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LP formulation for regional-optimal bounds

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Abstract

This technical report is written as a support material for the reader in [1], detailing the transformations to simplify an initial program to compute a tight bound for a $C$-optimal assignment into a linear program (LP).

In this technical report we show how the initial program to compute a tight bound for a $C$-optimal assignment, can be transformed into a linear program. Our departure point is the following program.

\[
\text{Find } R, x^C \text{ and } x^* \text{ that minimize } R(x^C) \text{ subject to } x^C \text{ being a } C\text{-optimal for } R
\]

We start by analyzing what exactly means saying that $x^C$ is $C$-optimal. The condition can be expressed as: for each $x$ inside region $C$ of $x^C$ we have that $R(x^C) \geq R(x)$. However, instead of considering all the assignments for which $x^C$ is guaranteed to be optimal, we consider only the subset of assignments such that the set of variables that deviate with respect to $x^C$ take the same value than in the optimal assignment. If we restrict to this subset of assignments, then each neighborhood covers a $2^{|C^\alpha|}$ assignments, one for each subset of variables in the neighborhood. Let $2^{C^\alpha}$ stand for the set of all subsets of the neighborhood $C^\alpha$. Then for each $A^k \subseteq 2^{C^\alpha}$ we can define an assignment $x^{\alpha_k}$ such that for every variable $x_i$ in a relation completely covered by $A^k$ we have that $x_i^{\alpha_k} = x_i^*$, and
for every variable $x_i$ that is not covered at all by $A^k$ we have that $x_i^C = x_i^*$. Then, we can write the value of $x_i^C$ as

$$R(x_i^C) = \sum_{S \in T(A^k)} S(x_i^C) + \sum_{S \in P(A^k)} S(x_i^a) + \sum_{S \in N(A^k)} S(x_i^a) \quad (1)$$

Now, the definition of $C$-optimal can be expressed as $A^k \in \{2C^\alpha_k | C^\alpha_k \in C\}$:

$$R(x^C) \geq R(x^a) = \sum_{S \in T(A^k)} S(x_i^C) + \sum_{S \in P(A^k)} S(x_i^a) + \sum_{S \in N(A^k)} S(x_i^a) \quad (2)$$

that, by setting partially covered relations to the minimum possible reward (0 assuming non-negative rewards), results in:

$$R(x^C) \geq \sum_{S \in T(A^k)} S(x_i^a) + \sum_{S \in N(A^k)} S(x_i^a) \quad \forall A^k \in \{2C^\alpha_k | C^\alpha_k \in C\} \quad (3)$$

where $T(A^k)$ is the set of completely covered relations, $P(A^k)$ the set of partially covered relations and $N(A^k)$ the set of relations not covered at all.

Given the definition of $C$-optimality of equation 3, we can proceed on specifying the linear programming formulation of the initial problem. First, we assume that $x^C = (0, \ldots, 0)$ and $x^* = (1, \ldots, 1)$ where 0 and 1 stand for the first and second value in each variable domain. This assumption can be made without losing generality. Second, we create two real positive variables for each relation $S \in R$, one representing $S(x^C)$, noted as $x_S$, and another one representing $S(x^*)$, noted as $y_S$. $x^C$ in $C^\alpha R(x^C) \geq R(x^a)$ using a single equation, concretely the one that sets every variable in $C^\alpha$ to 1.

Third, to obtain the LP we can normalize the rewards of our optimal to add up to one ($\sum_{S \in R} y_S = 1$). This is a common procedure for turning a linear fractional program into a linear program.

Fourth, we add all the constraints from equation 3, to guarantee the optimality of $x^C$.

The linear program resulting from these is as follows:

minimize $\sum_{S \in R} x_S$
subject to
$\sum_{S \in R} y_S = 1$
and for each $A^k \in \{2C^\alpha_k | C^\alpha_k \in C\}$ also subject to
$\sum_{S \in R} x_S \geq \sum_{S \in T(A^k)} y_S + \sum_{S \in N(A^k)} x_S$

where

- $x$ is a vector of positive real numbers representing the values for each relation of the C-optimal
- $y$ is a vector of real numbers representing the values for each relation of the optimal of the problem
- $T(A^k)$ contains the relations completely covered by $A^k$, and
- $N(A^k)$ contains the relations that are not covered by $C^\alpha$ at all.
References