Hilbert-style calculi for $\vdash_{\text{BPL}}$ and $\vdash_{\text{FPL}}$

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Abstract

A. Visser, in [Vis81], introduces the consequence relations $\vdash_{\text{BPL}}$ and $\vdash_{\text{FPL}}$. The latter is obtained by interpreting implication as a formal provability. He presents natural deduction calculi for both. Since then, some Gentzen-style calculi have also been given, but no Hilbert-style calculi. We present Hilbert-style calculi for both consequence relations.

Let us take the intuitionistic language $L_{\text{int}} = \langle \wedge, \vee, \rightarrow, \bot \rangle$ and the modal language $ML = \langle \wedge, \vee, \bot, \top, \square, \Box \rangle^a$. In the language $L_{\text{int}}$ we define the following three connectives: i) $\top = \bot \rightarrow \bot$, ii) $\neg \varphi = \varphi \rightarrow \bot$, iii) $\Box \varphi = \top \rightarrow \varphi$\(^b\). We define the map $\tau : FmL_{\text{int}} \rightarrow FmML$ as

\[
i) \tau(p_i) = \square p_i, \quad \text{ii) } \tau(\bot) = \Box \bot, \quad \text{iii) } \tau(\varphi \land \psi) = \tau(\varphi) \land \tau(\psi), \\
v) \tau(\varphi \rightarrow \psi) = \Box (\tau(\varphi) \supset \tau(\psi)).
\]

It is well known that $\tau$ is an embedding of

- the intuitionistic propositional logic $\vdash_{\text{IPL}}$ into $\vdash_{\text{S4}}$, i.e., $\Gamma \vdash_{\text{IPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\text{S4}} \tau(\varphi)$\(^d\),
- the intuitionistic propositional logic $\vdash_{\text{IPL}}$ into $\vdash_{\text{Grz}}$, i.e., $\Gamma \vdash_{\text{IPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\text{Grz}} \tau(\varphi)$,
- the classical propositional logic $\vdash_{\text{CPL}}$ into $\vdash_{\text{S5}}$, i.e., $\Gamma \vdash_{\text{CPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\text{S5}} \tau(\varphi)$.

The consequence relation $\vdash_{\text{FPL}}$ (formal propositional logic) in $L_{\text{int}}$ was first considered by A. Visser in [Vis81]. He introduced it as the consequence relation embeddable into the provability logic $\vdash_{\text{GL}}$ through the map $\tau$. That is, $\Gamma \vdash_{\text{FPL}} \varphi$ iff $\tau[\Gamma] \vdash_{\text{GL}} \tau(\varphi)$. Therefore, the consequence relation $\vdash_{\text{FPL}}$ can be interpreted inside the Peano Arithmetic.

In [Vis81] A. Visser also introduced the consequence relation $\vdash_{\text{BPL}}$ (basic propositional logic) in $L_{\text{int}}$. It is defined as the consequence relation embeddable into $\vdash_{\text{K4}}$ (sometimes

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\(^b\)In this paper the symbol $\rightarrow$ will refer to the intuitionistic implication (strict implication) and $\supset$ to the material implication.

\(^d\)The reason to adopt this abbreviation is given by the Kripke semantic for intuitionistic propositional logic; see [CZ97]. Under this semantic the validity of $\top \rightarrow \varphi$ in a world of a Kripke model is equivalent to the validity of $\varphi$ in all the worlds that are accessible from this one.

\(^d\)If $L$ is a normal modal logic, by $\vdash_L$ we understand the consequence relation that has the elements in $L$ as axioms, and modus ponens as its only proper rule (that is, we do not consider the necessitation rule). Therefore, the consequence relation $\vdash_L$ has the deduction-detachment theorem, i.e., $\Gamma, \varphi \vdash_L \psi$ iff $\Gamma \vdash_L \varphi \supset \psi$. All the modal consequence relations that are discussed in the present paper can be found elsewhere, for instance [CZ97]. For $\vdash_{\text{GL}}$ two excellent references are [Boo93, Smo85].
called basic modal logic, as in [Smo85], i.e., $\Gamma \vdash_{\text{BPL}} \varphi$ iff $\Gamma \vdash_{\text{K4}} \tau(\varphi)$. It is clear that $\vdash_{\text{FPL}}$ and $\vdash_{\text{BPL}}$ are extensions of $\vdash_{\text{BPL}}$. Although $\vdash_{\text{BPL}}$ was devised for technical reasons, over the last decade this consequence relation has acquired an interest of its own. It appears if we replace the Brouwer-Heyting-Kolmogorov interpretation of implication with the weaker interpretation

- a proof of $\varphi \rightarrow \psi$ is a construction that uses the assumption $\varphi$ to produce a proof of $\psi$.

Further explanations can be found in [Rui91, Rui93].

[Vis81] gives natural deduction calculi for $\vdash_{\text{FPL}}$ and $\vdash_{\text{BPL}}$; in [AR98] M. ARDESHIR and W. RUITENBURG present Gentzen-style calculi for both; and a different Gentzen-style calculus for $\vdash_{\text{BPL}}$ is presented by K. SASAKI in [Sas99]°. However, to our knowledge, no Hilbert-style calculus has been produced for either consequence (see [SWZ98, page 324]).

The difficulty lies in the fact that modus ponens does not hold in $\vdash_{\text{FPL}}$ (and therefore does not hold in $\vdash_{\text{BPL}}$). In fact, $\top, \top \rightarrow p_0 \not\vdash_{\text{FPL}} p_0$. In this paper we solve this problem. We define $\mathcal{H}_{\text{BPC}}$ as the following Hilbert-style calculus over $\mathcal{L}_{\text{int}}$:

\[
\begin{align*}
(Ru1) & \quad \frac{p_0 \land p_1}{p_0} , \quad (Ru2) \quad \frac{p_0 \land p_1}{p_1 \land p_0} , \quad (Ru3) \quad \frac{p_0 \land p_1}{p_0} , \quad (Ru4) \quad \frac{p_0}{p_0} , \\
(Ru5) & \quad \frac{p_0 \lor p_1}{p_1 \lor p_0} , \quad (Ru6) \quad \frac{p_0 \lor (p_1 \lor p_2)}{(p_0 \lor p_1) \lor p_2} , \quad (Ru8) \quad \frac{(p_0 \lor p_1) \land (p_0 \lor p_2)}{p_0 \lor (p_1 \land p_2)} , \quad (Ru9) \quad \frac{p_0 \lor p_0}{p_0} , \\
(T^*) & \quad \frac{(p_0 \rightarrow p_1) \land (p_1 \rightarrow p_2)}{(p_0 \rightarrow p_2) \lor p_3} , \quad (8^*) \quad \frac{(p_0 \rightarrow p_1) \land (p_0 \rightarrow p_2)}{(p_0 \rightarrow (p_1 \land p_2)) \lor p_3} , \\
(9^*) & \quad \frac{(p_0 \rightarrow p_2) \land (p_1 \rightarrow p_2)}{(p_0 \lor p_1) \rightarrow p_2) \lor p_3} , \quad (1^*) \quad \frac{p_0 \lor p_1}{p_0 \lor p_1} , \quad (N^*) \quad \frac{p_0 \lor p_1}{\Box p_0 \lor p_1} , \\
(Ax 1) & \quad p_0 \rightarrow p_0 , \\
(Ax 2) & \quad p_0 \rightarrow \top , \\
(Ax \, \bar{R}u1) & \quad (p_0 \land p_1) \rightarrow p_0 , \\
(Ax \, \bar{R}u2) & \quad (p_0 \lor p_1) \rightarrow (p_1 \land p_0) , \\
(Ax \, \bar{R}u4) & \quad p_0 \rightarrow (p_0 \lor p_1) , \\
(Ax \, \bar{R}u5) & \quad (p_0 \lor p_1) \rightarrow (p_1 \lor p_0) , \\
(Ax \, \bar{R}u8) & \quad ((p_0 \lor p_1) \land (p_0 \lor p_2)) \rightarrow (p_0 \lor (p_1 \land p_2)) , \\
(Ax \, \bar{7}) & \quad ((p_0 \rightarrow p_1) \land (p_1 \rightarrow p_2)) \rightarrow (p_0 \rightarrow p_2) , \\
(Ax \, \bar{8}) & \quad ((p_0 \rightarrow p_1) \land (p_0 \rightarrow p_2)) \rightarrow (p_0 \rightarrow (p_1 \land p_2)) , \\
(Ax \, \bar{9}) & \quad ((p_0 \rightarrow p_2) \land (p_1 \rightarrow p_2)) \rightarrow ((p_0 \lor p_1) \rightarrow p_2) , \\
(Ax \, \bar{I}) & \quad \bot \rightarrow p_0 , \\
(Ax \, \bar{N}) & \quad p_0 \rightarrow \Box p_0 .
\end{align*}
\]

The main result, Theorem 14, says i) $\mathcal{H}_{\text{BPC}}$ is a Hilbert-style calculus for $\vdash_{\text{BPL}}$, ii) $\mathcal{H}_{\text{BPC}} \cup \{(Ax \, \bar{L})\}$ is a Hilbert-style calculus for $\vdash_{\text{FPL}}$ where

\[
(Ax \, \bar{L}) \quad (\Box p_0 \rightarrow p_0) \rightarrow \Box p_0 .
\]

The first section introduces the notions and results that are already known and are necessary for the proof; and in the second section the proof is given.

°There the consequence relation $\vdash_{\text{BPL}}$ is called $\vdash_{\text{VPL}}$ (Visser’s propositional logic) in honor of Visser.
where $\Gamma \cup \{\varphi\}$ we understand by a Hilbert-style calculus. A Hilbert-style rule over $\langle H \rangle$ there exists a procedure to determine the elements in Hilbert-style calculus $H$. A Hilbert-style proper rules $\rho$ in both cases. That is, if $\varphi \in BPL$ and $\varphi \rightarrow \psi \in BPL$ then $\psi \in BPL$ (and the same for $FPL$). Thanks to this fact a Hilbert-style calculus with modus ponens and the same theorems as $\vdash_{BPL}$ has been presented; see [SO93, Sas01]. It is easy to see that if we add the axiom (Ax $\bar{L}$) we obtain a Hilbert-style calculus with modus ponens and the same theorems as $\vdash_{FPL}$.

Kripke semantics have been given for both $\vdash_{BPL}$ and $\vdash_{FPL}$. As we do not need them to prove Theorem 14 we will not introduce them here; the reader can find details in [Vis81, AR98, SWZ98]. From these semantics it is straightforward that the $\langle \land, \lor \rangle$-restriction of $\vdash_{BPL}$ coincides with the $\langle \land, \lor \rangle$-restriction of $\vdash_{CPL}$, and also with the $\langle \land, \lor \rangle$-restriction of $\vdash_{FPL}$.

1 Preliminaries

For the rest of the paper let us fix an infinite set $Var = \{p_n : n \in \omega\}$, whose elements are called variables, such that if $n \neq m$ then $p_n \neq p_m$. Given a propositional language $\mathcal{L}$ the set of $\mathcal{L}$-formulas is defined as usual and is denoted by $Fm\mathcal{L}$. Its elements will be denoted by $\varphi, \psi, \delta, \alpha, \ldots$; and the subsets of $\mathcal{L}$-formulas by $\Gamma, \Delta, \Sigma, \ldots$. For the rest of the paper we assume that an enumeration $\langle \rho_n : n \in \omega \rangle$ of the set $Fm\mathcal{L}$ is fixed. If $\land$ is a connective of our propositional language we will write $\varphi_1 \land \ldots \land \varphi_n$ as an abbreviation of the formula $((\varphi_1 \land \varphi_2) \land \varphi_3) \land \ldots \land \varphi_n$; and for every non-empty finite set $\Gamma$ of $\mathcal{L}$-formulas $\land \Gamma$ will be an abbreviation of the formula $\rho_{n_0} \land \ldots \land \rho_{n_m}$, where $\Gamma = \langle \rho_{n_0}, \ldots, \rho_{n_m} \rangle$, $n_0 < \ldots < n_m$.

We aim to give a Hilbert-style calculus for $\vdash_{BPL}$ and $\vdash_{FPL}$. First of all we need to define what we understand by a Hilbert-style calculus. A Hilbert-style rule over $\mathcal{L}$ is a pair $(\Gamma, \varphi)$ where $\Gamma \cup \{\varphi\}$ is a finite subset of $Fm\mathcal{L}$. Hilbert-style rules that have the form $\langle \emptyset, \varphi \rangle$ are called Hilbert-style axioms, and the ones that do not are called Hilbert-style proper rules. A Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}$ is a set of Hilbert-style rules over $\mathcal{L}$ that is recursive (i.e., there exists a procedure to determine the elements in $\mathcal{H}$). It can be infinite. A substitution instance of a Hilbert-style rule $\langle \Gamma, \varphi \rangle$ over $\mathcal{L}$ is a pair of the form $\langle s[\Gamma], s(\varphi) \rangle$ where $s$ is a $\mathcal{L}$-substitution. Given a Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}$ we can define the notion of a $\mathcal{H}$-proof of $\varphi$ from the assumptions $\Gamma$ as a finite sequence $\langle \varphi_0, \ldots, \varphi_n \rangle$, $n \in \omega$ such that $\varphi_n = \varphi$ and for each $i \in \{0, \ldots, n\}$ at least one of the following two conditions holds:

1. $\varphi_i \in \Gamma$,
2. there exists $\Delta \subseteq \{\varphi_j : j < i\}$ such that $\langle \Delta, \varphi_i \rangle$ is a substitution instance of a Hilbert-style rule in $\mathcal{H}$.

Given a Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}$ it is possible to define a consequence relation between the $\mathcal{L}$-formulas. Given $\Gamma \subseteq Fm\mathcal{L}$ and $\varphi \in Fm\mathcal{L}$, we define

$$\Gamma \vdash_{\mathcal{H}} \varphi \quad \text{iff} \quad \text{there exists a } \mathcal{H}\text{-proof of } \varphi \text{ from } \Gamma.$$

Therefore, when we say that we add a rule $\langle \Gamma, \varphi \rangle$ to a Hilbert-style calculus implicitly we are saying that we add all its substitution instances, that is, we treat the rules as schemata. When we say that a Hilbert-style calculus $\mathcal{H}$ over $\mathcal{L}$ is a calculus for a consequence relation $\vdash$ we mean that $\vdash = \vdash_{\mathcal{H}}$.

For many consequence relations finite Hilbert-style calculi are known. For instance, finite Hilbert-style calculi for $\vdash_{CPL}$ and $\vdash_{IPL}$ can be found in [CZ97]. These calculi use
modus ponens as their unique Hilbert-style proper rule. In most cases in which a Hilbert-style calculus is known for a consequence relation, modus ponens holds. The following example is one of the few exceptions.

**Example 1** Let Γ ⊢_{\text{CPL}} be the \(\langle\land, \lor\rangle\)-restriction of the classical propositional logic. And let the Hilbert-style calculus \(\mathcal{H}_{\text{CPL}}\) over \(\mathcal{L}_{\land\lor} = \langle\land, \lor\rangle\) be the one given by the first nine proper rules of \(\mathcal{H}_{\text{BPL}}\) on page 21, i.e., \(\mathcal{H}_{\text{CPL}} = \{(\text{Ru}1), (\text{Ru}2), \ldots, (\text{Ru}9)\}\) (there is no axiom in \(\mathcal{H}_{\text{CPL}}\)). It is known that \(\mathcal{H}_{\text{CPL}}\) is a Hilbert-style calculus for \(\vdash_{\text{CPL}}\) (see [DP80]; the remaining rules of [DP80] are proved to be derivable in [FGV91]).

**Lemma 2 (invariance under substitutions)** Let \(\mathcal{H}\) be a Hilbert-style calculus over a language \(\mathcal{L}\) such that \(\Gamma \vdash_{\mathcal{H}} \phi\). If \(s\) is a \(\mathcal{L}\)-substitution then \(s[\Gamma] \vdash_{\mathcal{H}} s(\phi)\).

**Proof:** Straightforward. 

To prove our main theorem we will use the Gentzen-style calculus given by M. ARDESHIR and W. RUITENBURG in [AR98] for \(\vdash_{\text{BPL}}\) and \(\vdash_{\text{FPL}}\). Thus, we need to introduce the notions of sequent and Gentzen-style calculus. We will give the definition only for the case of the intuitionistic language, the only case that we need.

A \(\mathcal{L}_{\text{int}}\)-sequent is a pair \(\langle\phi, \psi\rangle\) where \(\phi, \psi \in \text{Fm}_{\mathcal{L}_{\text{int}}}\). We will write them as \(\phi \Rightarrow \psi\), and the set of all \(\mathcal{L}_{\text{int}}\)-sequents will be denoted by \(\text{Seq}_{\mathcal{L}_{\text{int}}}\). The subsets of \(\text{Seq}_{\mathcal{L}_{\text{int}}}\) will be denoted by \(\Pi, \Upsilon \ldots\)

A Gentzen-style rule over \(\mathcal{L}_{\text{int}}\) is a pair \(\langle\Pi, \phi \Rightarrow \psi\rangle\) where \(\Pi \cup \{\phi \Rightarrow \psi\}\) is a finite subset of \(\text{Seq}_{\mathcal{L}_{\text{int}}}\). Gentzen-style rules that have the form \(\langle\emptyset, \phi \Rightarrow \psi\rangle\) are called Gentzen-style axioms, and the ones that do not are called Gentzen-style proper rules. A Gentzen-style calculus \(\mathcal{G}\) over \(\mathcal{L}_{\text{int}}\) is a recursive set of Gentzen-style rules over \(\mathcal{L}_{\text{int}}\). A substitution instance of a Gentzen-style rule \(\langle\Pi, \phi \Rightarrow \psi\rangle\) over \(\mathcal{L}_{\text{int}}\) is a pair of the form \(\langle s[\Pi], s(\phi) \Rightarrow s(\psi)\rangle\) where \(s\) is a \(\mathcal{L}_{\text{int}}\)-substitution. Given a Gentzen-style calculus \(\mathcal{G}\) over \(\mathcal{L}_{\text{int}}\) we can define the notion of a \(\mathcal{G}\)-proof of \(\phi \Rightarrow \psi\) as a finite sequence \(\langle\phi_0 \Rightarrow \psi_0, \ldots, \phi_n \Rightarrow \psi_n\rangle\), \(n \in \omega\) such that \(\phi_n = \phi\) and \(\psi_n = \psi\) and for each \(i \in \{0, \ldots, n\}\) there exists \(\Upsilon \subseteq \{\phi_j \Rightarrow \psi_j : j < i\}\) such that \(\langle \Upsilon, \phi_i \Rightarrow \psi_i\rangle\) is a substitution instance of a Gentzen-style rule in \(\mathcal{G}\). Given a Gentzen-style calculus \(\mathcal{G}\) over \(\mathcal{L}_{\text{int}}\) it is possible to define a consequence relation between the \(\mathcal{L}_{\text{int}}\)-formulas. Given \(\Gamma \subseteq \text{Fm}_{\mathcal{L}_{\text{int}}}\) and \(\phi \in \text{Fm}_{\mathcal{L}_{\text{int}}}\), we define

\[\Gamma \vdash_{\mathcal{G}} \phi \iff \begin{cases} \text{exists } \gamma_0, \ldots, \gamma_n \in \Gamma \cup \{\top\}, n \in \omega \text{ such that} \\ \text{there is a } \mathcal{G}\text{-proof of } \gamma_0 \land \ldots \land \gamma_n \Rightarrow \phi. \end{cases}\]

Therefore, when we say that we add a rule \(\langle\Pi, \phi \Rightarrow \psi\rangle\) to a Gentzen-style calculus implicitly we are saying that we add all its substitution instances, that is, we treat the rules as schemata. When we say that a Gentzen-style calculus \(\mathcal{G}\) over \(\mathcal{L}_{\text{int}}\) is a calculus for a consequence relation \(\vdash\) we mean that \(\vdash = \vdash_{\mathcal{G}}\).

**Example 3** Consider the Gentzen-style calculus over \(\mathcal{L}_{\text{int}}\) given by

\[\vdash_{\text{BPL}} \text{and } \vdash_{\text{FPL}}\] are finite. This does not hold if we adopt the other definition.
to the above calculus we obtain a calculus for \( \vdash_{BPL} \) ([AR98, Theorem 3.5]). It is also known that if we add the Gentzen-style axiom
\[
(\diamond) \quad \top \Rightarrow (\square p_0 \rightarrow p_0) \rightarrow \square p_0
\]
to the above calculus we obtain a calculus for \( \vdash_{FPL} \) ([AR98, Lemma 2.10]).

Example 4 Consider the Gentzen-style calculus over \( L_{int} \) given by
\[
\begin{align*}
\top & \Rightarrow p_0 \rightarrow p_0, \quad \top \Rightarrow p_0 \rightarrow (p_1 \rightarrow p_0), \quad \top \Rightarrow p_0 \rightarrow (p_1 \rightarrow (p_0 \land p_1)), \\
\top & \Rightarrow \bot \Rightarrow p_0, \quad \top \Rightarrow (p_0 \land p_1) \rightarrow p_0, \quad \top \Rightarrow (p_0 \land p_1) \rightarrow p_1, \\
\top & \Rightarrow p_0 \rightarrow (p_0 \lor p_1), \quad \top \Rightarrow p_1 \rightarrow (p_0 \lor p_1), \\
\top & \Rightarrow ((p_1 \rightarrow p_2) \land (p_0 \rightarrow p_1)) \rightarrow (p_0 \rightarrow p_2), \\
\top & \Rightarrow ((p_0 \rightarrow p_1) \land (p_0 \rightarrow p_2)) \rightarrow (p_0 \rightarrow (p_1 \land p_2)), \\
\top & \Rightarrow ((p_0 \rightarrow p_2) \land (p_1 \rightarrow p_2)) \rightarrow ((p_0 \lor p_1) \rightarrow p_2), \\
\top & \Rightarrow (p_0 \land (p_1 \lor p_2)) \rightarrow ((p_0 \land p_1) \lor (p_0 \land p_2)), \\
(Cut) \quad \frac{p_0 \Rightarrow p_1}{p_0 \Rightarrow p_2}, & \quad \frac{p_0 \Rightarrow p_1}{p_0 \Rightarrow p_2} \\
(\Rightarrow \land) \quad \frac{p_0 \Rightarrow p_1}{p_0 \Rightarrow p_1 \land p_2}, & \quad \frac{p_0 \Rightarrow p_1}{p_0 \Rightarrow p_1 \land p_2} \\
(mp) \quad \top \Rightarrow \frac{p_0 \Rightarrow p_1}{p_0 \Rightarrow p_1}. & \quad \frac{p_0 \Rightarrow p_1}{p_0 \Rightarrow p_1}. 
\end{align*}
\]
It is known that this Gentzen-style calculus is a calculus for \( \vdash_{BPL} \) ([Sas99, Theorem 2.2]). Although K. Sasaki says in [Sas01] that this is a Hilbert-style formalization for \( \vdash_{BPL} \) it is not a Hilbert-style calculus (the rule (mp) is a Gentzen-style rule that cannot easily be written as a Hilbert-style rule).

2 Proof

For our proof we will use the Gentzen-style calculi given in [AR98] with some simplifications. Using this fact, necessary and sufficient conditions for being a Hilbert-style calculus for \( \vdash_{BPL} \) and \( \vdash_{FPL} \) will be straightforwardly obtained in Proposition 6.

The Gentzen-style calculus \( G_{BPC} \) is defined as the calculus over \( L_{int} \) in Example 3 replacing the Gentzen-style rule \( (DT) \) with the following three Gentzen-style rules
\[
\begin{align*}
(10) \quad \frac{\emptyset}{p_0 \Rightarrow \top}, & \quad \frac{\emptyset}{p_0 \Rightarrow \square p_0}, & \quad (DT_0) \quad \frac{p_0 \Rightarrow p_1}{\top \Rightarrow p_0 \rightarrow p_1}.
\end{align*}
\]
It is straightforward to see that this replacement does not change the provable sequents.
Proposition 5

1. $G_{BPC}$ is a Gentzen-style calculus for $\vdash_{BPL}$.

2. $G_{BPC} \cup \{(L)\}$ is a Gentzen-style calculus for $\vdash_{FPL}$.

Proof: By Example 3 we only need to prove that these three rules are interderivable with the rule $(DT)$. It is really simple to observe that they can be obtained using $(DT)$. For the other direction here is given a sketch.

$$
\frac{}{\bot \vdash p_0 \rightarrow \top} \quad \frac{}{\top \vdash \bot \rightarrow p_0} \quad \frac{}{\top \vdash p_0 \rightarrow p_0} \quad \frac{}{\bot \vdash p_0 \rightarrow (p_0 \land p_1)} \quad \frac{}{p_0 \land p_1 \vdash p_2} \quad \frac{}{p_0 \vdash \bot \rightarrow (p_0 \land p_1 \rightarrow p_2) \rightarrow (p_0 \land p_1 \rightarrow p_2)}
$$

Let $\langle \phi_1 \Rightarrow \psi_1, \ldots, \phi_n \Rightarrow \psi_n \rangle$, $\varphi \Rightarrow \psi$ be a Gentzen-style rule over $L_{\text{int}}$. We will say that a Hilbert-style calculus $\mathcal{H}$ over $L_{\text{int}}$ is a model of this Gentzen-style rule iff for every $L_{\text{int}}$-substitution $s$ holds

$$
\text{if for every } i \in \{1, \ldots, n\} s(\phi_i) \vdash_{\mathcal{H}} s(\psi_i), \text{ then } s(\varphi) \vdash_{\mathcal{H}} s(\psi).
$$

For instance, the fact that $\mathcal{H}$ is a model of the Gentzen-style rule $(DT_0)$ means that for every $\varphi, \psi \in FmL_{\text{int}}$ it is the case that

$$
\text{if } \varphi \vdash_{\mathcal{H}} \psi \text{ then } \top \vdash_{\mathcal{H}} \varphi \rightarrow \psi.
$$

And the fact that $\mathcal{H}$ is a model of the Gentzen-style rule $(\lor \Rightarrow)$ means that for every $\varphi, \psi, \delta \in FmL_{\text{int}}$ holds

$$
\text{if } \varphi \vdash_{\mathcal{H}} \delta \text{ and } \psi \vdash_{\mathcal{H}} \delta \text{ then } \varphi \lor \psi \vdash_{\mathcal{H}} \delta.
$$

Given a Hilbert-style calculus $\mathcal{H}$, some properties are true and others false. We are interested in the following five properties.

$(P1)$ If $\langle \Gamma, \varphi \rangle \in \mathcal{H}$ then $\Gamma \vdash_{BPL} \varphi$.

$(P2)$ $\bot \vdash_{\mathcal{H}} p_0$, $\bot \land p_1 \vdash_{\mathcal{H}} \{p_0, p_1\}$, $p_0 \vdash_{\mathcal{H}} p_0 \lor p_1$, $p_1 \vdash_{\mathcal{H}} p_0 \lor p_1$, $p_0 \land (p_1 \lor p_2) \vdash_{\mathcal{H}} (p_0 \land p_1) \lor (p_0 \land p_2)$, $(p_0 \rightarrow p_1) \land (p_1 \rightarrow p_2) \vdash_{\mathcal{H}} p_0 \rightarrow p_2$, $(p_0 \rightarrow p_1) \land (p_0 \rightarrow p_2) \vdash_{\mathcal{H}} p_0 \rightarrow (p_1 \land p_2)$, $(p_0 \rightarrow p_2) \land (p_1 \rightarrow p_2) \vdash_{\mathcal{H}} (p_0 \lor p_1) \rightarrow p_2$, $p_0 \vdash_{\mathcal{H}} \top$, $p_0 \vdash_{\mathcal{H}} \bot$.

$(P3)$ $\mathcal{H}$ is a model of the Gentzen-style rules $(\lor \Rightarrow)$ and $(DT_0)$.

$(P4)$ $\emptyset \vdash_{\mathcal{H}} (\Box p_0 \rightarrow p_0) \rightarrow \Box p_0$.

$(P5)$ If $\langle \Gamma, \varphi \rangle \in \mathcal{H}$ then $\Gamma \vdash_{FPL} \varphi$.

A simple consequence of Proposition 5 (see [FJ96, Proposition 4.4(3)]) is the following.

**Proposition 6** Let $\mathcal{H}$ be a Hilbert-style calculus over $L_{\text{int}}$. Then,

1. $(P1)$, $(P2)$ and $(P3)$ holds in $\mathcal{H}$ if $\vdash_{\mathcal{H}} = \vdash_{BPL}$.

*This property $(P2)$ means that $\mathcal{H}$ is a model of all the rules in $G_{BPC} \setminus \{(\lor \Rightarrow), (DT_0)\}$. 

25
2. \((P2), (P3), (P4)\) and \((P5)\) holds in \(\mathcal{H}\) \iff \(\vdash_{\mathcal{H}} = \vdash_{\text{FPL}}\).

For each of these properties except \((P3)\) it is clear how to obtain a Hilbert-style calculus where the property holds. In the following two lemmas we will consider conditions about a Hilbert-style calculus \(\mathcal{H}\) that allow us to conclude \((P3)\). It is clear that the hypotheses of both lemmas are true if we replace \(\vdash_{\mathcal{H}}\) with \(\vdash_{\text{BPL}}\).

**Lemma 7** Let \(\mathcal{H}\) be a Hilbert-style calculus over \(\mathcal{L}_{\text{int}}\) such that
\[
\begin{align*}
(a) & \emptyset \vdash_{\mathcal{H}} p_0 \rightarrow p_0, & (b) & (p_0 \rightarrow p_1) \land (p_0 \rightarrow p_2) \vdash_{\mathcal{H}} p_0 \rightarrow (p_1 \land p_2), \\
(c) & \emptyset \vdash_{\mathcal{H}} p_0 \rightarrow \top, & (d) & (p_0 \rightarrow p_1) \land (p_1 \rightarrow p_2) \vdash_{\mathcal{H}} p_0 \rightarrow p_2, \\
(e) & p_0, p_1 \not\vdash_{\mathcal{H}} p_0 \land p_1, & (f) & p_0 \vdash_{\mathcal{H}} \Box p_0.
\end{align*}
\]

The following conditions are equivalent.

1. \(\mathcal{H}\) is a model of the Gentzen-style rule \((DT_0)\).
2. For every Hilbert-style proper rule \(\langle \Gamma, \varphi \rangle \in \mathcal{H}\) holds \(\emptyset \vdash_{\mathcal{H}} (\land \Gamma) \rightarrow \varphi\).

**Proof:** Straightforward.

**Lemma 8** Let \(\mathcal{H}\) be a Hilbert-style calculus over \(\mathcal{L}_{\text{int}}\) such that
\[
\begin{align*}
(a) & p_0 \vdash_{\mathcal{H}} p_0 \lor p_1, & (b) & p_0 \lor p_1 \vdash_{\mathcal{H}} p_1 \lor p_0, \\
(c) & p_0 \lor p_0 \vdash_{\mathcal{H}} p_0, & (d) & p_0 \lor p_2, p_1 \lor p_2 \vdash_{\mathcal{H}} (p_0 \land p_1) \lor p_2.
\end{align*}
\]

The following conditions are equivalent.

1. \(\mathcal{H}\) is a model of the Gentzen-style rule \((\lor \Rightarrow)\).
2. For every Hilbert-style proper rule \(\langle \{\gamma_1, \ldots, \gamma_n\}, \varphi \rangle \in \mathcal{H}\), \(n \geq 1\) holds \(\{\gamma_1 \lor p_k, \ldots, \gamma_n \lor p_k\} \vdash_{\mathcal{H}} \varphi \lor p_k\) where \(p_k\) is the first variable that doesn't appear in \(\{\gamma_1, \ldots, \gamma_n, \varphi\}\).

**Proof:** Straightforward.

From Lemmas 7 and 8 and Proposition 6 this corollary follows:

**Corollary 9** If \(\mathcal{H}\) is Hilbert-style calculus for \(\vdash_{\text{BPL}}\) then \(\mathcal{H} \cup \{\text{Ax } L\}\) is a Hilbert-style calculus for \(\vdash_{\text{FPL}}\).

Therefore, it only remains to obtain a Hilbert-style calculus for \(\vdash_{\text{BPL}}\), i.e., a Hilbert-style calculus satisfying \((P1), (P2)\) and \((P3)\). If we allow the use of an infinite Hilbert-style calculus it is possible to obtain a calculus for \(\vdash_{\text{BPL}}\) straightforwardly. Let us explain the method.

First of all we define two maps \(\text{der}_0, \text{der}_1\) from the set of Hilbert-style proper rules over \(\mathcal{L}_{\text{int}}\) to the set of Hilbert-style rules over \(\mathcal{L}_{\text{int}}\). Consider a Hilbert-style proper rule \(\langle \Gamma, \varphi \rangle\) over \(\mathcal{L}_{\text{int}}\). We define \(\text{der}_0(\langle \Gamma, \varphi \rangle)\) as the Hilbert-style axiom \(\langle \emptyset, (\land \Gamma) \rightarrow \varphi \rangle\); and \(\text{der}_1(\langle \Gamma, \varphi \rangle)\) is defined as \(\langle \{\gamma_1 \lor p_k, \ldots, \gamma_n \lor p_k\}, \varphi \lor p_k\rangle\) where \(\Gamma = \{\gamma_1, \ldots, \gamma_n\}\) and \(p_k\) is the first variable that does not appear in \(\{\gamma_1, \ldots, \gamma_n, \varphi\}\). If a Hilbert-style proper rule is called \((R)\) then in this paper we use the names \((\text{Ax } R)\) and \((R^*)\) to refer to \(\text{der}_0(\langle \Gamma, \varphi \rangle)\) and \(\text{der}_1(\langle \Gamma, \varphi \rangle)\) respectively\(^b\).

\(^b\)This method has been used for some of the names for the calculus \(\mathcal{H}_{\text{BFC}}\) on page 21; \((7^*), (8^*), (9^*), (N^*), (1^*), (\text{Ax } 7), (\text{Ax } 8), (\text{Ax } 9), (\text{Ax } 10), (\text{Ax } 11), (\text{Ax } N)\). The rules \((7), (8), (9), (1), (N)\) have been defined as Gentzen-style axioms, but it is clear that we can think of them as Hilbert-style proper rules (see these rules on page 29).
Suppose $\mathcal{H}$ is an arbitrary Hilbert-style calculus over $\mathcal{L}_{\text{int}}$. The sequence $(\mathcal{H}^n : n \in \omega)$ is defined recursively as $\mathcal{H}^0 = \mathcal{H}$, ii) $\mathcal{H}^{n+1} = \mathcal{H}^n \cup \{\text{der}_i(r) : i \in \{0, 1\}, r \text{ is a Hilbert-style proper rule in } \mathcal{H}^n\}$. And the Hilbert-style calculus $\mathcal{H}^{\text{der}}$ over $\mathcal{L}_{\text{int}}$ is defined as $\bigcup \{\mathcal{H}^n : n \in \omega\}$. It is clear that $\mathcal{H}^{\text{der}}$ is the closure of $\mathcal{H}$ under the maps $\text{der}_0$ and $\text{der}_1$. That is, $\mathcal{H}^{\text{der}}$ is the smaller extension of $\mathcal{H}$ such that if $r$ is a Hilbert proper rule in $\mathcal{H}^{\text{der}}$ then $\text{der}_0(r) \in \mathcal{H}^{\text{der}}$ and $\text{der}_1(r) \in \mathcal{H}^{\text{der}}$.

Take $\mathcal{H}_1$ as the Hilbert-style calculus over $\mathcal{L}_{\text{int}}$ given by the Hilbert-style rules in (P2) and in the hypotheses of lemmas 7 and 8. A moment of reflection allows us to say that $(\mathcal{H}_1)^{\text{der}}$ is a Hilbert-style calculus when (P1), (P2) and (P3) hold (for the last property use lemmas 7 and 8). Therefore, $(\mathcal{H}_1)^{\text{der}}$ is a Hilbert-style calculus for $\vdash_{\text{BPL}}$. The problem is that this is an infinite Hilbert-style calculus. How can we replace this calculus with a finite one? To answer this question let us see how our Hilbert-style calculus $\mathcal{H}_{\text{BPC}}$ has been obtained.

It is clear that if we consider a set of Hilbert-style rules $\mathcal{H}_2$ over $\mathcal{L}_{\text{int}}$ satisfying (P1) then $(\mathcal{H}_1 \cup \mathcal{H}_2)^{\text{der}}$ is also a Hilbert-style calculus for $\vdash_{\text{BPL}}$. Now, our idea is to look for a set of rules $\mathcal{H}_2$ satisfying (P1) such that for a certain $n \in \omega$ the Hilbert-style calculus $(\mathcal{H}_1 \cup \mathcal{H}_2)^n$ gives the same consequence relation as the Hilbert-style calculus $(\mathcal{H}_1 \cup \mathcal{H}_2)^{\text{der}}$. This holds if we consider the Hilbert-style calculus $\mathcal{H}_{\text{CPC}}$ described in Example 1. That is, it is easy to verify that $(\mathcal{H}_1 \cup \mathcal{H}_{\text{CPC}})^1$ gives the same consequence relation as $(\mathcal{H}_1 \cup \mathcal{H}_{\text{CPC}})^{\text{der}}$; the trick is based on the fact that we know $(p_0 \lor p_1) \lor p_2 \vdash_{\mathcal{H}_{\text{CPC}}} p_0 \lor (p_1 \lor p_2)$. Therefore, $(\mathcal{H}_1 \cup \mathcal{H}_{\text{CPC}})^1$ is a Hilbert-style calculus for $\vdash_{\text{BPL}}$ and is finite. This Hilbert-style calculus can be simplified. Some of its rules can be obtained from the others, and so can be removed. If we do so we obtain our calculus $\mathcal{H}_{\text{BPC}}$.

Let us see that the ideas explained before are adequate to solve the problem. That is, let us see that the Hilbert-style calculus $\mathcal{H}_{\text{BPC}}$ verifies (P1), (P2) and (P3).

**Lemma 10** $\mathcal{H}_{\text{BPC}}$ verifies (P1).

**Proof:** Straightforward. $\blacksquare$

**Lemma 11**

1. If $\Gamma \vdash_{\text{CPL}} \varphi$ with $\Gamma \cup \{\varphi\} \subseteq \text{Fm}\mathcal{L}_{\text{NV}}$ then $\Gamma \vdash_{\mathcal{H}_{\text{BPC}}} \varphi$

2. $\mathcal{H}_{\text{BPC}}$ verifies (P2).

**Proof:** 1) It is obvious using Example 1, $\mathcal{H}_{\text{CPC}} \subseteq \mathcal{H}_{\text{BPC}}$ and $\text{Fm}\mathcal{L}_{\text{NV}} \subseteq \text{Fm}\mathcal{L}_{\text{int}}$.

2) Since the first point holds, it only remains to prove that $\bot \vdash_{\mathcal{H}_{\text{BPC}}} p_0$, $(p_0 \to p_1) \land (p_1 \to p_2) \vdash_{\mathcal{H}_{\text{BPC}}} p_0 \to p_2$, $(p_0 \to p_1) \land (p_0 \to p_2) \vdash_{\mathcal{H}_{\text{BPC}}} p_0 \to (p_1 \land p_2)$, $(p_0 \to p_2) \land (p_1 \to p_2) \vdash_{\mathcal{H}_{\text{BPC}}} (p_0 \lor p_1) \to p_2$, $p_0 \vdash_{\mathcal{H}_{\text{BPC}}} \top$, $p_0 \vdash_{\mathcal{H}_{\text{BPC}}} \bot p_0$. Using (Ax 1) it is clear that $p_0 \vdash_{\mathcal{H}_{\text{BPC}}} \bot$.

All the other cases are proved in the same way. Suppose we are in a case $\varphi \vdash_{\mathcal{H}_{\text{BPC}}} \psi$. Then, it holds that $(\varphi \lor p_k, \psi \lor p_k) \in \mathcal{H}_{\text{BPC}}$ where $p_k$ is the first variable that doesn’t appear in $\{\varphi, \psi\}$. Thus, $\varphi \lor \varphi \vdash_{\mathcal{H}_{\text{BPC}}} \psi \lor \varphi$ and $\varphi \lor \psi \vdash_{\mathcal{H}_{\text{BPC}}} \psi \lor \psi$ by the invariance under substitutions. Therefore, using also (Ru4), (Ru5) and (Ru9),

$\varphi \vdash_{\mathcal{H}_{\text{BPC}}} \varphi \lor \varphi \vdash_{\mathcal{H}_{\text{BPC}}} \psi \lor \varphi \vdash_{\mathcal{H}_{\text{BPC}}} \varphi \lor \psi \vdash_{\mathcal{H}_{\text{BPC}}} \psi \lor \psi \vdash_{\mathcal{H}_{\text{BPC}}} \psi \lor \psi$

---

1A reason for considering this set of Hilbert-style rules is the fact that the $(\land, \lor)$-restriction of $\vdash_{\text{BPL}}$ coincides with the $(\land, \lor)$-restriction of $\vdash_{\text{CPL}}$ for which $\mathcal{H}_{\text{CPC}}$ is a Hilbert-style calculus.
Lemma 12 \( \mathcal{H}_{BPC} \) is a model of \((DT_0)\).

**Proof:** We know that the Hilbert-style calculus \( \mathcal{H}_{BPC} \) satisfies the hypotheses of Lemma 7 (remember Lemma 11). Therefore, we only need to see that for every Hilbert-style proper rule \((\Gamma, \varphi) \in \mathcal{H}_{BPC}\) it holds \( \emptyset \vdash_{\mathcal{H}_{BPC}} (\varphi \rightarrow \varphi) \). Let us look at all the Hilbert-style proper rules in \( \mathcal{H}_{BPC} \).

Suppose we consider a Hilbert-style rule \((X)\) in \{\((Ru1), (Ru2), (Ru4), (Ru5), (Ru8)\)\}. Then, it has the form \( \{\gamma, \varphi\} \). By the Hilbert-style axiom \((Ax \ X)\) we obtain \( \emptyset, \varphi \rightarrow \varphi \in \mathcal{H}_{BPC} \). Thus, \( \emptyset \vdash_{\mathcal{H}_{BPC}} \gamma \rightarrow \varphi \).

We consider the rule \((Ru3)\). We know \( \emptyset \vdash_{\mathcal{H}_{BPC}} (p_0 \land p_1) \rightarrow (p_0 \land p_1) \) and \( \emptyset \vdash_{\mathcal{H}_{BPC}} (p_1 \land p_0) \rightarrow (p_0 \land p_1) \). From this we obtain what we want.

For the rule \((Ru6)\) it is necessary to see that \( \emptyset \vdash_{\mathcal{H}_{BPC}} (p_0 \lor (p_1 \lor p_2)) \rightarrow ((p_0 \lor p_1) \lor p_2) \). An easy remark\(^1\) shows that \( \emptyset \vdash_{\mathcal{H}_{BPC}} p_0 \rightarrow ((p_0 \lor p_1) \lor p_2) \), \( \emptyset \vdash_{\mathcal{H}_{BPC}} p_1 \rightarrow ((p_0 \lor p_1) \lor p_2) \) and \( \emptyset \vdash_{\mathcal{H}_{BPC}} p_2 \rightarrow ((p_0 \lor p_1) \lor p_2) \). And by Lemma 11(2) we know that \( (p_0 \rightarrow p_2) \land (p_1 \rightarrow p_2) \vdash_{\mathcal{H}_{BPC}} (p_0 \lor p_1) \rightarrow p_2 \); i.e., \( \varphi \rightarrow \delta \land (\psi \rightarrow \delta) \vdash_{\mathcal{H}_{BPC}} (\varphi \lor \psi) \rightarrow \delta \) for every \( \varphi, \psi, \delta \in \text{Fm}_{\text{L}_\text{int}} \). Using the last two sentences we obtain what we want.

The cases in \{\((Ru7), (Ru9), (7^*), (8^*), (9^*), (1^*), (N^*)\)\} are treated in the same way than \((Ru6)\); i.e., we observe that the premise is a disjunction and we use the fact that \( \varphi \rightarrow \delta \land (\psi \rightarrow \delta) \vdash_{\mathcal{H}_{BPC}} (\varphi \lor \psi) \rightarrow \delta \) for every \( \varphi, \psi, \delta \in \text{Fm}_{\text{L}_\text{int}} \).

Lemma 13 \( \mathcal{H}_{BPC} \) is a model of \((\lor \Rightarrow)\).

**Proof:** We know that the Hilbert-style calculus \( \mathcal{H}_{BPC} \) satisfies the hypotheses of Lemma 8 (remember Lemma 11(1)). Therefore, we only need to see that for every Hilbert-style proper rule \( \{\gamma_1, \ldots, \gamma_n, \varphi\} \in \mathcal{H}_{BPC} \) \( n \geq 1 \) it holds that \( \{\gamma_1 \lor p_k, \ldots, \gamma_n \lor p_k\} \vdash_{\mathcal{H}_{BPC}} \varphi \lor p_k \) where \( p_k \) is the first variable that doesn’t appear in \( \{\gamma_1, \ldots, \gamma_n, \varphi\} \). Let us examine all the Hilbert-style proper rules in \( \mathcal{H}_{BPC} \).

Suppose \( \{\gamma_1, \ldots, \gamma_n, \varphi\} \in \mathcal{H}_{BPC} \) and \( p_k \) is the first variable that does not appear in \( \{\gamma_1, \ldots, \gamma_n, \varphi\} \). Then, \( \{\gamma_1 \lor p_k, \ldots, \gamma_n \lor p_k\} \vdash_{\text{CPL} \cdot} \varphi \lor p_k \). By Lemma 11(1) we conclude that \( \{\gamma_1 \lor p_k, \ldots, \gamma_n \lor p_k\} \vdash_{\mathcal{H}_{BPC}} \varphi \lor p_k \).

Suppose we consider a rule in \{\{(7^*), (8^*), (9^*), (1^*), (N^*)\}\}. Then, it has the form \( \{\gamma \lor p_j\} \lor (\varphi \lor p_j) \) for some \( \gamma, \varphi \in \text{Fm}_{\text{L}_\text{int}} \) and \( p_j \in \text{Var} \) which does not appear in \( \{\gamma, \varphi\} \). Let \( p_k \) be the first variable that does not appear in \( \{\gamma \lor p_j\} \lor (\varphi \lor p_j) \). We must prove that \( (\gamma \lor p_j) \lor (\varphi \lor p_j) \vdash_{\mathcal{H}_{BPC}} (\varphi \lor p_j) \lor p_k \). By Lemma 11(1) and the invariance under substitutions it is known that \( (\gamma \lor p_j) \lor p_k \vdash_{\mathcal{H}_{BPC}} (\gamma \lor (p_j \lor p_k)) \) and \( (\varphi \lor p_j) \lor p_k \vdash_{\mathcal{H}_{BPC}} (\varphi \lor (p_j \lor p_k)) \). Therefore, it only remains to prove \( \gamma \lor (p_j \lor p_k) \vdash_{\mathcal{H}_{BPC}} (\varphi \lor (p_j \lor p_k)) \). This is an immediate consequence of the Hilbert-style rule \( \{\gamma \lor p_j\}, \varphi \lor p_j \) that we are considering (use the invariance under substitutions).

Theorem 14

1. \( \mathcal{H}_{BPC} \) is a finite Hilbert-style calculus for \( \vdash_{\text{BPL}} \).

2. \( \mathcal{H}_{BPC} \cup \{\text{Ax L}\} \) is a finite Hilbert-style calculus for \( \vdash_{\text{FPL}} \).

---

\(^1\)Lemma 11(2) says that \( (p_0 \rightarrow p_1) \land (p_1 \rightarrow p_2) \vdash_{\mathcal{H}_{BPC}} p_0 \rightarrow p_2 \). Therefore, \( (\varphi \rightarrow \psi) \land (\psi \rightarrow \delta) \vdash_{\mathcal{H}_{BPC}} (\varphi \rightarrow \delta) \) for every \( \varphi, \psi, \delta \in \text{Fm}_{\text{L}_\text{int}} \).
Proof: The first point is deduced from Proposition 6 and Lemmas 10, 11, 12, 13. The second point is a consequence of Corollary 9.

A similar trick is used by J. Rebagliato and V. Verdú in [RV94] to obtain a Hilbert-style calculus over \((\land, \lor, \neg)\) for the \((\land, \lor, \neg)\)-restriction of \(\vdash_{\text{IPL}}\).

In the Hilbert-style calculus \(\mathcal{H}_{BP C}\) we have the rules \((7^*), (8^*), (9^*), (1^*), (N^*)\) in place of the simpler Hilbert-style rules

\[
\begin{align*}
(7) & \quad \frac{(p_0 \to p_1) \land (p_1 \to p_2)}{(p_0 \to p_2)}, \\
(8) & \quad \frac{(p_0 \to p_1) \land (p_0 \to p_2)}{p_0 \to (p_1 \land p_2)}, \\
(9) & \quad \frac{(p_0 \to p_2) \land (p_1 \to p_2)}{(p_0 \lor p_1) \to p_2}, \\
(1) & \quad \frac{\bot}{p_0}, \\
(N) & \quad \frac{p_0}{\Box p_0}.
\end{align*}
\]

We cannot simultaneously replace our five rules in \(\mathcal{H}_{BP C}\) with these simpler ones. In fact, if we take \((R)\) as one of the rules in \(\{(7), (1), (N)\}\) then the Hilbert-style calculus over \(\mathcal{L}_{\text{int}}\) obtained from \(\mathcal{H}_{BP C}\) replacing the Hilbert-style rule \((R^*)\) with \((R)\) is not a Hilbert-style calculus for \(\vdash_{\text{BPL}}\). If this holds for the rules in \(\{(8), (9)\}\) is still an open question. Let us see the proof for the case \((N)\). The same method, changing the algebra, can be used for the cases \((7)\) and \((1)\).

**Proposition 15** Let \(\mathcal{H}\) be the Hilbert-style calculus over \(\mathcal{L}_{\text{int}}\) obtained from \(\mathcal{H}_{BP C}\) replacing the Hilbert-style rule \((N^*)\) with \((N)\). Then, \(\mathcal{H}\) is not a Hilbert-style calculus for \(\vdash_{\text{BPL}}\).

**Proof:** We define the \(\mathcal{L}_{\text{int}}\)-algebra \(A = \langle \{0, a, b, 1\}, \land, \lor, \to, 0 \rangle\) where \(\land, \lor\) are the operations of the lattice in the picture, and \(\to\) is the operation defined below.

\[
\begin{array}{c|cccc}
    & 0 & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
ar & 1 & 1 & 1 & 1 \\
b & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 1
\end{array}
\]

An easy induction proves that if \(\Gamma \vdash_{\mathcal{H}} \varphi\) then

for every homomorphism \(h\) of \(\text{Fm}\mathcal{L}_{\text{int}}\) into \(A\) such that \(h[\Gamma] \subseteq \{1\}\) holds \(h(\varphi) = 1\).

Using that \(a \lor b = 1\) and \(\Box a \lor b = 0 \lor b = b \neq 1\) it is clear that \(p_0 \lor p_1 \not\vdash_{\mathcal{H}} \Box p_0 \lor p_1\). As \(p_0 \lor p_1 \vdash_{\text{BPL}} \Box p_0 \lor p_1\) we conclude that \(\mathcal{H}\) is not a calculus for \(\vdash_{\text{BPL}}\).

**References**


\footnote{In the terminology of Abstract Algebric Logic this means that the matrix \(<A, \{1\}>\) is a model for \(\vdash_{\text{BPL}}\); two good references in this subject are [Cze01, FJ96].}


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