represented as the union of truncated singletons, i.e. $A = \bigcup_{p\in\mathcal{L}} \{p\} \otimes A(p)$. Then it is clear that, for any $p \in \mathcal{L}$, $\tilde{C}(A) \supseteq \tilde{C}(\{p\} \otimes A(p))$. Then the proposition follows by just applying property $\tilde{C}4$ (see Proposition 1). □

Roughly speaking, this property is requiring that if q is a consequence of p to some degree, and p belongs to A also to some degree, then q must be also a consequence of A to a certain degree. In [6] it is proved that if a closure operator $\tilde{C}$ satisfies $\tilde{C}5$ then $\tilde{C}$ induces a *fuzzy preorder* $R_{\tilde{C}}$ on $\mathcal{L}$ defined by $R_{\tilde{C}}(p, q) = \tilde{C}(\{p\})(q)$. Then it is easy to prove the following proposition.

Proposition 8 *If $\tilde{C}$ is an implicative closure operator with generators $\{T_i\}_{i\in I}$ and $R_{\tilde{C}}$ is the fuzzy preorder defined as above, then it holds that*

$$R_{\tilde{C}}(p, q) = \inf_{i\in I}\{T_i(p) \Rightarrow T_i(q)\}.$$

Therefore this proposition establishes that the generators of an implicative closure operator $\tilde{C}$ are indeed generators of the induced fuzzy preorder $R_{\tilde{C}}$ as well. Conversely, given a fuzzy preorder R, the fuzzy closure operator $\tilde{C}_R$ satisfies the following property.

Proposition 9 *Let R be a fuzzy preorder. Then the closure system associated to the fuzzy closure operator $\tilde{C}_R$ is the set of all generators of the fuzzy preorder R.*

Proof: Taking into account that a fuzzy set T is a generator of the fuzzy preorder R if for all $p, q \in L$, then $R(p,q) \leq T(p) \to T(q)$, the following relations hold:

$$\begin{aligned} \tilde{C}_R(T)(q) &= \bigvee_{p\in\mathcal{L}} T(p) \otimes R(p,q) \\ &\leq \bigvee_{p\in\mathcal{L}} T(p) \otimes (T(p) \Rightarrow T(q)) \\ &= \bigvee_{p\in\mathcal{L}} T(p) \wedge T(q) \\ &= T(q) \ . \end{aligned}$$

Then $\tilde{C}_R(T) = T$ and thus T is closed with respect to $\tilde{C}_R$. On the other hand if T is closed by $\tilde{C}_R$, then $T(q) = \tilde{C}_R(T)(q) = \bigvee_p T(p) \otimes R(p,q)$ and thus, for all $p \in \mathcal{L}$, $(T(p) \otimes R(p,q)) \leq T(q)$ and, by residuation, this implies $R(p,q) \leq (T(p) \Rightarrow T(q))$. Therefore T is a generator of R. $\square$

As a consequence, and taking into account the result of [23] characterizing the generators of a fuzzy preorder, we have the following characterization of the closure system associated to an upper closure operator.

Proposition 10 *A set of fuzzy sets $\mathcal{C}_{\Rightarrow}$ is the closure system corresponding to an upper closure operator iff*

1) *it is closed under arbitrary unions, and*

2) *for any fuzzy set $F \in \mathcal{C}_{\Rightarrow}$ and for any $k \in L$, it holds that $\bar{k} \Rightarrow F$, $\bar{k} \otimes F$ and $F \Rightarrow \bar{k}$ belong to $\mathcal{C}_{\Rightarrow}$.*

Let us remark that a fuzzy preorder R can be generated by different families of generators. For each family we obtain, via the representation theorem, a possibly different implicative closure operator but whose associated fuzzy preorder is always R. It is an open problem to study under which conditions two families of generators of a fuzzy preorder induce the same implicative closure operator. Nevertheless, we have the following result.

Theorem 5 *The upper closure operator defined by a fuzzy preorder R is the* least *implicative closure operator $\tilde{C}$ such that $\tilde{C}(\{q\})(p) = R(q,p)$ for any $p, q \in \mathcal{L}$.*

Proof: Suppose that $\tilde{C}$ is an implicative closure operator and $R_{\tilde{C}}$ is the fuzzy preorder associated to it. Then the fuzzy closure operator defined by

$R_{\tilde{C}}$ satisfies

$$\tilde{C}_{R_{\tilde{C}}}(A)(q) = \bigvee_{p\in\mathcal{L}} A(p) \otimes R_{\tilde{C}}(p,q) = \bigvee_{p\in\mathcal{L}} A(p) \otimes \tilde{C}(\{p\})(q) \leq \tilde{C}(A)(q),$$

the last inequality being due to Proposition 7. □

Finally, observe that closure system for an implicative closure operator $\tilde{C}$ is just the complete inf-semilattice generated by the fuzzy sets $\{\bar{k} \Rightarrow T_i\}_{i\in I}$, with $k \in [0,1]$ and $\{T_i\}_{i\in I}$ being a set of generators for $\tilde{C}$. This is a consequence of Theorem 4, the fact that all generators T_i are closed and the fact that, for any A, $\tilde{C}(A)(q) = \inf_{i\in I}\{k_i \Rightarrow T_i(q)\}$, where $k_i = [A \sqsubseteq_\otimes T_i]$.

4 Relationships with other approaches

In this section we relate our approach to other approaches in the framework of fuzzy inference operators developed for some authors during the last years.

4.1 Extensional Inference Operators

In [5] Boixader and Jacas analyze approximate reasoning patterns through the notion of *extensionality* with respect to the so-called *natural $\otimes$-similarity functions*. For this purpose, they introduce a family of operators $I : [0,1]^U \longmapsto [0,1]^V$, where U and V are universes of discourse. These operators are called *extensional inference operators* if they preserve the point-wise order, i.e. if $A_1 \subseteq_U A_2$ then $I(A_1) \subseteq_V I(A_2)$, and satisfy $\otimes$-extensionality, i.e.

$$\bar{E}_U^\otimes(A_1, A_2) \leq \bar{E}_V^\otimes(I(A_1), I(A_2)) \ ,$$

where $\bar{E}_U^\otimes$ and $\bar{E}_V^\otimes$ are the natural similarity functions on fuzzy subsets of U and V respectively, defined by

$$\bar{E}_U^\otimes(A_1, A_2) = \inf_{x\in U}\{A_1(x) \vee A_2(x) \Rightarrow A_1(x) \wedge A_2(x)\}$$

and analogously for $\bar{E}_V^\otimes$. In this context, it is assumed that $\otimes$ and $\Rightarrow$ stand for continuous t-norm and its residuum respectively. They show that it is possible to associate to any fuzzy rule *"If A then B"* the so-called *natural inference operator*, which is the optimal one from the extensionality point of view.

Definition 7 (Natural inference operator) *Given a fuzzy rule "If A then B" with $A \in [0,1]^U$ and $B \in [0,1]^V$, the natural inference operator $\bar{I}_{AB} : [0,1]^U \longmapsto [0,1]^V$ associated to the rule is defined by $\bar{I}_{AB}(A')(v) = \inf_{u\in U}\{A'(u) \Rightarrow A(u)\} \Rightarrow B(v)$.*

Theorem 6 *[5, Theorem 15] The natural inference operator associated to the rule "If A then B" is the least specific extensional inference operator* $\bar{I}$ *satisfying:*

- $\bar{I}(A) = B$,
- $\bar{I}(A') \supseteq_V B$, *for any* $A' \in [0,1]^U$, *and moreover* $\bar{I}_{AB}(A') = B$ *if* $A' \subseteq_U A$.

Boixader and Jacas give a representation theorem for the extensional inference operators.

Theorem 7 *([5, Theorem 15])* $I : [0,1]^U \longmapsto [0,1]^V$ *is an extensional inference operator if, and only if, there exists a family of natural inference operators* $\{\bar{I}_{A_iB_i}\}_{i\in I}$ *such that* $I = \inf_{i\in I} \bar{I}_{A_iB_i}$.

We are now in position to show that implicative closure operators are a special class of extensional inference operators. To this end, and in order to consider a common definitional context, in the rest of this subsection we shall take the universes U and V to be same and equal to the propositional language $\mathcal{L}$, i.e. we take $U = V = \mathcal{L}$. Moreover, even Boixader and Jacas only consider fuzzy sets with values in [0, 1], we can safely extend their framework to consider fuzzy sets over an arbitrary BL-algebra $(L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$. Once settled all these preliminaries, we can reason as follows. Since for any implicative closure relation I_c there exists a family of fuzzy sets $\{T_i\}_{i\in I}$ such that

$$I_c(A)(p) = \inf_{i\in I}[A \sqsubseteq_\otimes T_i] \Rightarrow T_i(p)\} ,$$

it is obvious that, for each $i \in I$, the operator I_i defined as

$$I_i(A)(p) = [A \sqsubseteq_\otimes T_i] \Rightarrow T_i(p)$$

can be actually considered as the natural inference operator on $\mathcal{L}$ associated to the rule *"if T_i then T_i"*. Therefore, as an easy consequence of Theorem 7, the following theorem holds.

Theorem 8 *Implicative Closure Operators are Extensional Inference Operators.*

The question is then whether the reciprocal of this theorem is also true. The answer is negative since, in general, extensional inference operators satisfy neither fuzzy inclusion nor fuzzy idempotence. In the following we identify some conditions under which these properties are satisfied.

Lemma 1 *An extensional inference operator $\bar{I} : \mathcal{L} \to \mathcal{L}$ satisfies fuzzy inclusion if, and only if, the associated family of rules $\{$"If A_i then B_i"$\}_{i \in I}$ is such that $A_i \subseteq B_i$ for all $i \in I$.*

Proof: Suppose $A_i \subseteq B_i$ for all i. Then using BL-algebra properties we have $B_i(p) \geq A_i(p) \wedge A(p) = (A(p) \Rightarrow A_i(p)) \otimes A(p)$, which is equivalent to

$$A(p) \leq (A(p) \Rightarrow A_i(p)) \Rightarrow B_i(p).$$

Therefore $\bar{I}_{A_i B_i}(A)(p) = [A \sqsubseteq_\otimes A_i] \Rightarrow B_i(p) \geq (A(p) \Rightarrow A_i(p)) \Rightarrow B_i(p) \geq A(p)$. Hence, $I(A) = \bigcap_{i \in I} \bar{I}_{A_i B_i}(A) \supseteq A$.

Now suppose that $A \subseteq \bar{I}(A)$ for all A. Then, since $\bar{I}(A) \subseteq \bar{I}_{A_i B_i}(A)$ for each $i \in I$, we have

$$A \subseteq \bar{I}(A) \subseteq \bar{I}_{A_i B_i}(A) = [A \sqsubseteq_\otimes A_i] \Rightarrow B_i.$$

Taking $A = A_i$ we obtain $A_i = A \subseteq B_i$ and the lemma is proved. □

It is obvious that if $A = B$ then I_{AB} is an implicative consequence operator and, of course, it satisfies idempotence. In general, we can have I_{AB} idempotent without having necessarily $A = B$, as the following results will show. But first of all we state some properties of the degrees of inclusionship in order to simplify later proofs.

Lemma 2 *The following conditions hold:*
(i) $[C \sqsubseteq_\otimes D] \otimes [D \sqsubseteq_\otimes E] \leq [C \sqsubseteq_\otimes E]$,
(ii) $[(\bar{k} \Rightarrow C) \sqsubseteq_\otimes D] \geq k \otimes [C \sqsubseteq_\otimes D]$,
(iii) $[(\bar{k} \otimes C) \sqsubseteq_\otimes D] = k \Rightarrow ([C \sqsubseteq_\otimes D])$,
for any $k \in L$.

Proof: (i) $[C \sqsubseteq_\otimes D] \otimes [D \sqsubseteq_\otimes E] = (\inf_p C(p) \Rightarrow D(p)) \otimes (\inf_p D(p) \Rightarrow E(p)) \leq (C(p_0) \Rightarrow D(p_0)) \otimes (D(p_0) \Rightarrow E(p_0)) \leq C(p_0) \Rightarrow E(p_0)$, for each p_0. Therefore, $[C \sqsubseteq_\otimes D] \otimes [D \sqsubseteq_\otimes E] \leq \inf_p C(p) \Rightarrow E(p) = [C \sqsubseteq_\otimes E]$.
(ii) It is enough to prove that $(k \Rightarrow C(p)) \Rightarrow D(p) \geq k \otimes (C(p) \Rightarrow D(p))$ for each p. But this is equivalent to $D(p) \geq k \otimes (C(p) \Rightarrow D(p)) \otimes (k \Rightarrow C(p))$ and this obviously holds.
(iii) It is obvious since $(\bar{k} \otimes C) \Rightarrow D = \bar{k} \Rightarrow (C \Rightarrow D)$. □

Lemma 3 *If I_{AB} satisfies idempotence then $[B \sqsubseteq_\otimes A] \Rightarrow B = B$.*

Proof: Since $I_{AB}(A) = B$ holds true, we necessarily have $B = I_{AB}(A) = I_{AB}(I_{AB}(A)) = I_{AB}(B)$, i.e. $[B \sqsubseteq_\otimes A] \Rightarrow B = B$. □

Theorem 9 *I_{AB} is a closure operator iff $A \subseteq B$ and $[B \sqsubseteq_\otimes A] \Rightarrow B = B$.*

Proof:
$\Rightarrow$) It directly follows from Lemma 3.
$\Leftarrow$) Assume $A \subseteq B$ and $[B \sqsubseteq_\otimes A] \Rightarrow B = B$ and let us check that I_{AB} is a closure operator:

monotony: If $C \subseteq C'$, then $I_{AB}(C) = [C \sqsubseteq_\otimes A] \Rightarrow B \leq [C' \sqsubseteq_\otimes A] \Rightarrow B = I_{AB}(C')$.

inclusion: From $[C \sqsubseteq_\otimes B] \leq C(q) \Rightarrow B(q)$ for each q we have $[C \sqsubseteq_\otimes B] \otimes C(q) \leq B(q)$, and since $A \subseteq B$, we also have $[C \sqsubseteq_\otimes A] \otimes C(q) \leq B(q)$, and hence $C(q) \leq [C \sqsubseteq_\otimes A] \Rightarrow B(q)$, that is, $C \subseteq I_{AB}(C)$.

idempotence: We have $I_{AB}(I_{AB}(C)) = [I_{AB}(C) \sqsubseteq_\otimes A] \Rightarrow B = [[C \sqsubseteq_\otimes A] \Rightarrow B] \sqsubseteq_\otimes A] \Rightarrow B$. Now by (ii) of Lemma 2 we have $I_{AB}(I_{AB}(C)) \leq ([C \sqsubseteq_\otimes A] \otimes [B \sqsubseteq_\otimes A]) \Rightarrow B = [C \sqsubseteq_\otimes A] \Rightarrow ([B \sqsubseteq_\otimes A] \Rightarrow B) = [C \sqsubseteq_\otimes A] \Rightarrow B = I_{AB}(C)$.

Therefore I_{AB} is a closure operator. □

Corollary 2 *Let the BL-algebra L be defined over $[0,1]$ and $A \subseteq B$.*

- *If $\otimes = \min$, then I_{AB} is a closure operator if, and only if, $\inf\{A(q) \mid A(q) < B(q)\} > \sup\{B(p) \mid B(p) \neq 1\}$*[8]*.*
- *If $\otimes$ is isomorphic to the product t-norm then I_{AB} is a closure operator if, and only if, if there exists $q \in \mathcal{L}$ such that $A(q) < B(q)$ then B is a crisp set.*
- *If the t-norm $\otimes$ is isomorphic to the Lukasiewicz t-norm, $\max(x + y - 1, 0)$, then I_{AB} is a closure operator if, and only if, if there exists $q \in \mathcal{L}$ such that $A(q) < B(q)$ then $B(p) = 1$ for all $p \in \mathcal{L}$.*

Observe that in all these cases if there exists $q \in \mathcal{L}$ such that $A(q) < B(q)$ then $B(q) = 1$.

Theorem 10 *I_{AB} is a closure operator iff $I_{AB} = I_{BB}$.*

Proof:
$\Leftarrow$) Trivial
$\Rightarrow$) Assume I_{AB} is a closure operator. Then we have:

[8]Taking by convention, as usual, that $\inf \emptyset = 1$ and $\sup \emptyset = 0$

1. By (i) of Lemma 3, $A \subseteq B$, and thus for each C we have $[C \sqsubseteq_{\otimes} A] \leq [C \sqsubseteq_{\otimes} B]$, hence $I_{AB}(C) = [C \sqsubseteq_{\otimes} A] \Rightarrow B \supseteq [C \sqsubseteq_{\otimes} B] \Rightarrow B = I_{BB}(C)$.
2. By (i) of Lemma 2 and (ii) of Lemma 3, for each C we have $I_{AB}(C) = [C \sqsubseteq_{\otimes} A] \Rightarrow B \subseteq ([C \sqsubseteq_{\otimes} B] \otimes [B \sqsubseteq_{\otimes} A]) \Rightarrow B = [C \sqsubseteq_{\otimes} B] \Rightarrow ([B \sqsubseteq_{\otimes} A] \Rightarrow B) = [C \sqsubseteq_{\otimes} B] \Rightarrow B = I_{BB}(C)$.

Therefore we have seen that for each C, $I_{AB}(C) = I_{BB}(C)$. □

Until now we have shown that a natural inference operator I_{AB} defined on $U = V = \mathcal{L}$ can be a closure operator when $A \subseteq B$ with $A \neq B$, but in any case it must be $I_{AB} = I_{BB}$. Therefore, a natural inference operator which is a closure operator must be implicative as well. We can also prove this result for extensional inference operators.

Theorem 11 *Any extensional inference operator (defined on $U = V = \mathcal{L}$) which is a closure operator is implicative as well.*

Proof: Let $I = \inf_{i \in I} I_{A_i B_i}$ be an extensional inference operator such that it is a closure operator. Then we only need to prove that I fulfills property I_c4, i.e. that $I(\bar{k} \otimes A, p) \geq I(A, p) \otimes k$ hold true for every A, p and k. Actually we will prove that $I_{A_i B_i}(\bar{k} \otimes A, p) \geq k \otimes I_{A_i B_i}(A, p)$ for all $i \in I$. Namely, we have $I_{A_i B_i}(\bar{k} \otimes A, p) = [(\bar{k} \otimes A) \sqsubseteq_{\otimes} A_i] \Rightarrow B_i(p)$, and using (iii) of Lemma 2, this is equal to $(k \Rightarrow [A \sqsubseteq_{\otimes} A_i]) \Rightarrow B_i(p) \geq k \otimes ([A \sqsubseteq_{\otimes} A_i] \Rightarrow B_i(p)) = k \otimes I_{A_i B_i}(A, p)$. The reason for the last inequality is that the inequality $x \otimes (y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow z$ always holds in any BL-algebra. □

4.2 Approximate and proximity similarity-based entailments

Let $\otimes$ and $\Rightarrow$ be a continuous t-norm and its residuum respectively. In [13], given a $\otimes$-similarity relation $S : \Omega \times \Omega \to [0, 1]$ on the set Ω of Boolean interpretations of a propositional language $\mathcal{L}$, a fuzzy set p^* on Ω, is associated to each proposition $p \in \mathcal{L}$ in the following way:

$$p^*(w) = \sup_{w' \models p} S(w, w').$$

p^* can be interpreted as *approximately_p* since it defines the fuzzy set of interpretations which are close to some model of p. From this definition, Dubois et al. define in [11] two graded consequence relations on $\mathcal{L} \times \mathcal{L}$.

Definition 8 (Approximate Consequence Relation) *For each $p, q \in \mathcal{L}$ and $\alpha \in [0, 1]$, we define $p \models^{\alpha} q$ iff $I_S(q \mid p) = \inf_{w \models p} \sup_{w' \models q} S(w, w') \geq \alpha$.*

Definition 9 (Proximity Consequence Relation) *For each $p, q \in \mathcal{L}$, $K \subseteq \mathcal{L}$ and $\alpha \in [0,1]$, we define $p \models^{\alpha}_{K} q$ iff $J_{S,K}(q \mid p) = \inf_{w \models K}\{p^*(w) \Rightarrow q^*(w)\} \geq \alpha$.*

In [11] it is proved that $I_S(q \mid p) = J_{S,\top}(q \mid p)$, where $\top$ stands for a Boolean tautology, i.e. any formula whose set of models is the whole set Ω. In our framework, the approximate and proximity consequence relations can be obtained as implicative closure operators. Indeed, consider the family of fuzzy sets $F = \{\tilde{w}\}_{w\in\Omega}$, where for each $w \in \Omega$, the fuzzy set $\tilde{w} : \mathcal{L} \longmapsto [0,1]$ is defined by $\tilde{w}(q) = q^*(w)$. Now define a mapping $\tilde{C}_c : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$, where here $\mathcal{F}(\mathcal{L}) = [0,1]^{\mathcal{L}}$ by

$$\tilde{C}_c(A)(q) = \inf_{\tilde{w}\in F}[A \sqsubseteq_{\otimes} \tilde{w}] \Rightarrow \tilde{w}(q)),$$

for all $A \in \mathcal{F}(\mathcal{L})$. By construction, this is obviously an implicative closure operator. An easy computation shows that $\tilde{C}_c(\{p\})(q) = J_{S,\top}(q \mid p) = I_S(q \mid p)$, and hence we obtain the above approximate consequence relation. Moreover, if we consider now the family of fuzzy sets to be $F = \{\tilde{w}\}_{w\models K}$, for a subset $K \subseteq \mathcal{L}$, then what we get is $\tilde{C}_c(\{p\})(q) = J_{S,K}(q \mid p)$, that is, the proximity consequence Relation.

4.3 Gerla's Canonical Extensions

In [16] Gerla proposes a method to extend any classical closure operator C defined on $\mathcal{P}(\mathcal{L})$, i.e. on classical sets of formulas, into a fuzzy closure operator $\tilde{C}^*$ defined in $\mathcal{F}(\mathcal{L})$, i.e. on fuzzy sets of formulas. This approach is further delevoped in [18, Chap. 3, Sec. 6]. In the following, we assume $\mathcal{F}(\mathcal{L})$ to be fuzzy sets of formulas valued on a complete linearly-ordered Gödel BL-algebra $\mathbf{L}$, i.e. a linear BL-algebra $(L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ where $\otimes = \wedge$.

Definition 10 (Canonical Extension [16]) *Given a closure operator $C : \mathcal{P}(\mathcal{L}) \longmapsto \mathcal{P}(\mathcal{L})$, the canonical extension of C is the fuzzy operator $\tilde{C}^* : \mathcal{F}(\mathcal{L}) \longmapsto \mathcal{F}(\mathcal{L})$ defined by*

$$\tilde{C}^*(A)(p) = \sup\{\alpha \in L \mid p \in C(A_\alpha)\},$$

where A_α stands for the α-cut of A, i.e. $A_\alpha = \{p \in \mathcal{L} \mid A(p) \geq \alpha\}$.

The canonical extension $\tilde{C}^*$ is a fuzzy closure operator such that $\tilde{C}^*(A)(p) = 1$ if $p \in C(\emptyset)$ and $\tilde{C}^*(A)(p) \geq \sup\{A(q_1) \wedge \ldots \wedge A(q_n) \mid p \in C(\{q_1, \ldots, q_n\})\}$. If C is compact, then the latter inequality becomes an equality. Moreover, it holds that if a fuzzy set A is closed by $\tilde{C}^*$ then any α-cut of A is closed by $\tilde{C}$.

Furthermore, the following characterization of the canonical extension of a classical closure operator is given.

Theorem 12 *(cf. [18, Chap. 3, Th. 7.3]) A fuzzy closure operator $\tilde{C}$ is the canonical extension of a closure operator if, and only if, for every meet-preserving function $f : L \longmapsto L$ such that $f(1) = 1$, if $\tilde{C}(A) = A$ then $\tilde{C}(f \circ A) = f \circ A$*

In other words, this theorem says that if A belongs to the closure system defined by $\tilde{C}$, then so does $f \circ A$. Now, taking into account that, for all $k \in L$, the function $f_{\bar{k}} : L \longmapsto L$ defined by $f_{\bar{k}}(l) = \bar{k} \Rightarrow l$ is a meet preserving function with $f_{\bar{k}}(1) = 1$, the application of Theorem 4 leads us directly to the following result.

Theorem 13 *The canonical extension of any classical closure operator is an implicative closure operator.*

4.4 Biacino-Gerla-Ying's approach

Finally, we consider the connection with Ying's approach [30], further elaborataded in [3, 4]. Ying proposed a propositional calculus in which the reasoning may be approximate by allowing the antecedent of a rule to match a fact only approximately. More precisely, Ying considered a propositional language $\mathcal{L}$ built on a set Var of propositional variables, the constant F (for false) and an implication connective $\rightarrow$. He assumes a $\wedge$-similarity relation S on propositional variables is given. This relation is a mapping $S : Var \times Var \longmapsto L$, where $(L, \wedge, \vee, 0, 1)$ is a complete and infinitely distributive lattice, satisfying $S(p,p) = 1$, $S(p,q) = S(q,p)$ and $S(p,q) \wedge S(q,r) \leq S(p,r)$ for all $p, q, r \in \mathcal{L}$. Such a similarity S induces a similarity relation $\bar{S}$ on formulas defined in the following way:

$\bar{S}(p,q) = S(p,q)$, if p, q are propositional variables;

$\bar{S}(F,q) = \bar{S}(q,F) = 0$, if $q \neq F$;

$\bar{S}(p,q) = \bar{S}(x,x') \wedge \bar{S}(y,y')$, if $p = (x \rightarrow x')$ and $q = (y \rightarrow y')$;

$\bar{S}(F,F) = 1$;

and $\bar{S}(p,q) = 0$, otherwise.

Based on the similarity $\bar{S}$, Ying then considers a graded consequence relation $g_Y : \mathcal{P}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ by defining $g_Y(A, p) = \sup\{\hat{S}(A \cup \mathcal{AX}, B) \mid B \vdash p\}$, where $\hat{S}(X, Y) = \inf_{y \in Y} \sup_{x \in X} \bar{S}(x, y)$, $\mathcal{AX}$ is a set of logical axioms and $\vdash$ stands for deduction in classical propositional logic.

This graded consequence relation is extended by Biacino and Gerla in [3] to apply over fuzzy sets of formulas in the following way.

Definition 11 *([3]) Let $S : \mathcal{L} \times \mathcal{L} \longmapsto L$ be a $\wedge$-similarity relation on formulas. Then we define a fuzzy relation $\hat{S} : \mathcal{F}(\mathcal{L}) \times \mathcal{P}(\mathcal{L}) \longmapsto L$ between fuzzy sets and classical sets of formulas induced by S as follows:*

$$\hat{S}(A, B) = \inf_{q \in B} \sup_{p \in \mathcal{L}} \{S(q, p) \wedge A(p)\}.$$

$\hat{S}(A, B)$ gives the least degree at which each formula of B is equivalent to the fuzzy set of formulas A, and one can check that $\hat{S}(A, B) = \inf_{q \in B} \tilde{C}_S(A)(q)$, where $\tilde{C}_S$ is nothing but the upper closure operator defined by the similarity relation S, as defined in Definition 5. By using $\hat{S}$, a new fuzzy operator may be obtained.

Definition 12 *([3]) Let S be a $\wedge$-similarity on $\mathcal{L}$ as above and let C be a compact classical closure operator on $\mathcal{P}(\mathcal{L})$, respectively. Then we define the fuzzy operator $\tilde{C}_Y : \mathcal{F}(\mathcal{L}) \longmapsto \mathcal{F}(\mathcal{L})$ associated with S as*

$$\tilde{C}_Y(A)(p) = \sup\{\hat{S}(A \cup Taut, B) \mid B \in \mathcal{P}(\mathcal{L}), p \in C(B)\}$$

for each fuzzy set A and each formula p, where $Taut = C(\emptyset)$.

In [3], it is proved[9] that the new operator $\tilde{C}_Y$ is the composition of the canonical extension $\tilde{C}^*$ of the classical closure operator C and a slight modification of the upper closure operator $\tilde{C}_S$, namely $\tilde{C}_Y = \tilde{C}^* \circ \tilde{C}'_S$, where $\tilde{C}'_S(A) = \tilde{C}_S(A \cup Taut) = \tilde{C}_S(A) \cup \tilde{C}_S(Taut)$.

We extend now the complete and infinitely distributive lattice $\mathbf{L}$ to a complete Gödel BL-algebra by taking $\otimes = \wedge$ and defining its residdum as usual, i.e. $x \Rightarrow y = \sup\{z \in L \mid x \wedge z \leq y\}$. Then, we know from the last section that $\tilde{C}^*$ is an implicative closure operator, and from Section 3 so it is $\tilde{C}_S$. Moreover, one can easily check that $\tilde{C}'_S$ is also an implicative closure operator[10]. So we have that $\tilde{C}_Y$ is a composition of two implicative closure

[9] Assuming $L = [0, 1]$.

[10] Inclusion, monotony and idempotence are easy and $\tilde{C}4$, with $\otimes = \wedge$, is a consequence of property $C2$ of Proposition 5, which is verified by $\tilde{C}_S$.

operators, but this does not mean that $\tilde{C}_Y$ is necessarily an implicative closure operator, neither a closure operator. In general, this operator does not satisfy the fuzzy idempotence property, although it does satisfy property $\tilde{C}4$ (with $\otimes = \wedge$), since the composition of two implicative closure operators always satisfies property $\tilde{C}4$. Nevertheless, as was proved in [4], when C is the closure operator of classical propositional logic, such composition is a closure operator and then it is an implicative closure operator as well.

5 Conclusions and Open Problems

In the setting of a logical approach to approximate reasoning, we have introduced in this paper the class of implicative closure operators. They have been shown to provide with an unified view of many previous generalizations to the many-valued framework of closures operators assciated to classical deduction systems. These new operators deal with fuzzy sets of formulas whose membership functions take values on a BL-algebra. BL-algebras constitute the algebraic counterpart of Hájek's BL logic, the logic of continuous t-norms and their residua.

Although implicative closure operators are very general, and defined in the framework of BL-algebras, strangely enough they do not capture graded deduction (Pavelka-style) in any of the extensions of BL, except for Gödel's logic. Therefore, to come up with a suitable notion of fuzzy closure operator capturing graded deduction in BL axiomatic extensions requires further research.

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